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On the Willems closure with respect to $\mathcal{W}_{\mathfrak{s}}$

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We determine when a submodule is Willems with respect to the space of distributions that are tempered in the spatial direction.

1. Introduction

The behavioural theory of Willems exploits the correspondence between the algebraic properties of the module describing the behaviour and the properties of the behaviour. For excellent introductions to the behavioural theory in the 1 D case and the *n* D case, we refer the reader to Polderman & Willems (1998) and Pillai & Shankar (1998), respectively.

As opposed to the case of 1 D linear dynamical systems corresponding to a set of linear ODEs with constant coefficients, in the *n* D case there is a greater variety of possible solution spaces and the correspondence between modules and the associated behaviours may not be bijective: indeed, it depends on the solution space considered. There exists a bijective correspondence between modules and behaviours if one considers the space of smooth functions or distributions, and this was established in Oberst (1990). (In the 1 D case this was known, and it is the content of Theorem 3.6.2 on page 100 of Polderman & Willems, 1998.) However, this bijective correspondence does not go through for several classical spaces, such as the space of tempered distributions, $\mathcal{S}'(\mathbb{R}^{\mathsf{n}})$. This naturally brings one to the notions of a Willems module and the Willems closure of a module with respect to a given solution space, which were first introduced in the works of Pillai & Shankar (1998) and Shankar (1999, 2001). This is analogous to the definition of the radical of an ideal in a polynomial ring and the correspondence between affine varieties and radical ideals.

Roughly speaking, the notion of a Willems module can be explained as follows. Start with a given set of equations and find the corresponding behaviour in a certain solution space, say $\mathcal{W}$. Now find all the equations that this behaviour satisfies. If this set of equations turns out to be the same set one started off with, then the original set is said to be Willems with respect to the solution space under consideration. The Willems submodules play an important role in the behavioural theory and furthermore, from a purely mathematical point of view, the determination of Willems submodules is the Nullstellensatz for systems of PDEs, the analogue of Hilbert Nullstellensatz, where as opposed to looking at the zeros in $\mathbb{C}^{\mathsf{n}}$ of a set of polynomial equations, one now looks at the solutions of a set of linear PDEs with constant coefficients.

In Shankar (1999), it is determined when a module is Willems with respect to the Schwartz space of tempered distributions. In this paper, following Shankar (1999), we perform a similar calculation for another space, which we call ‘the space of distributions

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which are tempered in the spatial directions', and this space is denoted by $\mathcal{W}_{\mathfrak{s}}$. There are several reasons for being interested in the space $\mathcal{W}_{\mathfrak{s}}$, and these are discussed in detail in Section 2.

The organization of the paper is as follows. In Section 2 we introduce the space $\mathcal{W}_{\mathfrak{s}}$ by first giving motivating reasons that lead one to this space and subsequently defining it and giving examples of the other spaces it encompasses. Section 3 recalls some of the definitions from the behavioural theory of Willems. In particular, we recollect the notions of Willems module and Willems closure of a submodule with respect to a given solution space. We fix some algebraic notation in Section 4 and also, for the sake of completeness, we give a few algebraic definitions that may not be well known in the engineering community. Finally, in the last section, we prove our main theorem and consider a few examples.

2. The space $\mathcal{W}_{\mathfrak{s}}$

In this section, one might find that at certain instances the writing shows too little regard for concision—for which I apologize. I have made a point, rather, of explicitly formulating the thoughts that lie in the background of studying the space $\mathcal{W}_{\mathfrak{s}}$.

Motivation for the space $\mathcal{W}_{\mathfrak{s}}$

The diffusion equation

$$\left[\frac{\partial}{\partial t} - \left(\frac{\partial}{\partial x}\right)^2\right] w = 0$$

models the physical phenomenon of the diffusion of heat or the diffusion of matter. For example, in the case of diffusion of heat, one can imagine a hot rod which cools down as time progresses and the temperature satisfies the diffusion equation. Similarly, in the case of diffusion of matter, one can imagine a bucket of water with a drop of ink added to it; the ink diffuses in the water as time passes, and the density of ink satisfies the diffusion equation.

For either of these examples, if we assume the solution space to be the space of distributions $\mathcal{D}'(\mathbb{R}^2)$, then we run into difficulties regarding the notion of time autonomy[†]. Indeed, according to Theorem 3.4 in Sasane *et al.* (2002), the distributional behaviour corresponding to the diffusion equation is not time autonomous (since $\deg[p(\eta, \xi)] = \deg[\xi - \eta^2] = 2 \neq 1 = \deg[\xi] = \deg[p(0, \xi)]$), in contrast to our physical intuition. One expects, in the case of heat diffusion, that if we have a cold rod up to the time instant zero, and we do nothing to it, then in the future there cannot be a non-zero temperature profile. Similarly, in the case of diffusion of matter, non-time-autonomy would imply that if we have a bucket of clear water up to time zero, and we do nothing to it, there is still some ink in it in the future, whose density evolves in time. We now claim that this anomaly arises since we have assumed the solution space to be too general: in particular, we have not

[†] We quickly recall that the set of solutions of a PDE is said to be time autonomous if the only solution with zero past is the trivial solution.

imposed any growth restriction on the trajectories in the spatial direction. But before we elaborate on this, let us consider the following example of a trajectory which is zero in the past, non-zero in the future and which satisfies the diffusion equation. Let $w \in C^{\infty}(\mathbb{R}^2, \mathbb{C})$ be given by

$$w(x,t) = \sum_{k=0}^{\infty} f^{(k)}(t) \frac{x^{2k}}{(2k)!}, \quad -\infty < x, t < \infty, \tag{1}$$

with

$$f(t) = \begin{cases} e^{-\frac{1}{t^2}} & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t \leqslant 0. \end{cases}$$

Then it can be shown that (1) converges uniformly and it satisfies the diffusion equation. Furthermore, it can also be shown that for each $t > 0$, there do not exist constants M and A such that $|w(x,t)| \leqslant M\mathrm{e}^{Ax^2}$. (This example was constructed by A. N. Tychonov; see for instance Example 2 on pages 50–51 of Hellwig, 1964.) We claim that this is the reason for the lack of conformity with our physical intuition concerning time-autonomy. Indeed, in the case of the diffusion of heat one expects that at each point of time the temperature profile is an element in $L_{\infty}(\mathbb{R})$. Similarly, in the case of diffusion of matter, one expects that by the law of conservation of mass, the total amount of ink in the water remains the same, that is, the density of ink at each point of time is an integrable function (with a constant L_1-norm). And clearly if we have growth faster than e^{Ax^2}, then we fall outside either of the above solution spaces. So we search for the 'right' solution space; one which encompasses most natural solution spaces associated with PDEs, but excludes certain pathological solutions, such as the one demonstrated above. Furthermore, it is desirable that our solution space possess features that enable one to prove useful algebraic theorems in the context of linear control theory for n D systems, as pioneered in Pillai & Shankar (1998).

We purport that the space $\mathcal{W}_{\mathfrak{s}}$ which we define below is one such.

The space $\mathcal{W}_{\mathfrak{s}}$

If $\mathcal{V}_1$ and $\mathcal{V}_2$ are topological vector spaces, then we denote by $\mathcal{L}(\mathcal{V}_1, \mathcal{V}_2)$ the space of continuous linear maps from $\mathcal{V}_1$ to $\mathcal{V}_2$. For example, $\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{D}'(\mathbb{R}))$ denotes the space of continuous linear maps from $\mathcal{D}(\mathbb{R})$ to $\mathcal{D}'(\mathbb{R})$. For a short introduction about vector-valued distributions, we refer the reader to Carroll (1969). If $T \in \mathcal{D}'(\mathbb{R}^2)$, then one can associate a continuous linear map $\iota T : \mathcal{D}(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ as follows: $\langle (\iota T)(\varphi), \psi \rangle = \langle T, \psi \otimes \varphi \rangle$ for all $\varphi \in \mathcal{D}(\mathbb{R})$ and $\psi \in \mathcal{D}(\mathbb{R})$. We recall below the Schwartz kernel theorem (see for instance page 128, Theorem 5.2.1, Hörmander, 1990).

LEMMA 2.1 (The Schwartz kernel theorem.) The map $T \mapsto \iota T$ is an isomorphism from $\mathcal{D}'(\mathbb{R}^2)$ onto $\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{D}'(\mathbb{R}))$.

We note that $\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R})) \subset \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{D}'(\mathbb{R}))$ and $\iota^{-1}\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R})) \subset \mathcal{D}'(\mathbb{R}^2)$. In the sequel, we will denote the space $\iota^{-1}\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))$ by $\mathcal{W}_{\mathfrak{s}}$, and throughout this paper, we will study the behavioural trajectories that lie in this space $\mathcal{W}_{\mathfrak{s}}$. The space $\mathcal{W}_{\mathfrak{s}}$ is closed with respect to partial differentiation with respect to time and with respect to the

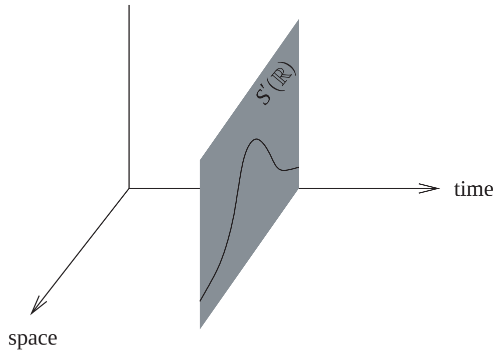

FIG. 1. The space $\mathcal{W}_{\mathfrak{s}}$.

spatial variable. The space $\mathcal{W}_{\mathfrak{s}}$ is furthermore isomorphic to the completed projective (or[†] epsilon) topological tensor product of the spaces $\mathcal{D}'(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$:

$$\mathcal{W}_{\mathfrak{s}} \simeq \mathcal{D}'(\mathbb{R}) \widehat{\otimes}_{\epsilon} \mathcal{S}'(\mathbb{R}) \simeq \mathcal{D}'(\mathbb{R}) \widehat{\otimes}_{\pi} \mathcal{S}'(\mathbb{R}).$$

Roughly speaking, one can think of $\mathcal{W}_{\mathfrak{s}}$ as comprising those maps for which if one freezes a time instant, then the resulting map (along the spatial axis) is in $\mathcal{S}'(\mathbb{R})$ (see Fig. 1). So a 'wild' growth in the spatial direction is ruled out. The space $\mathcal{W}_{\mathfrak{s}}$ is called *the space of distributions on* $\mathbb{R}^2$ *that are temperate in the spatial direction*. Finally, we mention that the choice of the notation $\mathcal{W}_{\mathfrak{s}}$ is motivated by the fact that the subscript $\mathfrak{s}$ serves the dual purpose of referring to *space* and *Schwartz*: in the *spatial* direction, one has a profile in the *Schwartz* space of tempered distributions $\mathcal{S}'$. The capital $\mathcal{W}$, on the other hand, is used simply because it is the set comprising little w's, the traditional choice of denoting trajectories in a behaviour.

Spaces contained in $\mathcal{W}_{\mathfrak{s}}$

Since all the L_p-spaces, $L_p(\mathbb{R})$, for $1 \leqslant p \leqslant \infty$, can be identified with subspaces of $\mathcal{S}'(\mathbb{R})$ it follows that the space $\mathcal{W}_{\mathfrak{s}}$ captures the situation when the spatial profile is a function in L_p. In particular, the case of the diffusion of heat ($p = \infty$) and the diffusion of matter ($p = 1$) are covered. Also, the case when the spatial profile is a square integrable function, is contained in $\mathcal{W}_{\mathfrak{s}}$. This is in conformity with the traditional infinite-dimensional system theoretic framework[‡] of handling some PDEs, where very often the state space is taken to be L_2 or a subspace thereof.

Moreover, since the space of compactly supported distributions $\mathcal{E}'(\mathbb{R})$ is also contained in $\mathcal{S}'(\mathbb{R})$, this includes the scenario when the physical domain is restricted, for instance when one has a heated rod of finite length and outside that length the temperature is zero.

[†] The projective tensor product topology and the epsilon tensor product topology coincide, since at least one of the two spaces (and in fact in our case, both $\mathcal{D}'(\mathbb{R})$ as well as $\mathcal{S}'(\mathbb{R})$!) is nuclear.

[‡] Here one looks at certain linear PDEs as if they were ODEs, but with an infinite-dimensional Hilbert space as the state space. See for instance Curtain & Zwart (1995).

In fact, all trajectories such that at each point of time their profile in the spatial direction is a locally integrable function with at most polynomial growth, are contained in $\mathcal{W}_{\mathfrak{s}}$.

What are examples of trajectories not in $\mathcal{W}_{\mathfrak{s}}$? Since $e^{\pm x} \notin \mathcal{S}'(\mathbb{R})$, whenever we have exponential growth in the spatial direction, the trajectory does not belong to $\mathcal{W}_{\mathfrak{s}}$. Also, certainly growth faster than e^{Ax^2} in the spatial direction is excluded and w given by (1) does not belong to $\mathcal{W}_{\mathfrak{s}}$.

Moreover, Fourier transformation in the spatial variable allows one to prove 'algebraic theorems' that characterize properties of the behaviour satisfying a set of PDEs in terms of properties of the polynomial matrix describing the behaviour (see for instance Theorems 3.1, 4.1 and 4.2 in Sasane, 2003). Here we recall the following weak version of the fundamental principle for $\mathcal{W}_{\mathfrak{s}}$ from Sasane (2003).

THEOREM 2.2 If $l \in \mathcal{W}_{\mathfrak{s}}$, $p \in \mathbb{C}[\eta] \setminus \{0\}$ and $q \in \mathbb{C}[\xi] \setminus \{0\}$, then there exists a $w \in \mathcal{W}_{\mathfrak{s}}$ such that

$$p\left(\frac{\partial}{\partial x}\right) q\left(\frac{\partial}{\partial t}\right) w = l.$$

3. Behaviours and the notion of Willems closure

We denote the polynomial ring $\mathbb{C}[\eta, \xi]$ by $\mathcal{A}$. We use the different symbols η and ξ to indicate that η corresponds to the spatial indeterminate (that is, it is replaced by $\frac{\partial}{\partial x}$ in order to obtain the corresponding differential map) and the ξ is the indeterminate corresponding to time (that is, it is replaced by $\frac{\partial}{\partial t}$, in order to obtain the corresponding differential map).

Let $\mathcal{W}$ be a subspace of $\mathcal{D}'(\mathbb{R}^2)$ that is closed under differentiation with respect to the spatial variable and time. An element $\chi = \begin{bmatrix} \chi_1 & \cdots & \chi_{\mathsf{w}} \end{bmatrix} \in \mathcal{A}^{\mathsf{w}}$ gives rise to a differential map $D_\chi : \mathcal{W}^{\mathsf{w}} \to \mathcal{W}$ as follows:

$$D_\chi \begin{bmatrix} w_1 \\ \vdots \\ w_{\mathsf{w}} \end{bmatrix} = \sum_{\mathsf{k}=1}^{\mathsf{w}} \chi_{\mathsf{k}} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) w_{\mathsf{k}}.$$

Given a submodule $\mathbf{R}$ of $\mathcal{A}^{\mathsf{w}}$, the *$\mathcal{W}$-behaviour given by* $\mathbf{R}$, denoted by the symbol $\mathfrak{B}_{\mathcal{W}}(\mathbf{R})$, is defined by

$$\mathfrak{B}_{\mathcal{W}}(\mathbf{R}) = \{w \in \mathcal{W}^{\mathsf{w}} \mid D_\chi w = 0 \text{ for all } \chi \in \mathbf{R}\}.$$

Given a $\mathcal{W}$-behaviour, say $\mathfrak{B}$, define

$$\mathfrak{R}(\mathfrak{B}) = \{\chi \in \mathcal{A}^{\mathsf{w}} \mid D_\chi w = 0 \text{ for all } w \in \mathfrak{B}\}.$$

It is clear that $\mathfrak{R}(\mathfrak{B})$ is a submodule of $\mathcal{A}^{\mathsf{w}}$. It was shown in Shankar (1999) that

$$\mathfrak{B}_{\mathcal{W}}(\mathfrak{R}(\mathfrak{B})) = \mathfrak{B}$$

for all $\mathcal{W}$-behaviours $\mathfrak{B}$. Also for any submodule $\mathbf{R}$ of $\mathcal{A}^{\mathsf{w}}$, there holds $\mathbf{R} \subset \mathfrak{R}(\mathfrak{B}_{\mathcal{W}}(\mathbf{R}))$. However, the equality $\mathfrak{R}(\mathfrak{B}_{\mathcal{W}}(\mathbf{R})) = \mathbf{R}$ does not hold in general, and it depends on the space $\mathcal{W}$ under consideration. Given any submodule $\mathbf{R}$ of $\mathcal{A}^{\mathsf{w}}$, $\mathfrak{R}(\mathfrak{B}_{\mathcal{W}}(\mathbf{R})) = \mathbf{R}$, if $\mathcal{W}$ is C^{∞} or $\mathcal{D}'$ (see Oberst, 1990). If $\mathcal{W} = \mathcal{D}(\mathbb{R}^2)$ or $\mathcal{S}(\mathbb{R}^2)$, then the equality $\mathfrak{R}(\mathfrak{B}_{\mathcal{W}}(\mathbf{R})) = \mathbf{R}$ does not hold for all submodules $\mathbf{R}$ of $\mathcal{A}^{\mathsf{w}}$. This motivates the following definition.

If $\mathfrak{R}(\mathfrak{B}_{\mathcal{W}}(\mathbf{R})) = \mathbf{R}$, then the module $\mathbf{R}$ is said to be *Willems with respect to* $\mathcal{W}$. This terminology has been used earlier, for example in Pillai & Shankar (1998). The module $\mathfrak{R}(\mathfrak{B}_{\mathcal{W}}(\mathbf{R}))$ is called the *Willems closure of* $\mathbf{R}$ *with respect to* $\mathcal{W}$. This definition is analogous to the classical definition of a radical ideal in algebraic geometry.

The following example illustrates that not every submodule of $\mathcal{A}^{\mathsf{w}}$ is Willems with respect to $\mathcal{W}_{\mathfrak{s}}$.

EXAMPLE If we take $\mathbf{R} = \langle 1 + \eta \rangle$, then the $\mathcal{W}_{\mathfrak{s}}$-behaviour given by $\mathbf{R}$ is zero, and so the set of all annihilators of this $\mathcal{W}_{\mathfrak{s}}$-behaviour is the full module $\mathbb{C}[\eta, \xi]$, which is not equal to $\langle 1 + \eta \rangle$.

Next we give an example of a module that is Willems with respect to $\mathcal{W}_{\mathfrak{s}}$.

EXAMPLE We show that the $\mathbb{C}[\eta, \xi]$-module $\langle \xi - \eta^2 \rangle$ is Willems with respect to $\mathcal{W}_{\mathfrak{s}}$. If w is an element in the $\mathcal{W}_{\mathfrak{s}}$-behaviour, we have

$$\frac{\partial}{\partial t} w = \left(\frac{\partial}{\partial x} \right)^2 w$$

and so if $q = q_0 + \cdots + q_{\mathsf{N}} \xi^{\mathsf{N}} \in \mathbb{C}[\eta, \xi]$ is such that $D_q w = 0$, where $q_0, \ldots, q_{\mathsf{N}} \in \mathbb{C}[\eta]$, then we have

$$0 = q_0 \left(\frac{\partial}{\partial x} \right) w + \cdots + q_{\mathsf{N}} \left(\frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} \right)^{\mathsf{N}} w = q_0 \left(\frac{\partial}{\partial x} \right) w + \cdots + q_{\mathsf{N}} \left(\frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial x} \right)^{2\mathsf{N}} w. \tag{2}$$

Define $\hat{w} \in \mathcal{D}'(\mathbb{R}^2)$ by

$$\langle \hat{w}, \psi \otimes \varphi \rangle = \langle \mathcal{F}[(\iota w)(\varphi)], \psi \rangle, \quad \text{for } \varphi, \psi \in \mathcal{D}(\mathbb{R}),$$

where $\mathcal{F} : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ denotes the Fourier transformation. (By the approximation lemma quoted in Sasane (2003) or Exercise 11 on page 56 of Carroll (1969), it follows that the above defines a distribution on $\mathbb{R}^2$.) Thus from (2), we have

$$\left[q_0(2\pi i y) + \cdots + q_{\mathsf{N}}(2\pi \mathrm{i} y)(2\pi \mathrm{i} y)^{2\mathsf{N}} \right] \hat{w} = 0.$$

Let us define

$$\alpha(y) = q_0(2\pi \mathrm{i} y) + \cdots + q_{\mathsf{N}}(2\pi \mathrm{i} y)(2\pi \mathrm{i} y)^{2\mathsf{N}}$$

and let $R > 0$ be large enough such that the roots of α are all contained in the ball $B(0, R) = \{s \in \mathbb{C} \mid |s| < R\}$. Clearly the non-zero trajectory

$$w_0 = \mathrm{e}^{2\pi \mathrm{i} R x} \otimes \mathrm{e}^{-4\pi^2 R^2 t} \in \mathcal{W}_{\mathfrak{s}}$$

belongs to the behaviour. Furthermore,

$$\widehat{w_0} = \delta_R(y) \otimes \mathrm{e}^{-4\pi^2 R^2 t}$$

satisfies $0 = \alpha(y)\widehat{w_0} = \alpha(R)\widehat{w_0}$. Since $\alpha(R) \neq 0$, we have $\widehat{w_0} = 0$, which contradicts the fact that $w_0 \neq 0$. Consequently the polynomial $\alpha \in \mathbb{C}[y]$ must be zero. Consider the unique ring homomorphism $\Theta : \mathbb{C}[\eta, \xi] \to \mathbb{C}[y]$ such that $\xi \mapsto (2\pi \mathrm{i} y)^2$ and $\eta \mapsto 2\pi \mathrm{i} y$. We claim that $\ker \Theta = \langle \xi - \eta^2 \rangle$. Clearly, $\langle \xi - \eta^2 \rangle \subset \ker(\Theta)$. To show the reverse inclusion, let $r \in \ker(\Theta)$, and let us write $r = r_0 + r_1\xi + \cdots + r_{\mathsf{M}}\xi^{\mathsf{M}}$, for some $r_0, r_1, \ldots, r_{\mathsf{M}} \in \mathbb{C}[\eta]$. Since $\xi = \xi - \eta^2 + \eta^2$, we have $\xi^{\mathsf{k}} = (\xi - \eta^2 + \eta^2)^{\mathsf{k}} = (\xi - \eta^2)h_{\mathsf{k}} + \eta^{2\mathsf{k}}$, where $h_{\mathsf{k}} \in \mathbb{C}[\eta, \xi]$, $\mathsf{k} \in \mathbb{N}$. Hence we obtain $r = \beta + \gamma(\xi - \eta^2)$ for some $\beta \in \mathbb{C}[\eta]$ and $\gamma \in \mathbb{C}[\eta, \xi]$. Finally, since $r \in \ker(\Theta)$, we have $\beta(2\pi \mathrm{i} y) = 0$ and so $\beta = 0$. Consequently $r \in \langle \xi - \eta^2 \rangle$. Hence we have shown that q is divisible by p, that is $q \in \langle p \rangle$. So the module $\langle \xi - \eta^2 \rangle$ is Willems with respect to $\mathcal{W}_{\mathfrak{s}}$.

Finally, we recall the definition of time controllability with respect to a space $\mathcal{W} \subset \mathcal{D}'(\mathbb{R}^2)$: A $\mathcal{W}$-behaviour, say $\mathfrak{B}$, is said to be *time controllable* (*with respect to the space* $\mathcal{W}$) if for any w_1 and w_2 in $\mathfrak{B}$, there exists a $w \in \mathfrak{B}$ and a $\tau \geqslant 0$ such that

$$\langle w, \varphi \rangle = \begin{cases} \langle w_1, \varphi \rangle & \text{for all } \varphi \in \mathcal{D}\left(\mathbb{R}^2\right) \text{ with } \operatorname{supp}(\varphi) \subset \mathbb{R} \times (-\infty, 0), \\ \langle \sigma_\tau w_2, \varphi \rangle & \text{for all } \varphi \in \mathcal{D}\left(\mathbb{R}^2\right) \text{ with } \operatorname{supp}(\varphi) \subset \mathbb{R} \times (\tau, \infty), \end{cases}$$

where σ_τ denotes translation by $(0, \tau) \in \mathbb{R}^2$: $\langle \sigma_\tau w_2, \varphi \rangle = \langle w_2, \varphi(\bullet, \bullet + \tau) \rangle$ for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$.

4. Algebraic preliminaries and notation

We assume familiarity with rings, ideals and modules.

Let $\mathcal{A}$ be the ring $\mathbb{C}[\eta, \xi]$. Given a subset E of $\mathcal{A}$, the *affine variety of* E in $\mathbb{C}^2$ is denoted by $V(E)$ and it is defined as follows:

$$V(E) = \{\zeta \in \mathbb{C}^2 \mid p(\zeta) = 0 \text{ for all } p \in E\}.$$

Let M be an $\mathcal{A}$-module. An element $p \in \mathcal{A}$ defines an endomorphism Φ_p of M, namely $\chi \mapsto p \cdot \chi$. The element p is said to be a *zero divisor in M* if Φ_p is not injective. The element p is said to be *nilpotent in M* if Φ_p is nilpotent. A submodule Q of M is *primary in M* if

(1) $Q \neq M$ and
(2) every zero divisor in M/Q is nilpotent.

(If $M = \mathcal{A}$ and Q is an ideal in $\mathcal{A}$, then we simply call Q a *primary ideal*.) The *radical of* (*a submodule*) Q *in* M is

$$\mathfrak{r}_M(Q) = \{p \in \mathcal{A} \mid p^{\mathsf{n}} M \subset Q \text{ for some } \mathsf{n} > 0\}.$$

(If $M = \mathcal{A}$ and Q is the ideal I, then we simply call it the *radical of the ideal I*, and denote it by $\mathfrak{r}(I)$. More generally, the radical of a subset E of $\mathcal{A}$ is defined as follows:

$$\mathfrak{r}(E) = \{p \in \mathcal{A} \mid p^{\mathsf{n}} \in E \text{ for some } \mathsf{n} > 0\}.$$

Clearly $V(\mathfrak{r}(E)) = V(E)$.) If P, Q are submodules of M, we define $(P : Q)$ to be the set of all $p \in \mathcal{A}$ such that $pQ \subset P$; it is an ideal of $\mathcal{A}$. If Q is primary in M, then $(Q : M)$ is a primary ideal and (hence) $\mathfrak{r}_M(Q)$ is a prime ideal, say $\mathfrak{p}$. We then say that *Q is $\mathfrak{p}$-primary* (*in M*). A *primary decomposition of* (*a submodule*) *Q in M* is a representation of Q as an intersection

$$Q = \cap_{i=1}^{n} Q_i \tag{3}$$

of primary submodules of M. If, moreover,

(1) the ($\mathfrak{p}_i :=$) $\mathfrak{r}_M(Q_i)$ are all distinct, and
(2) none of the components Q_i can be omitted from the intersection, that is, $\cap_{j \neq i} Q_j \subsetneq Q_i$ $(1 \leqslant i \leqslant n)$,

then the primary decomposition (3) is said to be *irredundant*. Such an irredundant primary decomposition always exists for any given proper submodule Q of $M = \mathcal{A}^w$, where $\mathcal{A} = \mathbb{C}[\eta, \xi]$ (see, for instance, Eisenbud, 1995). In fact there also exist algorithms to find them out using Gröbner bases (see Eisenbud *et al.*, 1992).

If $p \in \mathbb{C}[\eta, \xi]$, then p can be expressed as $p = a_0 + a_1\xi + \cdots + a_N\xi^N \in \mathbb{C}[\eta][\xi]$, where $a_0, \ldots, a_N \in \mathbb{C}[\eta]$ and $a_N \neq 0$. Then the *η-content of p*, denoted by $C_\eta(p)$, is defined to be the ideal generated by $a_0, \ldots, a_N$. From Gauss's lemma (see for instance, Reid, 1995), it follows that if p and q are polynomials in $\mathbb{C}[\eta, \xi]$, then

$$C_\eta(pq) = C_\eta(p)C_\eta(q), \tag{4}$$

where $C_\eta(p)C_\eta(q)$ denotes the product of the ideals $C_\eta(p)$ and $C_\eta(q)$, that is, it is the set of all finite sums $\sum_i a_i b_i$, with each $a_i \in C_\eta(p)$ and each $b_i \in C_\eta(q)$. In the sequel, $\mathcal{A}$ always refers to the ring $\mathbb{C}[\eta, \xi]$. Given an ideal I in $\mathcal{A}$, the *η-content of the ideal I* is

$$C_\eta(I) = \cup_{p \in I} C_\eta(p).$$

If I and J are ideals such that $I \subset J$, then clearly $C_\eta(I) \subset C_\eta(J)$. Also, for any ideal I, using (4), it can be seen that

$$C_\eta(\mathfrak{r}(I)) \subset \mathfrak{r}(C_\eta(I)).$$

Finally, we give the notion of the determinantal ideal of a given submodule $\mathbf{R}$ of $\mathcal{A}^w$. First of all, given a submodule $\mathbf{R}$ of $\mathcal{A}^w$, it can be generated by a finite number, say g, of elements in $\mathcal{A}^w$ (since $\mathcal{A}^w$ is Noetherian). Thus $\mathbf{R}$ can be represented by a $g \times w$ matrix with entries in $\mathcal{A}$, where the g rows (as elements of $\mathcal{A}^w$) generate $\mathbf{R}$. For the definition of the (wth[†]) determinantal ideal, we require that $g \geqslant w$, and this can be arranged, for instance, by augmenting the matrix with zero rows. Consider now the *wth determinantal ideal* of this matrix, that is, the ideal generated by the determinants of all $w \times w$ minors of the matrix. Then it can be seen that this ideal, denoted by $I_w(\mathbf{R})$, depends only on the submodule $\mathbf{R}$, and not on the choice of the generators above. Given a submodule $\mathbf{R}$ of $\mathcal{A}^w$, $I_w(\mathbf{R})$ is called the *determinantal ideal of* $\mathbf{R}$.

[†]Although we do not need it in this article, in general, one speaks of the kth determinantal ideal of a submodule $\mathbf{R}$ of $\mathcal{A}^w$.

5. Main result

First we will prove the following useful result about the support of the Fourier transform (only in the spatial variable) of trajectories that satisfy a single scalar PDE, which will be used in proving our main result (Theorem 5.2).

LEMMA 5.1 If $p \in \mathbb{C}[\eta, \xi]$ and $w \in \mathcal{W}_{\mathfrak{s}}$ is such that $D_p w = 0$ and $\langle w, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$ with $\text{supp}(\varphi) \subset \mathbb{R} \times (-\infty, 0)$, then

$$\text{supp}(\hat{w}) \subset \left(\frac{1}{2\pi \mathrm{i}} \left[V(C_\eta(p)) \cap \mathrm{i}\mathbb{R} \right] \right) \times [0, \infty),$$

where $\hat{w} \in \mathcal{D}'(\mathbb{R}^2)$ denotes the *Fourier transform of w in the spatial direction*, defined† by

$$\langle \hat{w}, \psi \otimes \varphi \rangle = \langle \mathcal{F}[(\iota w)(\varphi)], \psi \rangle \quad \text{for} \quad \varphi \in \mathcal{D}(\mathbb{R}), \quad \psi \in \mathcal{D}(\mathbb{R}). \tag{5}$$

Proof. Let the η-content of p be generated by $a \in \mathbb{C}[\eta]$. Thus $p = ap_1$, where $p_1 = a_0 + a_1\xi + \cdots + a_{\mathsf{N}}\xi^{\mathsf{N}}$, $a_0, \ldots, a_{\mathsf{N}} \in \mathbb{C}[\eta]$, with $C_\eta(p_1) = \langle 1 \rangle$. We have

$$D_p w = D_{p_1}(D_a w) = D_{p_1} w_1 = 0,$$

where $w_1 := D_a w \in \mathcal{W}_{\mathfrak{s}}$ has zero past, that is, $\langle w_1, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$ with $\text{supp}(\varphi) \subset \mathbb{R} \times (-\infty, 0)$.

First we will show that $w_1 = 0$. We have

$$a_0 \left(\frac{\partial}{\partial x} \right) w_1 + a_1 \left(\frac{\partial}{\partial x} \right) \frac{\partial}{\partial t} w_1 + \cdots + a_{\mathsf{N}} \left(\frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} \right)^{\mathsf{N}} w_1 = 0.$$

Thus for all $\varphi \in \mathcal{D}(\mathbb{R})$ we obtain

$$a_0 \left(\frac{\partial}{\partial x} \right) (\iota w_1)(\varphi) + a_1 \left(\frac{\partial}{\partial x} \right) \left(\iota \frac{\partial}{\partial t} w_1 \right)(\varphi) + \cdots + a_{\mathsf{N}} \left(\frac{\partial}{\partial x} \right) \left(\iota \left(\frac{\partial}{\partial t} \right)^{\mathsf{N}} w_1 \right)(\varphi) = 0.$$

Upon Fourier transformation we get

$$a_0(2\pi \mathrm{i} y)\mathcal{F}[(\iota w_1)(\varphi)] + \cdots + a_{\mathsf{N}}(2\pi \mathrm{i} y)\mathcal{F}\left[\left(\iota \left(\frac{\partial}{\partial t} \right)^{\mathsf{N}} w_1 \right)(\varphi) \right] = 0. \tag{6}$$

From (6), we obtain that the Fourier transform of w_1 in the spatial direction, namely $\widehat{w_1}$, satisfies

$$a_0(2\pi \mathrm{i} y)\widehat{w_1} + a_1(2\pi \mathrm{i} y)\frac{\partial}{\partial t}\widehat{w_1} + \cdots + a_{\mathsf{N}}(2\pi \mathrm{i} y)\left(\frac{\partial}{\partial t} \right)^{\mathsf{N}} \widehat{w_1} = 0.$$

Applying the uniqueness theorem of Holmgren (see Theorem 5.3.1 on page 125 of Hörmander, 1969), with $\Omega := \complement \left\{ (x, t) \in \mathbb{R}^2 \mid a_{\mathsf{N}}(2\pi \mathrm{i} x) = 0 \right\}$, $\Phi \in C^1(\Omega)$ defined by $\Phi : (x, t) \mapsto -t$ and x_0 any point in $\Omega \cap (\mathbb{R} \times \{0\})$, we have

$$\text{supp}(\widehat{w_1}) \subset \left\{ (x, t) \in \mathbb{R}^2 \mid a_{\mathsf{N}}(2\pi \mathrm{i} x) = 0, \ t \geqslant 0 \right\}$$

†By the approximation lemma (see, for instance Exercise 11 on page 56 of Carroll, 1969), it follows that (5) defines a distribution on $\mathbb{R}^2$.

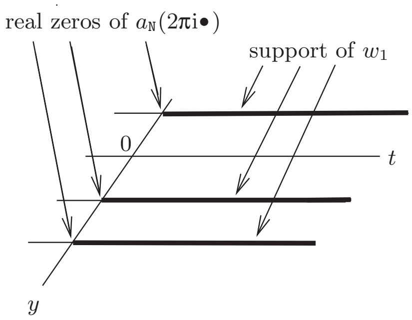

FIG. 2. The support of $\widehat{w_1}$.

(see Fig. 2). If a_N is a constant ($\neq 0$), then we obtain $\widehat{w_1} = 0$ and so $w_1 = 0$, and we are done. If a_N is not a constant then there exists a $k < N$ such that $(a_k, a_N) = 1$, that is, the greatest common divisor of a_k and a_N is 1. Each half-line in $\complement\Omega \cap (\mathbb{R} \times [0, \infty))$ carries a solution of the differential equation with polynomial coefficients and $\widehat{w_1}$ is the sum of these. We prove our claim for each of these summands, because each of these has support on the corresponding half-line. By means of a translation, we may assume that the half-line is $\{0\} \times [0, \infty)$. Let $T \in (0, \infty)$. Then $\widehat{w_1}$ is a distribution of finite order in $\mathbb{R} \times (-T, T)$. Applying Theorem 2.3.5 (Hörmander, 1990, p. 47), it follows that there exist distributions $T_1, \dots, T_J \in \mathcal{D}'(\mathbb{R})$, with $T_J \neq 0$ such that

$$\widehat{w_1} = \sum_{j=1}^{J} \left(\frac{\partial}{\partial y}\right)^{j} \delta_y \otimes T_j \tag{7}$$

in the strip $\mathbb{R} \times (-T, T)$. Then it can be seen that $\text{supp}(T_J|_{(-T,T)}) \subset [0, T)$, since $\text{supp}(\widehat{w_1}) \subset \mathbb{R} \times [0, \infty)$. From (7), we have

$$w_1 = \sum_{j=1}^{J} (2\pi i x)^{j} \otimes T_j$$

in $\mathbb{R} \times (-T, T)$. Since $D_{p_1} w_1 = 0$, we have

$$D_{p_1} \left(\frac{\partial}{\partial y}\right)^{J} w_1 = 0,$$

and so $D_{p_1}(1_x \otimes T_J) = 0$, where 1_x denotes the regular distribution corresponding to the constant function taking value 1 everywhere. But now we can drop all the terms in D_{p_1} which contain $\frac{\partial}{\partial x}$, leaving a linear ordinary differential operator, say D_{p_0}, in only t with constant coefficients. Owing to our assumption that $(a_k, a_N) = 1$, we obtain that $p_0 \neq 0$. From the fact that $D_{p_0}(1_x \otimes T_J) = 0$ in $\mathbb{R} \times (-T, T)$, we have $D_{p_0} T_J = 0$ in $(-T, T)$. Since T_J is zero in $(-T, 0)$, it follows that $T_J = 0$ in $(-T, T)$ (this follows from Theorem 8.6.8 on page 312 of Hörmander (1990) applied to p_0). So $\widehat{w_1} = 0$ in $\mathbb{R} \times (-T, T)$. But we recall that the choice of T was arbitrary. Hence $\widehat{w_1} = 0$ and finally $w_1 = 0$.

Finally, we observe that since $D_a w$ $(= w_1)$ is zero, upon Fourier transformation in the spatial variable, we get $a(2\pi \mathrm{i} y)\hat{w} = 0$, and so

$$\operatorname{supp}(\hat{w}) \subset \{y \in \mathbb{R} \mid a(2\pi \mathrm{i} y) = 0\} \times \mathbb{R}.$$

But we know that $w|_{(-\infty,0)\times\mathbb{R}} = 0$ and so

$$\operatorname{supp}(\hat{w}) \subset \{y \in \mathbb{R} \mid a(2\pi \mathrm{i} y) = 0\} \times [0, \infty).$$

Consequently, $\operatorname{supp}(\hat{w}) \subset \frac{1}{2\pi \mathrm{i}} \left[V(C_\eta(p)) \cap \mathrm{i}\mathbb{R}\right] \times [0, \infty)$, and this completes the proof.□

REMARK This result is analogous to the observation that if a tempered distribution $w \in \mathcal{S}'$ satisfies a PDE corresponding to a polynomial p, then the support of its Fourier transform is contained in the intersection of the variety of the polynomial p with the imaginary axes. Of course, in the case of $\mathcal{W}_{\mathfrak{s}}$, since we have a tempered profile only in the spatial direction, we get a result that is not quite symmetric with respect to the indeterminates in the polynomial.

We now give our main result. The proof is similar to the proof of Theorem 2.3 of Shankar (1999), in which it is determined when a given submodule is Willems with respect to the space of tempered distributions, $\mathcal{S}'$. In the following theorem, we characterize the submodules that are Willems with respect to the space $\mathcal{W}_{\mathfrak{s}}$ of distributions that are tempered *only* in the spatial direction.

THEOREM 5.2 Let $\mathbf{R}$ be a submodule of $\mathcal{A}^{\mathtt{w}}$ such that the corresponding $\mathcal{W}_{\mathfrak{s}}$-behaviour, $\mathfrak{B}_{\mathcal{W}_{\mathfrak{s}}}(\mathbf{R})$, is time controllable. Let $\mathbf{R} = \cap_{\mathtt{i}=1}^{\mathtt{r}} \mathbf{Q}_{\mathtt{i}}$ be an irredundant primary decomposition of $\mathbf{R}$, where $\mathbf{Q}_{\mathtt{i}}$ is $\mathfrak{p}_{\mathtt{i}}$-primary in $\mathcal{A}^{\mathtt{w}}$. Let $\mathtt{r}_0$ be the integer satisfying $1 \leqslant \mathtt{r}_0 \leqslant \mathtt{r}$ for which

(1) $V(\mathfrak{p}_{\mathtt{i}}) \cap (\mathrm{i}\mathbb{R} \times \mathbb{C}) \neq \emptyset$, for all $\mathtt{i} \in \{1, \ldots, \mathtt{r}_0\}$, and
(2) $V\left(\cap_{\mathtt{i}=\mathtt{r}_0+1}^{\mathtt{r}} \mathfrak{p}_{\mathtt{i}}\right) \cap (\mathrm{i}\mathbb{R} \times \mathbb{C}) = \emptyset$.

Then $\mathfrak{R}\left(\mathfrak{B}_{\mathcal{W}_{\mathfrak{s}}}(\mathbf{R})\right) = \cap_{\mathtt{i}=1}^{\mathtt{r}_0} \mathbf{Q}_{\mathtt{i}}$. In particular, $\mathbf{R}$ is Willems with respect to $\mathcal{W}_{\mathfrak{s}}$ iff $\mathtt{r}_0 = \mathtt{r}$.

Proof. Let us denote $\cap_{\mathtt{i}=1}^{\mathtt{r}_0} \mathbf{Q}_{\mathtt{i}}$ by $\mathbf{R}_0$. Then $\mathbf{R}_0$ is independent of the primary decomposition of $\mathbf{R}$. Indeed, let I be the ideal $\cap_{\mathtt{i}=\mathtt{r}_0+1}^{\mathtt{r}} \mathfrak{p}_{\mathtt{i}}$. Then I is an ideal such that $I \subset \mathfrak{p}_{\mathtt{i}}$, $\mathtt{i} = \mathtt{r}_0 + 1, \ldots, \mathtt{r}$ and that is not contained in the other $\mathfrak{p}_{\mathtt{i}}$. (This is because if $\cap_{\mathtt{i}=\mathtt{r}_0+1}^{\mathtt{r}} \mathfrak{p}_{\mathtt{i}} \subset \mathfrak{p}_{\mathtt{i}}$ for some $\mathtt{i} \in \{1, \ldots, \mathtt{r}_0\}$, then we would have

$$\emptyset \neq V(\mathfrak{p}_{\mathtt{i}}) \cap (\mathrm{i}\mathbb{R} \times \mathbb{C}) \subset V\left(\cap_{\mathtt{i}=\mathtt{r}_0+1}^{\mathtt{r}} \mathfrak{p}_{\mathtt{i}}\right) \cap (\mathrm{i}\mathbb{R} \times \mathbb{C}) = \emptyset,$$

a contradiction!) Consider the ascending chain of submodules

$$(\mathbf{R} : I) \subset (\mathbf{R} : I^2) \subset \ldots.$$

Then this chain stabilizes to the submodule $\cap_{\mathtt{i}=1}^{\mathtt{r}_0} \mathbf{Q}_{\mathtt{i}}$, and this submodule is therefore independent of the primary decomposition.

Part 1. We first show that the $\mathcal{W}_{\mathfrak{s}}$-behaviour of $\mathbf{R}_0$ equals that of $\mathbf{R}$. As $\mathbf{R} \subset \mathbf{R}_0$, it suffices to show that

$$\mathfrak{B}_{\mathcal{W}_{\mathfrak{s}}}(\mathbf{R}) \subset \mathfrak{B}_{\mathcal{W}_{\mathfrak{s}}}(\mathbf{R}_0). \tag{8}$$

Clearly $w_1 = 0$ belongs to the $\mathcal{W}_\mathfrak{s}$-behaviour of $\mathbf{R}$. If (8) is not true, then there exists a $w_2 \in \mathfrak{B}_{\mathcal{W}_\mathfrak{s}}(\mathbf{R})$ and a $\chi \in \mathbf{R}_0 \setminus \mathbf{R}$ such that $D_\chi w_2 \neq 0$.

Since $\mathfrak{B}_{\mathcal{W}_\mathfrak{s}}(\mathbf{R})$ is time controllable, there exists a trajectory $w \in \mathfrak{B}_{\mathcal{W}_\mathfrak{s}}(\mathbf{R})$ that patches up w_1 and w_2 along the time direction. If necessary, by shifting w_2 beforehand, we can ensure that $D_\chi w \neq 0$. But for every $p \in (\mathbf{R} : \chi)$, we have

$$D_p(D_\chi w) = D_{p\cdot\chi} w = 0,$$

and so it follows from Lemma 5.1 that

$$\operatorname{supp}(\widehat{D_\chi w}) \subset \left(\frac{1}{2\pi \mathrm{i}}\left[V(C_\eta(p)) \cap \mathrm{i}\mathbb{R}\right]\right) \times \mathbb{R}.$$

Since this is true for each $p \in (\mathbf{R} : \chi)$, we obtain

$$\operatorname{supp}(\widehat{D_\chi w}) \subset \left(\frac{1}{2\pi \mathrm{i}}\left[V(C_\eta(\mathbf{R} : \chi)) \cap \mathrm{i}\mathbb{R}\right]\right) \times \mathbb{R}. \tag{9}$$

As $\mathbf{R} = \cap_{\mathtt{i}=1}^{\mathtt{r}} \mathbf{Q}_\mathtt{i}$, $(\mathbf{R} : \chi) = \cap_{\mathtt{i}=1}^{\mathtt{r}} (\mathbf{Q}_\mathtt{i} : \chi)$, and as χ is in every one of $\mathbf{Q}_1, \dots, \mathbf{Q}_{\mathtt{r}_0}$ and not in at least one of the other $\mathbf{Q}_\mathtt{i}$, it follows that $\mathfrak{r}((\mathbf{R} : \chi))$ is equal to the intersection of a subset (say, $\mathfrak{p}_{\mathtt{i}_\mathtt{k}}$, $\mathtt{k} \in \{1, \dots, \mathtt{K}\}$) of $\mathfrak{p}_{\mathtt{r}_0+1}, \dots, \mathfrak{p}_\mathtt{r}$ ($\mathbf{Q}_\mathtt{i}$ is $\mathfrak{p}_\mathtt{i}$-primary). Thus it follows that

$$C_\eta\left(\cap_{\mathtt{k}=1}^{\mathtt{K}} \mathfrak{p}_{\mathtt{i}_\mathtt{k}}\right) = C_\eta(\mathfrak{r}((\mathbf{R} : \chi))) \subset \mathfrak{r}(C_\eta(\mathbf{R} : \chi))$$

and so

$$V(C_\eta(\mathbf{R} : \chi)) = V(\mathfrak{r}(C_\eta(\mathbf{R} : \chi))) \subset V\left(C_\eta\left(\cap_{\mathtt{k}=1}^{\mathtt{K}} \mathfrak{p}_{\mathtt{i}_\mathtt{k}}\right)\right) \subset V\left(C_\eta\left(\cap_{\mathtt{i}=\mathtt{r}_0+1}^{\mathtt{r}} \mathfrak{p}_\mathtt{i}\right)\right).$$

Consequently, from (9), we obtain

$$\operatorname{supp}(\widehat{D_\chi w}) \subset \left(\frac{1}{2\pi \mathrm{i}}\left[V\left(C_\eta\left(\cap_{\mathtt{i}=\mathtt{r}_0+1}^{\mathtt{r}} \mathfrak{p}_\mathtt{i}\right)\right) \cap \mathrm{i}\mathbb{R}\right]\right) \times \mathbb{R}.$$

But by assumption,

$$V\left(\cap_{\mathtt{i}=\mathtt{r}_0+1}^{\mathtt{r}} \mathfrak{p}_\mathtt{i}\right) \cap (\mathrm{i}\mathbb{R} \times \mathbb{C}) = \emptyset.$$

This implies that

$$\operatorname{supp}(\widehat{D_\chi w}) \subset \emptyset$$

and so $\widehat{D_\chi w} = 0$. Consequently, we obtain $D_\chi w = 0$, in contradiction to the assumption above.

Part 2. Now we show that $\mathbf{R}_0$ is the largest submodule of $\mathcal{A}^\mathtt{w}$ with the same $\mathcal{W}_\mathfrak{s}$-behaviour as that of $\mathbf{R}$. So let

$$\chi = \begin{bmatrix} \chi_1 & \cdots & \chi_\mathtt{w} \end{bmatrix}$$

be any element of $\mathcal{A}^{\mathtt{w}} \setminus \mathbf{R}_0$, and consider the exact sequence

$$0 \longrightarrow \mathcal{A}/(\mathbf{R}_0 : \chi) \xrightarrow{\Psi_\chi} \mathcal{A}^{\mathtt{w}}/\mathbf{R}_0 \xrightarrow{\pi} \mathcal{A}^{\mathtt{w}}/(\mathbf{R}_0 + \langle \chi \rangle) \longrightarrow 0,$$

where the morphism Ψ_χ above maps the class of p to the class of $p \cdot \chi$, and π is the canonical surjection. Since C^∞ is an injective $\mathcal{A}$-module (see for instance Theorem 3 on page 305 of Palamodov, 1970), it follows that the sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{\mathtt{w}}/(\mathbf{R}_0 + \langle \chi \rangle), C^\infty) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{\mathtt{w}}/\mathbf{R}_0, C^\infty) \\ &\xrightarrow{D_\chi} \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}/(\mathbf{R}_0 : \chi), C^\infty) \longrightarrow 0 \end{aligned}$$

is exact. Observe that the above sequence is, by Malgrange†,

$$0 \longrightarrow \mathfrak{B}_{C^\infty}(\mathbf{R}_0 + \langle \chi \rangle) \longrightarrow \mathfrak{B}_{C^\infty}(\mathbf{R}_0) \xrightarrow{D_\chi} \mathfrak{B}_{C^\infty}((\mathbf{R}_0 : \chi)) \longrightarrow 0.$$

As $\mathbf{R}_0 = \cap_{\mathtt{i}=1}^{\mathtt{r}_0} \mathbf{Q}_{\mathtt{i}}$, χ is not in at least one of these $\mathbf{Q}_{\mathtt{i}}$, so that $V(\mathbf{R}_0 : \chi)$ is the union of some of the $V(\mathfrak{p}_1), \ldots, V(\mathfrak{p}_{\mathtt{r}_0})$. From the proof of Theorem 2.2 on page 1823 of Shankar (2001), it follows that the union of these varieties of the $\mathfrak{p}_{\mathtt{i}}$ is contained in the variety of $I_{\mathtt{w}}(\mathbf{R}_0)$, where $I_{\mathtt{w}}(\mathbf{R}_0)$ denotes the ($\mathtt{w}$th) determinantal ideal of $\mathbf{R}_0$; namely, we have $\cup_{\mathtt{i}=1}^{\mathtt{r}_0} V(\mathfrak{p}_{\mathtt{i}}) \subset V(I_{\mathtt{w}}(\mathbf{R}_0))$.

Each of the varieties $V(\mathfrak{p}_1), \ldots, V(\mathfrak{p}_{\mathtt{r}_0})$ intersects $\mathrm{i}\mathbb{R} \times \mathbb{C}$, and hence so does the variety of $(\mathbf{R}_0 : \chi)$. Let (x_0, t_0) be some point in this intersection. Consider the smooth function $w_0 : (x, t) \mapsto \mathrm{e}^{\langle (x,t),(x_0,t_0) \rangle}$ in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}/(\mathbf{R}_0 : \chi), C^\infty)$, that is the C^∞-behaviour of the ideal $(\mathbf{R}_0 : \chi)$. As the spatial coordinate of (x_0, t_0), namely x_0, is purely imaginary, it follows that w_0 belongs to $\mathcal{W}_{\mathfrak{s}}$. From the last part of the proof of Theorem 2.3 on page 371 of Shankar (1999)), w_0 is the image of an element in the C^∞-behaviour of $\mathbf{R}_0$, which is of the form $u(x, t)\mathrm{e}^{\langle (x,t),(x_0,t_0) \rangle}$, where the components of u are polynomials. Thus, in fact, w_0 is the image of an element in the $\mathcal{W}_{\mathfrak{s}}$-behaviour of $\mathbf{R}_0$, that is an element in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{\mathtt{w}}/\mathbf{R}_0, \mathcal{W}_{\mathfrak{s}})$. By exactness of the sequence above, it then follows that this element in the $\mathcal{W}_{\mathfrak{s}}$-behaviour of $\mathbf{R}_0$ cannot be in the $\mathcal{W}_{\mathfrak{s}}$-behaviour of $\mathbf{R}_0 + \langle \chi \rangle$. This proves that $\mathbf{R}_0$ is Willems with respect to $\mathcal{W}_{\mathfrak{s}}$. □

REMARK We note that we made the assumption that $\mathfrak{B}_{\mathcal{W}_{\mathfrak{s}}}(\mathbf{R})$ is time controllable. Such a corresponding assumption is not present in Theorem 2.3 of Shankar (1999). Indeed, it is certainly desirable to have a test for the Willems-ness of a submodule purely in terms of $\mathbf{R}$, that is, purely in terms of algebraic computations with $\mathbf{R}$. However, in the case of the space $\mathcal{W}_{\mathfrak{s}}$ we suspect this task to be a formidable one, if at all possible. The reason is that there is no simple relation available between the support of the Fourier transform (in the spatial direction) of w and the algebraic properties of the polynomial p such that $D_p w = 0$. This relation lies at the very core of Theorem 2.3 in Shankar (1999), where indeed owing to the choice of the space, namely $\mathcal{S}'(\mathbb{R}^{\mathtt{n}})$, an elegant such relation exists. In our case, for the space $\mathcal{W}_{\mathfrak{s}}$, a relation exists, provided we assume that w has, for instance, past equal to zero. (This is precisely the content of Lemma 5.1 above.) Because of this

† He observed that $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{\mathtt{w}}/\mathbf{R}, \mathcal{W}) \simeq \mathfrak{B}_{\mathcal{W}}(\mathbf{R})$. See Malgrange (1963).

lacuna in the knowledge of algebraic-analytic results, one makes the assumption of time controllability with respect to $\mathcal{W}_{\mathfrak{s}}$. Of course, an algebraic test on $\mathbf{R}$ that characterizes the property of time controllability of the corresponding behaviour with respect to $\mathcal{W}_{\mathfrak{s}}$ will yield purely algebraic assumptions in the theorem above. However, an algebraic test on $\mathbf{R}$ characterizing timecontrollability is presently not known and it is an open problem (see Sasane, 2003).

EXAMPLES

(1) Let $\mathbf{R} = \langle 1+\eta \rangle$. Then the $\mathcal{W}_{\mathfrak{s}}$-behaviour given by $\mathbf{R}$ is zero, and so it is trivially time controllable. Clearly with $\mathbf{Q}_1 := \mathbf{R}$, it follows that $\mathbf{R} = \mathbf{Q}_1$ is an irredundant primary decomposition of $\mathbf{R}$, with $\mathbf{Q}_1$ being ($\mathfrak{p}_1 =$) $\langle \eta+1 \rangle$-primary in $\mathcal{A}$. Since $V(\mathfrak{p}_1) \cap (\mathrm{i}\mathbb{R} \times \mathbb{C}) = (-1 \times \mathbb{C}) \cap (\mathrm{i}\mathbb{R} \times \mathbb{C}) = \emptyset$, $\mathbf{R}$ is not Willems with respect to $\mathcal{W}_{\mathfrak{s}}$. We have $\Re(\mathfrak{B}_{\mathcal{W}_{\mathfrak{s}}}(\mathbf{R}))$ (being an empty intersection) is equal to $\langle 1 \rangle = \mathcal{A}$.

(2) Let $\mathbf{R} = \left\langle \begin{bmatrix} \xi - \eta^2 & -\eta \end{bmatrix} \right\rangle$. Then the corresponding $\mathcal{W}_{\mathfrak{s}}$-behaviour, $\mathfrak{B}_{\mathcal{W}_{\mathfrak{s}}}(\mathbf{R})$, is the set of all $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathcal{W}_{\mathfrak{s}}^2$ such that

$$\left[\frac{\partial}{\partial t} - \left(\frac{\partial}{\partial x} \right)^2 \right] w_1 = \frac{\partial}{\partial x} w_2.$$

First of all we will show that

$$\mathfrak{B}_{\mathcal{W}_{\mathfrak{s}}}(\mathbf{R}) = \left\{ w \in \mathcal{W}_{\mathfrak{s}}^2 \mid \text{there exists an } l \in \mathcal{W}_{\mathfrak{s}} \text{ such that } w = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial t} - \left(\frac{\partial}{\partial x}\right)^2 \end{bmatrix} l \right\}.$$

From Theorem 2.2 it follows that there exists an $l_0 \in \mathcal{W}_{\mathfrak{s}}$ such that

$$w_1 = \frac{\partial}{\partial x} l_0.$$

Let us define $w_{2,0}$ to be $\left[\frac{\partial}{\partial t} - \left(\frac{\partial}{\partial x}\right)^2 \right] l_0$. Then we have

$$\frac{\partial}{\partial x} w_{2,0} = \left[\frac{\partial}{\partial t} - \left(\frac{\partial}{\partial x} \right)^2 \right] \frac{\partial}{\partial x} l_0 = \left[\frac{\partial}{\partial t} - \left(\frac{\partial}{\partial x} \right)^2 \right] w_1.$$

Consequently

$$\frac{\partial}{\partial x}(w_2 - w_{2,0}) = 0,$$

and so from Lemma 2.5 (Cotroneo & Sasane, 2002), it follows that there exists a $T \in \mathcal{D}'(\mathbb{R})$ such that $w = w_{2,0} + \mathbb{1}_x \otimes T$, where $\mathbb{1}_x$ denotes the regular distribution corresponding to the constant function taking value 1 everywhere. Let $S \in \mathcal{D}'(\mathbb{R})$ such that $S' = T$ that is, S is a primitive of T (that such a primitive exists follows for instance from the fundamental principle for $\mathcal{D}'(\mathbb{R})$). Define

$$l = l_0 + \mathbb{1}_x \otimes S \in \mathcal{W}_{\mathfrak{s}}.$$

We have

$$\frac{\partial}{\partial x} l = \frac{\partial}{\partial x}(l_0 + 1_x \otimes S) = \frac{\partial}{\partial x} l_0 = w_1,$$

and

$$\begin{aligned} w_2 &= w_{2,0} + 1_x \otimes T \\ &= \left[\frac{\partial}{\partial t} - \left(\frac{\partial}{\partial x}\right)^2\right] l_0 + \left[\frac{\partial}{\partial t} - \left(\frac{\partial}{\partial x}\right)^2\right](1_x \otimes S) \\ &= \left[\frac{\partial}{\partial t} - \left(\frac{\partial}{\partial x}\right)^2\right](l_0 + 1_x \otimes S) \\ &= \left[\frac{\partial}{\partial t} - \left(\frac{\partial}{\partial x}\right)^2\right] l. \end{aligned}$$

Now from Theorem 4.1 in Sasane (2003), it follows that $\mathfrak{B}_{\mathcal{W}_{\mathfrak{s}}}(\mathbf{R})$ is time controllable.

It can be checked that with $\mathbf{Q}_1 := \mathbf{R}$, $\mathbf{R} = \mathbf{Q}_1$ is an irredundant primary decomposition of $\mathbf{R}$, with $\mathbf{Q}_1$ being ($\mathfrak{p}_1 =$) 0-primary in $\mathcal{A}^2$. Thus

$$V(\mathfrak{p}_1) \cap (\mathrm{i}\mathbb{R} \times \mathbb{C}) = \mathbb{C}^2 \cap (\mathrm{i}\mathbb{R} \times \mathbb{C}) = \mathrm{i}\mathbb{R} \times \mathbb{C} \neq \emptyset.$$

Thus $\mathsf{r}_0 = \mathsf{r} = 1$, and consequently $\mathbf{R}$ is Willems with respect to $\mathcal{W}_{\mathfrak{s}}$.

(3) Let $\mathbf{R} = \langle [\ (\eta + 1)(\xi - \eta^2) \quad -(\eta + 1)\eta\ ] \rangle$. In light of the previous example, it is easy to see that

$$\mathfrak{B}_{\mathcal{W}_{\mathfrak{s}}}(\mathbf{R}) = \left\{ w \in \mathcal{W}_{\mathfrak{s}}^2 \mid \text{there exists an } l \in \mathcal{W}_{\mathfrak{s}} \text{ such that } \left(\frac{\partial}{\partial x} + 1\right) w = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial t} - \left(\frac{\partial}{\partial x}\right)^2 \end{bmatrix} l \right\}.$$

Hence from Theorem 4.1 in Sasane (2003), it follows that $\mathfrak{B}_{\mathcal{W}_{\mathfrak{s}}}(\mathbf{R})$ is time controllable.

It can be checked that if $\mathbf{Q}_1 := \langle [\ \xi - \eta^2 \quad -\eta\ ] \rangle$ and $\mathbf{Q}_2 := \langle [\ \eta + 1 \quad 0\ ], [\ 0 \quad \eta + 1\ ] \rangle$, then $\mathbf{R} = \mathbf{Q}_1 \cap \mathbf{Q}_2$ is an irredundant primary decomposition of $\mathbf{R}$, with $\mathbf{Q}_1$ being ($\mathfrak{p}_1 =$) 0-primary in $\mathcal{A}^2$, and $\mathbf{Q}_2$ ($\mathfrak{p}_2 =$) $\langle \eta + 1 \rangle$-primary in $\mathcal{A}^2$. Thus

$$V(\mathfrak{p}_1) \cap (\mathrm{i}\mathbb{R} \times \mathbb{C}) = \mathbb{C}^2 \cap (\mathrm{i}\mathbb{R} \times \mathbb{C}) = \mathrm{i}\mathbb{R} \times \mathbb{C} \neq \emptyset$$

while

$$V(\mathfrak{p}_2) \cap (\mathrm{i}\mathbb{R} \times \mathbb{C}) = (-1 \times \mathbb{C}) \cap (\mathrm{i}\mathbb{R} \times \mathbb{C}) = \emptyset.$$

Thus $\mathsf{r}_0 = 1 < 2 = \mathsf{r}$, and consequently $\mathbf{R}$ is not Willems with respect to $\mathcal{W}_{\mathfrak{s}}$. In fact, we have $\mathfrak{R}(\mathfrak{B}_{\mathcal{W}_{\mathfrak{s}}}(\mathbf{R})) = \mathbf{Q}_1$.

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