Invention and Bounded Learning by Doing
Author(s): Alwyn Young
Source: *Journal of Political Economy*, Vol. 101, No. 3 (Jun., 1993), pp. 443-472
Published by: The University of Chicago Press
Stable URL: http://www.jstor.org/stable/2138771

Invention and Bounded Learning by Doing

Alwyn Young

Massachusetts Institute of Technology

This paper attempts to integrate traditional models of invention and learning by doing, developing a model that emphasizes the interdependence between research activity in the laboratory and production experience on the factory floor. Learning depends on invention in that learning by doing is viewed as the exploration of the finite and bounded productive potential of invented technologies. At the same time, the profitability of costly invention is dependent on learning in that costs of production depend on cumulative learning experience. The model is a true hybrid, allowing for circumstances in which either the incentives to engage in research and/or the incentives to produce different goods are the binding constraints on growth.

I. Introduction

Models of endogenous technical change fall into two broad, and yet surprisingly disjoint, categories. On the one hand, there are models of "invention" (e.g., Romer 1990; Segerstrom, Anant, and Dinopoulous 1990; Grossman and Helpman 1991), in which technical change is the outcome of costly and deliberative research aimed at the development of new technologies. On the other hand, there are models of "learning by doing" (e.g., Arrow 1962; Lucas 1988), in which technical change is the serendipitous by-product of experience gained in the production of goods. Models of invention generally focus on the factors that influence the incentive to consciously innovate, such as the institutional framework and market size, whereas models of learning focus on factors that influence the incentives to produce different types of goods, such as the pattern of comparative advantage. This

[*Journal of Political Economy*, 1993, vol. 101, no. 3]

paper seeks to integrate these two literatures, developing a model in which sustained technical progress involves an interaction between deliberative invention and serendipitous learning. The model provides some preliminary insights into the conditions under which either the incentives to innovate or the pattern of demand and production might be the binding constraints on growth.

One of the most troubling features of existing learning-by-doing models is the assumption that the potential productivity gains from learning are essentially unbounded.[1] This formulation implies that fairly steady technical progress should have occurred throughout human history since goods have always been produced. The historical record indicates, however, that the premodern world was characterized by extraordinarily long periods of technical stagnation. Furthermore, the solutions to some surprisingly simple technical problems appear to have eluded producers, despite centuries of repetitive activity. Thus, the Europeans strangled their horses with the throat and girth harness for millennia, until the invading Avars introduced the vastly superior trace harness in the sixth century A.D. For their part, the Chinese experienced extraordinary technical progress up until the end of the Sung dynasty in the mid–thirteenth century, but then suffered from almost total technological stagnation (if not regression) until the nineteenth century, when they began to imitate European technology en masse. If one views technical change as the steady and never ending serendipitous by-product of production experience, then one is completely at a loss to explain the recurring pattern of technological improvement and stagnation apparent in premodern history.[2]

Intuitively, it seems reasonable to assume that the potential for

[1] The usual formulation involves a finite number of goods, with productivity in each industry either a linear (e.g., Lucas 1988) or a log-linear (e.g., Arrow 1962; Bardhan 1970; Krugman 1987) function of cumulative production or investment experience, both of which imply that experience alone can lead to unbounded productivity improvements. Young (1991) introduces a bound on learning in each good, allowing for unbounded growth by taking as given the existence of an infinite continuum of potentially producible goods. Bounds on learning have frequently been introduced in partial equilibrium analyses, as in Spence (1981), Fudenberg and Tirole (1983), and Stokey (1986).

[2] For details on the Chinese and European experiences mentioned above, see Elvin (1973), Temple (1986), and Mokyr (1990, chaps. 2, 9). Jones (1988) provides a general survey of periods of growth and stagnation. I should emphasize that concavity alone (as opposed to boundedness) in the learning process is not sufficient to explain asymptotic stagnation. For example, if $Q_t = A_t L_t$, $A_t = E_t^b$ $(1 > b > 0)$, $\dot{E}_t = Q_t$, and $L_t = L_0 e^{nt}$, then $\dot{A}_t/A_t$ converges to $bn/(1 - b)$. Thus, population growth in the typical log-linear model overcomes the concavity in learning and prevents asymptotic stagnation. In the 600 years following the Sung dynasty, China experienced an enormous increase in its population. Contrary to the prediction of the log-linear learning model, this increase did not prevent technological stagnation.

learning in the production of any particular good, using any particular process, is in fact finite and bounded. When a new technical process is first invented, rapid learning occurs as, by virtue of experience, the productive potential of that process is explored. After some time, however, the inherent (physical) limit on the productivity of a technology will be approached and learning will slow and, perhaps, ultimately stop. In the absence of the introduction of new technical processes, it is unlikely that learning by doing can be sustained. Put differently, it seems likely that in any given environment there is simply a finite amount of knowledge to be serendipitously acquired from experience in productive activity, as opposed to purposeful investigation.[3]

For their part, models of invention make the surprising assumption that new technologies attain their full productive potential at the moment of their invention and are, at that point in time, superior (or at

[3] Many of the early empirical studies of learning (e.g., Carr 1946; Asher 1956; Conway and Schultz 1959; Baloff 1966) argued that learning was bounded but did not test this proposition formally. Much of the empirical work since then (e.g., Wright 1936; Hirsch 1956; Alchian 1963; Lieberman 1984) has focused on the log-linear model, which allows for unbounded learning. Levhari and Sheshinski (1973), however, found that a formulation in which the elasticity of output with respect to experience was a (concave) quadratic function of the level of experience provided as good a fit as the log-linear model. Similarly, Epple, Argote, and Devadas (1991) found significant evidence of a slowdown in learning in a log-quadratic model. Head (1991) found that the learning elasticity was not significantly different from zero for the last 25 of his 50 years of data on the British steel rails industry.

Some readers might object to my emphasis on the serendipity of learning by doing. Clearly, not all the knowledge generated by production experience is unappropriable. To the degree that it is appropriable, learning will give rise to costly attempts to acquire knowledge. In this sense, a model with appropriable learning is conceptually and technically analogous to a model of invention, with current utility sacrificed in pursuit of the rents from knowledge. As an example, consider the case of a firm that initially produces a product of such poor quality that the market value of its output is near zero. As a result of its experience, however, the firm acquires the (appropriated) knowledge of how to produce a high-quality and valuable product. One might term the firm's investment in improving the quality of its product learning by doing or, equivalently, research and development.

Technological change may be the serendipitous by-product of other activities, the outcome of conscious attempts to acquire knowledge, or the result of a combination of both factors. Models of invention focus exclusively on the incentives to invest in costly research, whereas models of learning (with the exception of Spence [1981]) focus exclusively on the manner in which technical change emerges as the serendipitous by-product of other activities. Both approaches are extreme, and yet both focus one's attention on important aspects of the process of technical change. My argument above is that the amount of knowledge to be serendipitously acquired in any given environment, while positive, is finite and bounded. I am quite willing to believe that appropriable learning could give rise to sustained technical progress since I basically consider such activity to be conceptually indistinguishable from research. Like all research activity, appropriable learning will respond to the incentives (such as market size) to invest in knowledge and, consequently, generate the type of threshold/stagnation effects that I believe are necessary to explain premodern economic history.

least equal) to the older technologies for which they substitute (see, e.g., Romer 1990; Grossman and Helpman 1991; Aghion and Howitt 1992). The history of technical change suggests, however, that most new technologies are, in fact, initially broadly inferior to the older technologies they seek to replace and are actually competitive in only a narrow range of specialized functions. Incremental improvements over time, however, allow new technologies to ultimately dominate older systems of production across a wide variety of activities. Thus, the steam engine, even after the invention of the separate condenser by James Watt in 1765, was still merely a crudely engineered piston, which was principally used as a water pump in mines. At that point in time, it posed no threat whatsoever to the dominance of the waterwheel in the provision of power. Only after John Wilkinson's boring mill (in 1776) eliminated gaps between pistons and cylinders (which had previously been stuffed with rags) and William Murdock's sun and planet gearing system (in 1781) provided a means of converting vertical motion into rotary force did the steam engine become a generally useful source of power. Even so, as late as 1869 fully 48.2 percent of all primary power in U.S. manufacturing was still provided by waterwheels and turbines, which provided a more regular speed, required less maintenance, and used a cheaper fuel than steam engines. Similarly, mid-nineteenth-century steamships, because of frequent fouling of their iron bottoms and the need to carry on board large quantities of coal fuel, on long voyages were actually slower, less reliable, and more costly to operate than the more primitive wooden sailing ships. Although as early as 1850 steamships dominated sailing ships on short routes, it was not until the mid-1880s, when, after a variety of design changes and metallurgical advances, the fuel-efficient, high-pressure compound steam engine had been perfected, that they universally dominated the older technology.[4]

A recognition that most products of research, although brimming with potential, at the moment of their invention may be broadly inferior to more mature technologies alerts one to the important role of both production experience and complementary inventive activity in actualizing the enormous productive potential of new technologies. In this paper I focus on the role of experience, leaving an analysis of complementarity in inventive activity to another paper (Young 1993). A new technology, when invented, is capable of providing only limited utility services (per unit of factor input) to consumers and, hence,

[4] See Graham (1956), Harley (1971), Rosenberg (1976, chaps. 10, 11), and Burke (1985, chap. 6). Jewkes, Sawers, and Stillerman (1958) and Rosenberg (1976) provide additional examples of the initial inadequacies and subsequent improvements of new technologies.

only limited profits to the holder of its patent. Experience in production increases the productivity of the new technology, which allows it to supplant older technologies and increases the flow of profits to the holder of its patent. In time, additional inventions will appear, which, as production experience accumulates, in turn will supplant the now mature technology. Thus, both the rate of invention and the rate at which production experience accumulates will determine the life cycle, and hence discounted profitability, of new technologies. Just as a recognition of potential bounds on serendipitous learning makes the rate of learning dependent on the rate of invention, so a recognition of the need to actualize the productive potential of newly invented technologies makes the incentives to engage in costly invention, and hence the rate of invention, dependent on the rate of learning.

My objective in this paper is to develop a model that retains the main themes of traditional models of invention and learning while simultaneously introducing assumptions that capture some of the characteristics of the history of technical change noted above. Following models of invention, I assume that the invention of new products requires the costly sacrifice of current consumption as resources are reallocated from the production of goods to research. With each successful innovator receiving an infinitely lived patent to the product she invents, this aspect of the model emphasizes the incentives to engage in costly research. Following models of learning by doing, I assume that production experience generates new knowledge on how to produce goods more efficiently. Since the knowledge generated by this production experience is not appropriated by any economic actor, this aspect of the model emphasizes the incentives to produce different types of goods, with productivity improvements emerging as the serendipitous by-product of this goods production. Following many learning-by-doing models and numerous examples in the history of technical change, I also assume that the productivity gains generated by learning spill over across sectors.[5] Assuming that these spillovers are symmetric across sectors allows me to derive an analytically convenient measure of the society's cumulative learning experience.

To incorporate the arguments presented above, I shall assume that

[5] For formal models with learning spillovers across sectors, see Succar (1987), Boldrin and Scheinkman (1988), and Stokey (1988). Empirically, it is notable that clock makers played a critical role in the construction of early British industrial machinery, where they could make use of their experience with gearing mechanisms, metalworking, dynamics, and other problems in mechanical engineering. Similarly, the experience of fourteenth-century European craftsmen in the casting of church bells turned out to be very useful in constructing bronze cannons, which were decidedly superior to cannons constructed from wrought iron (Musson and Robinson 1960; Cipolla 1965, chap. 1; Rosenberg 1976, p. 328).

the potential for learning-induced productivity improvements in each good is finite and bounded. Consequently, the measure of the society's cumulative learning experience will, at any point in time, be bounded from above by the total number of goods the society has invented. I shall also assume that although newly invented goods have the potential to provide more utility per unit of factor input than older goods, they are initially inferior to mature technologies that have attained their maximum productivity (i.e., their learning bound), with the margin of inferiority increasing in the degree to which the newly invented goods move beyond the society's cumulated learning experience. In general equilibrium, learning and invention will be mutually interdependent, with sustained invention necessary to allow the continuation of (otherwise) bounded aggregate learning and sustained learning necessary to allow the continued invention of (otherwise) increasingly unproductive new technologies.

As we shall see, the model of this paper behaves like a true hybrid. With small markets, large rates of time preference, or relatively costly invention, the profitability of inventive activity is so low that no invention takes place. In this equilibrium, the model behaves exactly like a traditional model of invention (emphasizing the incentives to withdraw resources from production), with, in particular, the parameter governing the rate of learning having no effect. With very large markets or relatively inexpensive invention, inventive activity is extraordinarily profitable and tends to pull ahead of the society's learning experience, leading to an equilibrium with the rate of invention (and growth) paced by the society's rate of learning. In this equilibrium, the model behaves exactly like a model of learning in that the pattern of production is the sole constraint on growth, with, for example, a subsidy to inventive activity having no effect on the economy's steady-state growth rate. For intermediate parameter values, both invention and learning are important constraints, and policies aimed at either activity will influence the economy's growth rate.

Section II presents the formal structure of the model. Sections III and IV develop the instantaneous and intertemporal aspects of the general equilibrium. Section V analyzes the steady state, and Section VI concludes the paper.

II. A Model of Invention and Learning

Imagine that all the goods that have ever been or ever will be invented can be arranged, in order of increasing technical sophistication, along the real line. At any point in time, however, a society will know how to produce only a subset of this real line, goods in $[0, N(t)]$, where $N(t)$ naturally denotes the most sophisticated good the society is currently able to produce.[6] Labor is the sole factor of production, and the function $a(s, t)$ describes the amount of labor necessary to produce one unit of good s at time t:[7]

$$a(s,t) = \bar{a}e^{-s} \quad \forall s \in [0, T(t)],$$
$$a(s,t) = \bar{a}e^{-T(t)}e^{s-T(t)} \quad \forall s \in [T(t), N(t)], \tag{1}$$

with $T(t)$ evolving according to the learning-by-doing equation

$$\dot{T}(t) = \int_{T(t)}^{N(t)} \psi L(s,t)\, ds, \tag{2}$$

where ψ denotes the rate at which each worker learns.

This economy experiences bounded learning by doing with symmetric spillovers across goods. Learning is bounded in that the amount of labor required to produce each good s cannot fall below $\bar{a}e^{-s}$. Since all goods will enter into utility symmetrically (see below), the fact that this lower bound is downward sloping reflects the notion that the ultimate productivity of labor (in units of utility) is increasing in the technical sophistication of the production processes involved. There are spillovers in learning across goods, with technical improvements that originate in any particular industry s having applications in other industries. This is encompassed in the formulation for $\dot{T}(t)$, which essentially implies that these spillovers are symmetric across all sectors. Once an industry has reached its lower bound, $\bar{a}e^{-s}$, there is nothing left to learn in that industry. It seems logical to conclude that further experience in the production of that good will not contribute to productivity increases in the rest of the economy. Hence, the economywide learning-by-doing equation (2) includes only the labor devoted to the production of goods in which learning has yet to be exhausted.[8] Given the symmetric nature of learning-by-doing spillovers, learning is exhausted sequentially in goods, and $T(t)$, the most recent good to reach the lower bound $\bar{a}e^{-s}$, constitutes a state variable that summarizes the society's cumulative learning experience.[9]

New goods are invented through the creative efforts of entrepreneurs/firms, which acquire an infinitely lived patent on each good they invent. The rate of invention is linear in the aggregate amount

[6] Throughout this paper, the notation with respect to time denotes an implicit, rather than explicit, dependence that emerges from the general equilibrium behavior of the economic actors. Overdots will denote time derivatives.

[7] The function $a(s, t)$ is defined only on the domain $[0, N(t)]$.

[8] Thus, as in the typical learning-by-doing model, there is a constant rate of learning, ψ, but only until the industry has exhausted its potential stock of knowledge.

[9] As can be seen from (2), many different historical production paths could lead to the same $T(t)$.

of labor devoted to research:

$$\dot{N}(t) = \frac{L_R}{a_R}, \tag{3}$$

and there is free entry into the process of invention. Firms finance their research and development efforts by selling shares, which are traded in a capital market. After a good is invented, the firm that owns the patent will engage in monopolistic competition with all other patent holders, distributing any profits to its shareholders. The assumption that the unit labor requirements curve $a(s, t)$ is upward sloping beyond $T(t)$ reflects the notion that new technologies may initially be inferior to mature technologies (which have attained their maximum productivity). The further invention progresses beyond the society's cumulative learning experience; that is, the larger $N(t) - T(t)$, the larger the initial margin of inferiority.

In this economy there are L representative consumers, each of whom inelastically supplies one unit of labor at all times and seeks to maximize the present discounted value of the logarithm of a time-separable utility functional:

$$\max \mathcal{P} = \int_t^\infty e^{-\rho(v-t)} \log(U\{C(\cdot, v)\})\, dv \tag{4}$$

subject to the intertemporal budget constraint

$$\int_t^\infty e^{-R(v)+R(t)} E(v)\, dv = A(t) + \int_t^\infty e^{-R(v)+R(t)} w(v)\, dv, \tag{5}$$

where $R(t)$ denotes the cumulative interest factor up to time t, and $w(t)$ and $A(t)$ denote the nominal wage and individual assets at time t. Instantaneous consumer expenditure, $E(t)$, equals

$$E(t) = \int_0^{N(t)} p(s, t) C(s, t)\, ds, \tag{6}$$

with $p(\cdot, t)$ and $C(\cdot, t)$ describing goods prices and individual consumption along $[0, N(t)]$. The utility functional, $U\{C(\cdot, t)\}$, is given by[10]

$$U\{C(\cdot, t)\} = \int_0^{N(t)} \|C(t)\| g\left(\frac{C(s, t)}{\|C(t)\|}\right) ds, \tag{7}$$

where

$$\|C(t)\| = \int_0^{N(t)} C(s, t)\, ds \tag{8}$$

[10] Preferences are actually defined over all current and future goods along the real line, $[0, \infty)$. Since, at any point in time t, consumption of any good $x > N(t)$, which has yet to be invented, is trivially zero, for ease of exposition I restrict the upper bound on the integral to $N(t)$.

and $g(\cdot)$ is strictly concave and continuously differentiable, with

$$g(0) = 0, \quad g'(0) < \infty. \tag{9}$$

For the purposes of this paper, it is necessary to pick a particular functional form for $g(\cdot)$, and to that end I choose the quadratic[11]

$$g(x) = x - \frac{x^2}{2}. \tag{10}$$

Clearly, the presence of $\|C(t)\|$ in $g(\cdot)$ ensures that, for any given $p(\cdot, t)$ function, consumer demand for each good s is unit income elastic.[12] At the same time, the concavity of $g(\cdot)$, combined with the restriction on $g'(0)$, indicates a strong, but *not unbounded,* preference for variety. Changes in $p(\cdot, t)$ will lead to changes in the set of goods consumed, with, over time, new and more advanced goods replacing older, more primitive goods. Thus, these preferences retain a rich structure while remaining tractable enough to handle nontrivial intertemporal optimization.[13]

III. Instantaneous Equilibrium

I begin with the analysis of the instantaneous equilibrium at each time t. Let labor be the numeraire. Thus, the flow of each consumer's labor income equals one, and all prices and values are denominated in units of labor. To simplify the notation, in what follows I shall frequently suppress the notation denoting the implicit dependence of the variables on time.

Given the time separability of the consumer's utility function, the consumer's optimal consumption and expenditure program can be broken down into a two-stage analysis: first maximizing instantaneous utility subject to instantaneous expenditure and then, with $U\{C(\cdot, t)\}$ defined as a function of $E(t)$ and $p(\cdot, t)$, maximizing total intertemporal utility subject to the intertemporal budget constraint. With respect to the maximization of the instantaneous utility functional, $U\{C(\cdot)\}$,

[11] Absolutely none of the results of this paper is dependent on the choice of the quadratic functional form, which simply eases the analysis. To avoid the possibility of negative utility (due to insufficient variety) and to simplify the analysis, throughout this paper I assume that the initial stock of goods is sufficiently large that good 0 is no longer consumed. One can assume that this initial variety of goods was provided by nature.

[12] One can see that if one multiplies the consumption of each good s by a constant λ, the marginal utilities g' do not change.

[13] The notion of these types of preferences is borrowed from Wan (1975). For examples of preferences with a bounded desire for variety, but without unitary income elasticities, see Stokey (1988, 1991) and Young (1991). Despite the nonunitary income elasticities, Stokey (1991) handily solves the consumer's intertemporal maximization problem in the steady state.

the solution to this problem is best understood by characterizing $C(s)/\|C\|$ as a consumption density $f(s)$, which integrates to one. The consumer's problem is to allocate this consumption density across goods and then adjust the consumption scaling factor ($\|C\|$) on the basis of the desired level of instantaneous expenditure, that is,

$$\max U\{C(\cdot)\} = \int_0^N \|C\| g(f(s))\, ds \tag{11}$$

subject to

$$1 = \int_0^N f(s)\, ds \tag{12}$$

and

$$E = \int_0^N p(s) f(s) \|C\| ds. \tag{13}$$

Appendix A explores the mathematical details of the consumer's optimization problem. For our purposes here, it is sufficient to proceed intuitively.[14] Clearly, since all goods enter symmetrically into her utility, the consumer will choose to consume the cheapest goods. Similarly, symmetry implies that if the consumer consumes any good z, she will also consume all goods s cheaper than z. Given that the consumer has a bounded desire for variety, it follows that there exists some limit good Z such that the consumer consumes all goods that are cheaper than Z and consumes no goods that are more expensive than Z.[15] The density of consumption allocated to each good s (cheaper than Z) should, intuitively, depend in some fashion on the difference between the price of good Z and the price of good s. As it so happens, with quadratic utility this dependence is linear, with the optimal $f(s)$ being given by

$$f(s) = \lambda[p(Z) - p(s)], \tag{14}$$

where λ is the marginal utility derived from an additional unit of expenditure, E, at the consumer optimum. Good Z is determined by the requirement that the integral of the consumption density, $f(s)$, equal one (eq. [12]). The consumer's budget constraint (13) then determines the consumption scaling factor, $\|C\|$.

Since the output of each individual firm (of measure zero on the real line) makes no significant contribution to economywide learning, the current behavior of any particular firm does not influence its

[14] The reader is encouraged to become familiar with the intuitive exposition before turning to App. A.

[15] Given sufficient variability in goods prices. If all goods shared the same equilibrium price, the consumer would obviously choose to consume them all.

future profitability. Consequently, all firms find it optimal to maximize current profits:

$$\max_{p(s)} \pi(s) = LC(s)[p(s) - a(s)] = L\|C\|f(s)[p(s) - a(s)]. \quad (15)$$

The profit-maximizing equilibrium price of each good s is given by[16]

$$p(s) = \frac{p(Z) + a(s)}{2}. \quad (16)$$

Since (16) holds for firm Z as well, it follows that $p(Z) = a(Z)$. Consequently,

$$p(s) = \frac{a(Z) + a(s)}{2}. \quad (16')$$

The instantaneous equilibrium is illustrated in figure 1. Consumers consume a range of goods $\tau = T - Z$ and $\eta = A - T$ on either side of T. Given the unitary income elasticities, τ and η are invariant with respect to T and E and are determined by $N - T$ alone.[17] If $N - T$ is sufficiently small, $A = N$ (as drawn in fig. 1*a*). An increase in $N - T$ will lead to a substitution away from goods below T ($d\tau/d[N - T] < 0$) to those above T ($\eta = N - T$), with the total variety ($\tau + \eta$) of goods consumed increasing. If $N - T$ is larger than some critical value η^*, then, as illustrated in figure 1*b*, consumption is a symmetric around T ($\tau = \eta = \eta^*$), and the most recently invented goods, in $[T + \eta^*, N]$, are not consumed. Although the blueprints to produce these goods exist, their costs of production have not yet fallen to a level at which they can be profitably marketed. This is a case in which basic research has outstripped economywide learning, producing technologies that are "ahead of their time."[18]

Given the pattern of consumer expenditure determined by (14) and (16′), it is possible to compute the (consumer) price of a unit of instantaneous utility, $U\{C(\cdot, t)\}$:[19]

$$P_U = \bar{a}e^{-T}\left[\frac{(\tau + \eta)e^{\tau} + 2 - e^{\tau} - e^{\eta}}{2}\right], \quad (17)$$

[16] Since the individual firm is of insignificant measure on the real line, $\partial\lambda/\partial p(s) = \partial\|C\|/\partial p(s) = \partial p(Z)/\partial p(s) = 0$.

[17] The consumption scaling factor, however, is linear in Ee^T.

[18] An example is Charles Babbage's analytical engine (c. 1832), which would have performed many of the basic operations of the modern computer but required precision engineering far beyond the capabilities of nineteenth-century British craftsmen. Similarly, in the early eighteenth century, Christopher Polham devised a variety of techniques for the use of machinery in the construction of metal products that could not be implemented given the power sources and wooden machinery of his time (Hollingdale and Tootill 1965, pp. 42–49; Rosenberg 1976, chap. 11).

[19] Once again, see App. A for computational details.

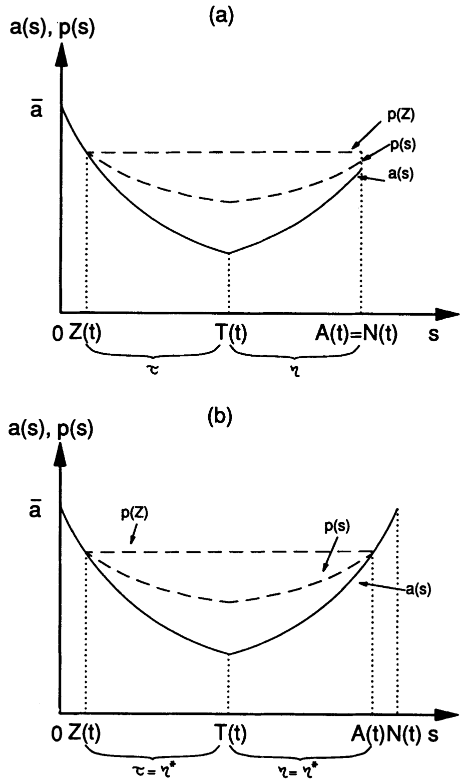

FIG. 1.—Static equilibrium

whereas the actual (labor) cost of producing this unit of utility is

$$MC_U = \frac{\bar{a}e^{-T}(2 + e^{2\tau} - 4e^{\tau} + 2e^{\eta}e^{\tau} - e^{2\eta})[e^{\tau}(\tau + \eta) + 2 - e^{\tau} - e^{\eta}]}{e^{2\tau}(2\tau + 2\eta) + 2 - e^{2\tau} - e^{2\eta}}. \quad (18)$$

Thus, overall, in this monopolistically competitive setting there is a markup (relative to costs) of

$$\frac{P_U}{MC_U} = \frac{e^{2\tau}(\tau + \eta) + 1 - (e^{2\tau}/2) - (e^{2\eta}/2)}{2 + e^{2\tau} - 4e^{\tau} + 2e^{\eta}e^{\tau} - e^{2\eta}} = m(N - T). \quad (19)$$

This economywide markup is decreasing in $N - T$.[20] An increase in $N - T$ provides the consumer with more symmetrically priced

[20] For $N - T < \eta^*$. For $N - T \geq \eta^*$, it is obviously invariant with respect to $N - T$. Appendix A provides a sketch of the proof.

substitutes, which, given that no good is essential in consumption, increases the elasticity of demand and squeezes the markups of all producers in the economy.[21] As will be seen further below, this result holds interesting implications for the relationship between rents and growth.

Finally, labor market equilibrium requires that the demand for labor in manufacturing (L_M) and research (L_R) equal the total supply:

$$L_M + L_R = L. \tag{20}$$

The amount of labor in final goods production is equal to aggregate consumer expenditure divided by the price of a unit of utility times the amount of labor required to produce a unit of utility, or, more simply, consumer expenditure divided by the economywide markup over costs:

$$L_M = \left(\frac{EL}{P_U}\right) MC_U = \frac{EL}{m(N - T)}, \tag{21}$$

and the distribution of that labor between learning and nonlearning industries is determined by $N - T$:

$$\begin{aligned} L_{LBD} &= L_M h(N - T), \\ L_{NLBD} &= L_M[1 - h(N - T)], \end{aligned} \tag{22}$$

where[22]

$$h(N - T) = \frac{2e^{\tau}e^{\eta} - 2e^{\tau} - e^{2\eta} + 1}{2 + e^{2\tau} - 4e^{\tau} + 2e^{\eta}e^{\tau} - e^{2\eta}} \leq 1/2, \quad h'(N - T) > 0. \tag{23}$$

IV. Intertemporal Equilibrium

Having derived the equilibrium price of a unit of utility, we can consider the consumer's dynamic optimization problem as one of picking an expenditure plan, $E(t)$, so as to maximize

$$\mathcal{P} = \int_t^{\infty} e^{-\rho(v-t)}\{\log[E(v)] - \log[P_U(v)]\}dv \tag{24}$$

subject to the intertemporal budget constraint (5) above. This leads

[21] This can be seen in fig. 1, where, for given T, an increase in N lowers $p(Z)$ (increases Z), reducing the gap between $p(s)$ and $a(s)$ for all firms above Z.

[22] When $N - T \geq \eta^*$, $h(N - T) = 1/2$, since, in that case, demand is distributed symmetrically between learning and nonlearning industries (examine fig. 1b). For a full derivation of $h(N - T)$, see App. A.

to the familiar optimality condition for the time path of expenditure $E(t)$:

$$\frac{\dot{E}(t)}{E(t)} = \dot{R}(t) - \rho. \tag{25}$$

Turning to the behavior of firms, let $V(s, t)$ equal the asset market value of holding the patent to good s at time t:

$$V(s, t) = \int_t^\infty e^{-R(v)+R(t)} \pi(s, v)\, dv, \tag{26}$$

where $\pi(s, t)$ denotes the profits of firm s at time t. Differentiating (26) with respect to time yields an expression for the interest rate:

$$\dot{R}(t) = \frac{\dot{V}(s, t)}{V(s, t)} + \frac{\pi(s, t)}{V(s, t)} \quad \forall s, \tag{27}$$

which simply states that, in this deterministic environment, asset market equilibrium requires that the return to holding the patent to any good s (profits plus patent value appreciation) must equal the risk-free rate of return. Free entry into the inventive process will ensure that the present discounted value of the profits of firm $N(t)$ will be less than or equal to the cost of invention:[23]

$$a_R \geq V(N(t), t) = \int_t^\infty e^{-R(v)+R(t)} \pi(N(t), v)\, dv \quad (= \text{if } \dot{N}(t) > 0). \tag{28}$$

Assuming that $\dot{N}(t) > 0$ and thus that (28) holds with equality, we can differentiate to derive an expression for $\dot{R}(t)$ as a function of easily determinable values:[24]

$$\begin{aligned} \dot{R}(t) &= \frac{\pi(N(t), t) - \dot{N}(t)\left[\int_t^\infty e^{-R(v)+R(t)} \pi_1(N(t), v)\, dv\right]}{a_R} \\ &= \frac{\pi(N(t), t) - \dot{N}(t) V_1(N(t), t)}{a_R}. \end{aligned} \tag{29}$$

Figure 2 helps explain equilibrium condition (29). At any point in time, the value of each firm depends on its position along the real line. Firms $s \leq Z(t)$ have a value of zero since, as demand has moved

[23] Consider that by devoting labor l_R to invention for an infinitesimally small period of time dt, a firm can, at cost $l_R dt$, acquire patents to goods in $(N(t), N(t) + (l_R dt/a_R)]$, which, again for infinitesimally small dt, have value $V(N(t), t) l_R dt/a_R$. With free entry, profits in the process of invention are driven to zero, and hence $a_R = V(N(t), t)$.

[24] If $\dot{N} = 0$, then $L_R = 0$ and the equilibrium interest rate can be determined by differentiating (21) with respect to time and applying (25). The subscript 1 in (29) denotes the derivative with respect to the first argument of the function.

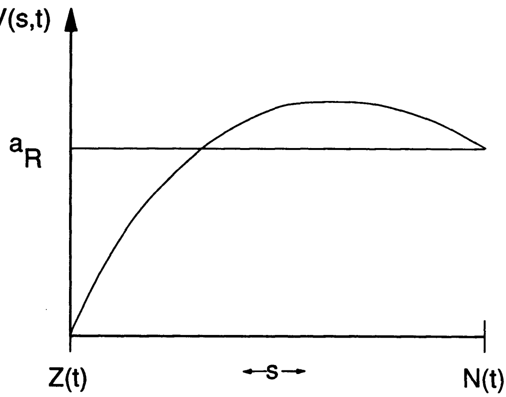

FIG. 2.—Market value of firm s

to the right, they will never again earn any profits.[25] Free entry into the inventive process ensures that $V(N(t), t) = a_R$. Thus, we know that there exists a differentiable curve[26] describing $V(s, t)$ linking the coordinates $(Z(t), 0)$ and $(N(t), a_R)$, as drawn in the figure. This curve need not be monotonic, nor is it stationary through time. Comparing (27) and (29), we see that we need to show that $V_2(N(t), t) = -\dot{N}(t)V_1(N(t), t)$. At time t, the value of firm $N(t)$, $V(N(t), t)$, equals a_R. However, at that same time t, as a result of invention, firm $N(t)$ is being pushed to the left in figure 2 (since some other firm is becoming $N(t + dt)$). The rate of change of the value of firm $N(t)$, $V_2(N(t), t)$, depends on the derivative of the $V(s, t)$ function at the point $N(t)$, $V_1(N(t), t)$, times the rate at which firm $N(t)$ is being pushed to the left, $-\dot{N}(t)$. Hence, $V_2(N(t), t) = -\dot{N}(t)V_1(N(t), t)$, which explains (29).

To summarize, consumers, maximizing current utility subject to current expenditure, choose a distribution of expenditure across goods that depends only on $N(t) - T(t)$, as illustrated in figure 1, with the density of expenditure given by (14). Maximizing intertemporal utility, consumers find it optimal to set the growth rate of expenditure equal to the interest rate minus their rate of time discount, equation (25). Existing firms, competing monopolistically, set their current price as the average of their costs of production and those of the limit good Z, (16′). Free entry into invention, combined with asset market equilibrium, determines the interest rate as a function of the profits of the most advanced firm ($N(t)$) and the rate of change of the value of that firm, (29). The current level of consumer expenditure, as well

[25] Increases in either T or N move Z to the right. Hence, once a firm s has become $Z(t)$, there will never again be any demand for its product.

[26] Examine (26), (15), (16), and (14).

as the economy's current structure (as given by $N - T$), determines the amount of labor allocated to industries in which learning continues and industries in which learning has been exhausted, (21)–(22). Labor market clearing requires that this labor, plus the labor allocated to research, equal the total labor force, (20). Given these relations, the intertemporal equilibrium then consists of dynamic paths for $T(t)$, $N(t) - T(t)$, and $E(t)$, which satisfy equations (2), (3), and (5).

V. Steady-State Behavior

In the steady state, E and $N - T$ must be constant. This implies that the real rate of interest ($\dot{R}$) equals the rate of time discount (ρ) and that the rate of invention ($\dot{N}$) equals the rate of learning ($\dot{T}$). With a constant steady-state level of expenditure, E, and an exponentially declining price of a unit of $U\{C(\cdot)\}$,[27] the proportional rate of growth of $U\{C(\cdot)\}$ is equal to the equilibrium rate of learning and invention, which I shall call g.[28]

It is easiest to begin our analysis by examining two extreme types of equilibria. This model, as in typical models of invention, allows for a stagnant steady state with zero growth. In that case, $\dot{T} = \dot{N} = 0$ and $N - T = 0$, with all the firms in $(Z, T]$ earning an infinitely lived stream of constant profits. This will constitute an equilibrium if and only if the present discounted value of the flow of profits to firm T is less than or equal to the cost of invention.[29] Intuitively, the flow of profits to each firm in this equilibrium should be proportional to the aggregate market size, a measure of which is the total labor in manufacturing ($L_M = L$). As it so happens, for the functional forms chosen in this paper, the instantaneous flow of profits to firm T in this stagnant steady state is exactly equal to L.[30] Thus, for stagnation to occur, it is necessary and sufficient that

$$a_R \geq \frac{L}{\rho}. \tag{30}$$

Condition (30) states that if the aggregate market is small enough, the cost of invention (a_R) high enough, or the steady-state rate of interest (ρ) large enough, firms will not find it profitable to invent, and the economy will stagnate. It is interesting to note that the rate of labor learning, ψ, has no effect on the existence of this equilibrium,

[27] Recall that $P_U = \bar{a}e^{-T}[(\tau + \eta)e^{\tau} + 2 - e^{\tau} - e^{\eta}]/2$.

[28] Since instantaneous utility is actually the log of $U\{C(\cdot)\}$, for $\rho > 0$ total intertemporal utility will always be bounded, no matter how large g.

[29] Otherwise, entrepreneurs would find it profitable to invent products infinitesimally to the right of T.

[30] For mathematical details, see App. B.

for there must be at least some minimal level of invention for learning to occur and have an impact on growth. This contrasts with the typical learning-by-doing model, in which there is always some growth (even at small scales of production) and in which the rate of growth is always strictly increasing in the rate of learning. Thus, for small market sizes, this model behaves *exactly* like an endogenous growth model based solely on invention (with no modeling of learning). In this equilibrium, the incentives to withdraw resources from goods production for use in the research and development of new technologies are the sole constraints on the growth process.

For L, ρ, and a_R such that (30) does not hold, the steady-state growth rate is positive, with $N - T > 0$. An increase in L relative to a_R raises the profitability of invention, leading to an endogenous increase in $N - T$, which lowers markups, squeezes profitability, and reestablishes an equilibrium. For large enough L or small enough a_R, the steady-state equilibrium $N - T$ exceeds η^* (recall fig. 1*b*). In this case, potential profits are so large (relative to the cost of invention) that firms find it optimal to invent products before they are even marketable, holding the patents until aggregate production experience brings their costs of production down to acceptable levels.[31] The rate of growth in this type of equilibrium is easily computed. With $N - T > \eta^*$, the demand for manufactured goods will be distributed symmetrically between learning and nonlearning industries. Consequently, $\dot{T} = g = \psi L_M/2$. With $\dot{N} = g = L_R/a_R$ and labor market clearing requiring that $L_M + L_R = L$, it follows that

$$g = \frac{\psi L}{2 + \psi a_R}. \tag{31}$$

Although both the cost of invention and the learning parameter influence the rate of growth, it is apparent that, in this equilibrium, the growth rate is (locally) independent of the incentives for invention.

[31] Although a small ρ is sufficient to guarantee an equilibrium with positive rates of invention, it is not sufficient to guarantee an equilibrium with $N - T > \eta^*$. With $N - T > 0$ in the steady state, the economy experiences a positive rate of growth and transformation, with each firm passing through a life cycle from invention at time t, $N(t)$, to maturity at time t', $T(t')$, to obsolescence at t'', $Z(t'')$. Consequently, each firm accumulates positive profits over a finite time horizon. For a small enough market size relative to the cost of invention, there does not exist any $\rho \geq 0$ sufficient to justify an equilibrium with $N - T > \eta^*$. Similarly, changes in ψ can change the economy's growth rate and rate of transformation and, hence, the rate at which firms transit from one state to another. Nevertheless, if L is sufficiently small relative to a_R, given the small markups in a state with $N - T > \eta^*$, there exists no rate of transition sufficient to justify the cost of invention. However, a value of ρ near zero (almost no discounting) and a value of ψ near zero (slow transition/long life cycle) are jointly sufficient to guarantee the existence of an equilibrium with $N - T > \eta^*$. Proofs of these statements are available on request from the author.

For example, a subsidy to invention would increase only $N - T$ (which increases the time spent before the firm begins to earn positive profits and thereby reestablishes an equality between the private costs and benefits of inventive activity), without influencing the steady-state growth rate in any way.[32] Thus, for large market sizes and low costs of invention, this model behaves *exactly* like a simple learning-by-doing model with a constant learning parameter (equal to $\psi/[2 + \psi a_R]$). In this equilibrium, as in traditional learning-by-doing models, it is the static pattern of demand and production alone, that is, the incentives to produce different types of goods, rather than the incentives to engage in research and development, that determines the economy's growth rate.

Outside of these two types of equilibria, the steady-state growth rate is positive, with $\eta^* > N - T > 0$. The determination of the equilibrium growth rate can be analyzed using some simple graphical tools. One can think of this economy as having two sectors, a manufacturing (consumption) goods sector and an invention (growth) goods sector. Using the equation

$$L_M + ga_R = L, \tag{PPF}$$

one can draw a production possibilities frontier illustrating the potential trade-offs between the size of the consumption goods sector (as measured by L_M) and the rate of invention, g (fig. 3*a*). Clearly, a rise in L expands the total resource base of the economy and shifts the curve out, whereas an increase in a_R rotates it clockwise.

In the steady state the rate of learning must equal the rate of invention. This "balanced growth" relation can be written as

$$\psi h(N - T)L_M = g, \tag{BG}$$

which, for given $N - T$, can be drawn (in g, L_M space) as a ray emanating from the origin. As $N - T$ increases, a greater proportion of any manufacturing labor force is allocated to learning industries ($h' > 0$), and thus the curve rotates clockwise, reaching the limiting locus $\psi L_M/2 = g$ for $N - T \geq \eta^*$. An increase in ψ rotates the BG curve clockwise.

Finally, free entry into invention defines a factor market equilibrium relation, which states that the return on devoting a unit of labor to invention, that is, $V(N(t), t)/a_R$, must be less than or equal to the real return to labor in manufacturing, that is, one:

$$1 \geq \frac{V(N(t), t)}{a_R}. \tag{FME}$$

[32] In fact, during the transition dynamics, in which $N - T$ increases, it would actually lower the growth rate by drawing labor out of the learning sector.

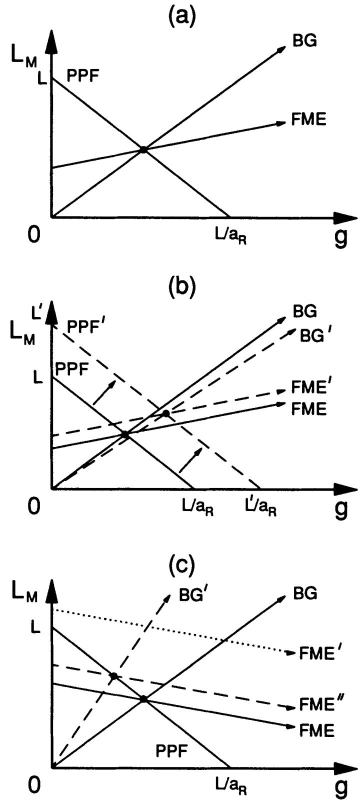

FIG. 3.—Graphical determination of the steady state

It is easily shown that $V(N(t), t)$ depends on the steady-state size of the final goods market, as measured by L_M, and lifetime profits per unit of market size, which depend on $N - T$, ρ, and g:[33]

$$V(N(t), t) = L_M T(N - T, \rho, g). \tag{32}$$

A rise in L_M, for given $N - T$, ρ, and g, will raise the value of firm $N(t)$. The partial effect of an increase in the steady-state rate of invention and learning, g, is more ambiguous. With a more rapid rate of learning, firms find that the society's production basket is moving to the right more rapidly; thus the firm transits more quickly from being firm $N(t)$ at time t, to being firm $T(t')$ at time t', to being firm $Z(t'')$ at time t''. As a firm goes from being $N(t)$ to $T(t')$, its profitability rises (see fig. 1), but its profitability falls as it transits from $T(t')$ to $Z(t'')$. In addition, although a more rapid rate of transit puts a firm in a

[33] For details, see App. B.

more desirable state ($T(t')$) at an earlier date, the more rapid rate of transit also ensures that the firm will spend less time in each such state and ultimately move on to less desirable states. Whether the net effect is positive or negative depends on the distance between N and T as well as the discount rate ρ.[34] Thus, in (g, L_M) space, the factor market equilibrium relation may slope upward or downward (as drawn in fig. 3*a* and *c*).[35] For any given market size and rate of growth, an increase in $N - T$ (which squeezes markups), ρ, or a_R will lower the relative profitability of invention, shifting the entire FME curve up.

The steady-state size of the final goods market L_M, rate of learning and invention g, and level of $N - T$ are determined by the joint intersection of these three curves.[36] For example, consider an expansion in the resource base, L, that shifts out the PPF curve in figure 3*b*. At the original growth rate g, invention is now more profitable, leading to a surge in inventive activity that outpaces the rate of learning, which increases $N - T$. As $N - T$ rises, the balanced growth equation rotates down (more of any given manufacturing labor force is allocated to learning industries) and the factor market equilibrium relation shifts upward (lower return to invention requiring an increase in market size), establishing a new equilibrium, with increased

[34] For example, for a discount rate of zero, the net effect is always negative since there is no benefit to arriving at a more favorable state earlier in time, but the higher rate of transit ensures that the time spent in each state is shorter. Similarly, as $N - T$ goes to zero, the net effect becomes negative since, along $[Z, T]$, the firm finds itself moving more rapidly into less profitable states. For large ρ and large $N - T$, however, it can be shown that the net effect is unambiguously positive.

[35] Clearly, there is no intuitive reason why the FME curve cannot be steeper than the PPF curve (whose slope is given by technological constraints). The FME curve must, however, be flatter than the BG curve. The intuition is as follows: The FME curve slopes upward whenever the partial effect of an increase in g on the net present value of profits is negative because an increase in the growth rate results in a more rapid rate of transit from good to bad states. As one moves up along the BG curve, the final goods market size is increasing in proportion to the growth rate. Thus, if one moves up along the BG curve from an initial intersection with the FME, one finds that although firms are transiting through states at a faster rate, the flow of profits in each such state (which is linear in L_M) has increased in proportion to the increased rate of transit, which exactly cancels the negative effect of the increased rate of transit. In essence, the firm enjoys the same (undiscounted) integral value of profits. However, with a positive rate of discount and the flow of profits compressed into a shorter time frame, the value of the firm actually rises. A reduction in L_M is necessary to reestablish an equality between the return to labor in invention and the return to labor in manufacturing. Consequently, the FME curve must be flatter than the BG curve. An appendix, available on request from the author, proves formally that the FME curve is always flatter than the BG curve and, when negatively sloped, may be either more or less steep than the PPF curve.

[36] I have focused my analysis on L_M rather than E since the steady-state value of the latter depends on the overall markup and is therefore harder to interpret. As can be seen from (21), L_M and $N - T$ jointly determine E.

levels of L_M, g, and $N - T$. Alternatively, an increase in ρ shifts the FME curve up (fig. 3c, drawn for a downward-sloping FME curve). As invention becomes less profitable, the rate of learning will begin to outstrip the rate of invention, leading to a drop in $N - T$. Consequently, the FME curve will shift back down and the BG curve rotate up, reestablishing a steady-state equilibrium with a larger manufacturing labor force (L_M) and lower levels of $N - T$ and g.

Using similar graphical analyses, one can derive the following results, most of which are fairly intuitive:[37]

$$\frac{\partial g}{\partial L} > 0, \quad \frac{\partial g}{\partial a_R} < 0, \quad \frac{\partial g}{\partial \psi} > 0, \quad \frac{\partial g}{\partial \rho} < 0,$$

$$\frac{\partial L_M}{\partial L} > 0, \quad \frac{\partial L_M}{\partial a_R} \gtreqless 0,^{38} \quad \frac{\partial L_M}{\partial \psi} < 0, \quad \frac{\partial L_M}{\partial \rho} > 0, \tag{33}$$

$$\frac{\partial N - T}{\partial L} > 0, \quad \frac{\partial N - T}{\partial a_R} < 0, \quad \frac{\partial N - T}{\partial \psi} \gtreqless 0,^{39} \quad \frac{\partial N - T}{\partial \rho} < 0.$$

Finally, it is interesting to note that this model has implications for the relationship between growth rates and the share of rents in national income. Presumably, most economies face the same underlying technical opportunities, that is, have similar a_R's and ψ's, but may vary in their resource base/market size, L, or the rate at which inventive

[37] The graphical analysis is more difficult when the downward-sloping FME curve is steeper than the PPF curve. In that case, an increase in $N - T$ moves both the FME and BG curves to the right. An appendix, available on request from the author, shows that in this case an increase in $N - T$ always moves the FME curve further along the PPF than the BG curve. Thus, e.g., in the case of an increase in ρ analyzed in fig. 3c, if the FME curve were steeper than the PPF curve, then the initial upward shift in the FME curve would put its intersection with the PPF curve to the right of that of the BG curve. A fall in $N - T$, however, would move the FME curve to the left (along the PPF curve) faster than it would the BG curve, reestablishing a three-way intersection at a lower $N - T$ and higher L_M. The same appendix also establishes that the steady state is always unique.

[38] An increase in a_R lowers the rate of invention but increases the amount of labor required for any given rate of invention. The net effect on L_M depends on the type of equilibrium. When $N - T$ is near zero, an increase in a_R shuts down all invention and unambiguously increases L_M. When $N - T > \eta^*$, the total labor in invention equals $a_R\psi L/(2 + a_R\psi)$, which is unambiguously increasing in a_R.

[39] A rise in ψ increases the rate of learning and transformation in the steady state. To match the increased rate of learning associated with a rise in ψ, more labor is drawn into invention ($\partial L_M/\partial\psi < 0$). The reduction in market size lowers the return to invention, but, as discussed before, the effect of the increased rate of transit on the return to invention is ambiguous. Generally, the market reduction effect dominates, and the return to steady-state equilibrium requires a reduction in $N - T$. However, when the FME curve is steeper than the PPF curve, the effect of an increase in the rate of transit is strongly positive and dominates the loss associated with a reduction in market size. In this case, a return to steady-state equilibrium actually requires an increase in $N - T$.

profits must be discounted, ρ.[40] An increase in L or a fall in ρ leads to a steady-state rise in $N - T$, lowering the economywide markup. Consequently, economies that grow faster will tend to have lower markups and, thus, a lower share of rents in national income.[41] It is a common belief that in slow-growth economies a larger proportion of income is derived from rents, with the usual explanation focusing on the perceived detrimental effects of rent seeking on growth. The model of this paper suggests that the direction of causality might in fact be the reverse. In economies with low growth rates, holders of privilege reap large rents because there are few emerging competitors. If the economy were growing faster, then, even though each new entrant also acquired privilege, the intensified competition could lower the overall proportion of rents in national income. If one wants to argue that rent seeking reduces growth rates, the argument should perhaps rest not on the detrimental effects of rent seeking per se, but rather on the attempts by existing rentiers to bar other claimants to government-decreed privilege. Thus, rent seeking by economic actors in a fairly open political system, such as that of the United States, need not have detrimental effects on growth and might, in fact, encourage growth by allowing innovators to reap rewards greater than those that could be achieved under a free-market system.[42]

VI. Conclusion

This paper has made a first attempt at integrating models of invention and models of learning by doing. In part, I have followed the

[40] I should note that in a more complex model, such as Romer's (1990) multifactor model, the appropriate measure of resource base or market size is not simply the aggregate labor force. Similarly, in a model with risk the rate at which inventive profits are discounted would exceed the rate of time preference, ρ, and could vary from country to country.

[41] Total profit income (TPI) equals total sales times profits per sale:

$$\text{TPI} = \frac{EL}{P_U}(P_U - MC_U) = EL\frac{m(N-T)-1}{m(N-T)} = L_M[m(N-T)-1].$$

Using the PPF and BG relations, one can solve for TPI in the steady state:

$$\text{TPI}_{SS} = L\left[\frac{m(N-T)-1}{1+\psi a_R h(N-T)}\right].$$

The share of rents in national income equals TPI/(L + TPI), which, for given $N - T$, is clearly homogeneous of degree zero in L. As $m'(\cdot) < 0$ and $h'(\cdot) > 0$, while $N - T$ is increasing in L and decreasing in ρ, it follows that increases in the steady-state growth rate brought about by changes in L or ρ will lower the share of rents in national income.

[42] The positive welfare effects of granting innovators temporary monopolies are cleanly illustrated in Krugman (1990).

assumptions typical to these models, modeling invention as the costly and conscious attempt to develop new technologies (for which inventors are rewarded with patents) and learning by doing as a process whereby technical improvements are generated as the serendipitous (and unappropriated) by-product of goods production. I have, however, modified these models with assumptions that, I believe, are motivated by the history of technological change, arguing that, in the absence of further costly invention, learning by doing is fundamentally bounded and that, in the absence of further development, most of the new technologies developed by research are broadly inferior to existing productive techniques.

The resulting model is a true hybrid. When the cost of invention is large relative to market size, the profitability of invention is low, and hence the rate of invention becomes the constraining factor in growth, with the learning parameter having no effect. In this type of "invention-constrained" equilibrium, the incentives to withdraw resources from current production for use in costly research are the sole constraint on the growth process. In contrast, when the cost of invention is small relative to market size, invention is extremely profitable and tends to run ahead of the society's cumulated learning experience. This leads to a "learning-constrained" equilibrium in which the incentives to produce different types of goods, that is, the pattern of consumer demand, become the relevant constraint on the growth process.

These results are suggestive of the types of issues different types of endogenous growth models might most appropriately address. For the analysis of the industrial revolution or the historical periods of growth and stagnation alluded to in the Introduction, models of invention, with their emphasis on minimum market size (i.e., threshold effects) and the incentives to engage in costly research, would seem to be the most appropriate. For the analysis of the interaction of modern trading economies, given the large size of the international market, it might easily be the case that the incentives to produce different types of goods are the principal constraint on the growth process, suggesting that the simple insights into the dynamic effects of static comparative advantage provided by many learning-by-doing models provide a useful means of thinking about the postwar growth process.

Appendix A

Mathematical Details of the Instantaneous Equilibrium

A. Consumer's Optimization Problem

Combining (11)–(13), one can form the Lagrangian:

$$\mathscr{L} = \int_0^{N(t)} \|C\| g(f(s))\,ds + \lambda\left[E - \int_0^{N(t)} \|C\| p(s) f(s)\,ds\right] + \zeta\left[\|C\| - \int_0^{N(t)} \|C\| f(s)\,ds\right]. \tag{A1}$$

The conditions necessary and sufficient for $f(\cdot)$, ζ, λ, and $\|C\|$ to maximize this Lagrangian are[43]

$$g'(f(s)) \leq \zeta + \lambda p(s) \quad (= \text{if } f(s) > 0), \tag{A2}$$

$$1 = \int_0^{N(t)} f(s)\,ds, \tag{A3}$$

$$E = \int_0^{N(t)} \|C\| p(s) f(s)\,ds, \tag{A4}$$

$$\lambda \int_0^{N(t)} p(s) f(s)\,ds = \int_0^{N(t)} g(f(s))\,ds. \tag{A5}$$

Since $g'(f(s)) = 1 - f(s)$ and ζ and λ are constants, it is apparent from (A2) that the consumer will consume all goods cheaper than some good Z, with $f(s) \to 0$ as $p(s) \to p(Z)$ from below. Consequently, $\zeta = 1 - \lambda p(Z)$. Substituting back into (A2) yields equation (14) in the text. Maximization on the part of firms then gives (16). As highlighted in figure 1, the consumer enjoys all goods in $(Z, A]$, where

$$A - T = N - T \quad \text{or} \quad A - T = T - Z. \tag{A2$'$}$$

Substituting (14) and (16) from the text into (A3)–(A4) and integrating (using the notation $\eta = A - T$ and $\tau = Z - T$), we get

$$\begin{aligned} 1 &= \int_0^N f(s)\,ds = \int_Z^A \lambda[p(Z) - p(s)]\,ds = \int_Z^A \frac{\lambda}{2}[a(Z) - a(s)]\,ds \\ &= \frac{\lambda}{2}\left(\int_Z^T \bar{a}e^{-Z} - \bar{a}e^{-s}\,ds + \int_T^A \bar{a}e^{-Z} - \bar{a}e^{-T}e^{s-T}\,ds\right) \\ &= \frac{\lambda}{2}[(\tau + \eta)\bar{a}e^{-Z} + 2\bar{a}e^{-T} - \bar{a}e^{-Z} - \bar{a}e^{-T}e^{A-T}] \\ &= \frac{\lambda \bar{a}e^{-T}}{2}[(\tau + \eta)e^{\tau} + 2 - e^{\tau} - e^{\eta}], \end{aligned} \tag{A3$'$}$$

[43] Concavity of $g(\cdot)$ ensures concavity of the integrand in $f(s)$. Thus the inequality-constrained Euler equation (A2) is both necessary and sufficient for a path $f(s)$ to maximize (A1). If the optimal $f(s)$ is taken as given, since the Lagrangian is negative semidefinite in $\|C\|$, ζ, and λ, the first-order conditions (A3)–(A5) are necessary and sufficient for an optimum, where (1) I treat the problem as one of unconstrained optimization, since (A4) will ensure that $\|C\|$ is nonnegative; and (2) the envelope

$$E = \int_0^N \|C\| p(s) f(s)\,ds = \|C\| \int_Z^A \left[\frac{a(Z) + a(s)}{2}\right] \lambda \left[\frac{a(Z) - a(s)}{2}\right] ds$$

$$= \frac{\lambda\|C\|}{4}\left(\int_Z^T \bar{a}^2 e^{-2Z} - \bar{a}^2 e^{-2s} ds + \int_T^A \bar{a}^2 e^{-2Z} - \bar{a}^2 e^{-2T} e^{2s-2T} ds\right) \quad \text{(A4}')$$

$$= \frac{\lambda\|C\|\bar{a}^2 e^{-2T}}{4}\left[e^{2\tau}(\tau + \eta) + 1 - \frac{e^{2\tau}}{2} - \frac{e^{2\eta}}{2}\right].$$

Since (A4) implies that $\lambda \int_0^N p(s) f(s)\,ds = \lambda E/\|C\|$, we also have

$$\frac{\lambda E}{\|C\|} = \int_0^N g(f(s))\,ds = \int_0^N f(s) - \frac{f(s)^2}{2} ds$$

$$= 1 - \frac{1}{2}\int_0^N f(s)^2 ds = 1 - \frac{\lambda^2}{8}\int_Z^A [a(Z) - a(s)]^2 ds \quad \text{(A5}')$$

$$= 1 - \frac{\lambda^2 \bar{a}^2 e^{-2T}}{8}\left[\left(\tau + \eta - \frac{3}{2}\right)e^{2\tau} - 1 + 4e^{\tau} + \frac{e^{2\eta}}{2} - 2e^{\tau}e^{\eta}\right].$$

Combining (A3′)–(A5′), one can derive a single equation in τ and η alone:

$$\Phi(\tau, \eta) = 2[(\tau + \eta)e^{\tau} + 2 - e^{\tau} - e^{\eta}]^2 - 3e^{2\tau}(\tau + \eta) - 1 + {}^5\!/\!_2 e^{2\tau} + {}^1\!/\!_2 e^{2\eta} + 2e^{\tau}e^{\eta} - 4e^{\tau} = 0. \quad \text{(A6)}$$

It is easily shown that there exists an $\eta^* > 0$ such that $\Phi(\eta^*, \eta^*) = 0$. Furthermore,[44]

$$\Phi_\tau|_{\Phi=0} = \frac{e^{\tau}[(\tau + \eta)(2 + 3e^{2\tau} - 8e^{\tau} + 4e^{\eta}e^{\tau} - e^{2\eta}) + 8e^{\tau} + 8e^{\eta} - 2e^{2\tau} - 2e^{2\eta} - 4e^{\tau}e^{\eta} - 8]}{(\tau + \eta)e^{\tau} + 2 - e^{\tau} - e^{\eta}}$$
$$> 0 \quad \forall \tau > \eta \geq 0,$$

$$\Phi_\eta|_{\Phi=0} = \frac{(e^{\tau} - e^{\eta})[e^{2\tau}(3\tau + 3\eta) - 2e^{2\tau} + 2e^{\tau} - 2e^{\eta} - e^{\eta}e^{\tau}(\tau + \eta) + 2]}{(\tau + \eta)e^{\tau} + 2 - e^{\eta} - e^{\tau}} > 0$$
$$\forall \tau > \eta \geq 0. \quad \text{(A7)}$$

Recall the restriction imposed by (A2′). Consequently, for $N - T \geq \eta^*$, $\tau = \eta = \eta^*$. For $N - T < \eta^*$, Φ determines a unique $\tau > \eta^* > \eta = N - T$, with $d\tau/d(N - T) = d\tau/d\eta|_{\Phi=0} < 0$. In each case, $\|C\|$ and λ are given by

$$\|C\| = \frac{2E[(\tau + \eta)e^{\tau} + 2 - e^{\tau} - e^{\eta}]}{\bar{a}e^{-T}[e^{2\tau}(\tau + \eta) + 1 - (e^{2\tau}/2) - (e^{2\eta}/2)]} \quad \text{(A8)}$$

theorem ensures that the partial of $f(s)$ with respect to $\|C\|$, ζ, and λ does not appear in these first-order conditions.

[44] Take the derivative and substitute for $e^{\tau}(\tau + \eta) + 2 - e^{\tau} - e^{\eta}$ using (A6).

and

$$\lambda^{-1} = \frac{\bar{a}e^{-T}}{2}[(\tau + \eta)e^{\tau} + 2 - e^{\tau} - e^{\eta}]. \tag{A9}$$

B. P_U and MC_U

From (A1) and (A5′), it can be seen that λ is the utility of an additional unit of expenditure at the consumer optimum. Consequently, λ^{-1} is the price of a unit of utility, which is the origin of equation (17) in the text. Define $\|C\|_1$ as the $\|C\|$ necessary to enjoy one unit of utility at the consumer optimum. This is easily determined using $U\{C(\cdot)\} = \lambda E = 1$ and (A4′). The total labor used in producing a unit of U equals $\|C\| \int_0^N f(s)a(s)\,ds$, which, with (A4′) and (A9), yields (18) in the text:

$$\begin{aligned} MC_U &= \|C\|_1 \int_0^N f(s)a(s)\,ds \\ &= \|C\|_1 \left(\frac{\lambda}{2}\right)\bar{a}^2 e^{-2T}(1 - 2e^{\tau} + \tfrac{1}{2}e^{2\tau} + e^{\tau}e^{\eta} - \tfrac{1}{2}e^{2\eta}) \\ &= \frac{2 - 4e^{\tau} + e^{2\tau} + 2e^{\tau}e^{\eta} - e^{2\eta}}{\lambda[e^{2\tau}(\tau + \eta) + 1 - \tfrac{1}{2}e^{2\tau} - \tfrac{1}{2}e^{2\eta}]} \\ &= \frac{\bar{a}e^{-T}[(\tau + \eta)e^{\tau} + 2 - e^{\tau} - e^{\eta}](2 - 4e^{\tau} + e^{2\tau} + 2e^{\tau}e^{\eta} - e^{2\eta})}{e^{2\tau}(2\tau + 2\eta) + 2 - e^{2\tau} - e^{2\eta}}. \end{aligned} \tag{A10}$$

C. $m'(N - T)$

For $N - T < \eta^*$, $\eta = N - T$. Consequently,

$$m'(\cdot) = \frac{e^{\tau}(e^{\tau} - e^{\eta})[(\tau + \eta)A(\tau, \eta) + B(\tau, \eta)]}{\Phi_{\tau}[(\tau + \eta)e^{\tau} + 2 - e^{\tau} - e^{\eta}](2 + e^{2\tau} - 4e^{\tau} + 2e^{\eta}e^{\tau} - e^{2\eta})^2}, \tag{A11}$$

where

$$\begin{aligned} A(\tau, \eta) = {} & 12e^{2\tau} - 4e^{\tau} - 28e^{3\tau} + 20e^{4\tau} + 4e^{\tau}e^{\eta} - 16e^{2\tau}e^{\eta} + 32e^{3\tau}e^{\eta} \\ & + 2e^{2\tau}e^{2\eta} - 4e^{3\tau}e^{2\eta} - 20e^{4\tau}e^{\eta} + 2e^{2\eta}e^{\tau} + 2e^{3\eta}e^{2\tau} - 2e^{3\eta}e^{\tau}, \\ B(\tau, \eta) = {} & 22e^{\tau} - 17e^{4\tau} - 6e^{3\eta} - 50e^{2\tau} + 53e^{3\tau} + 12e^{\eta} - 8 - 42e^{\tau}e^{\eta} \\ & + 84e^{2\tau}e^{\eta} - 71e^{3\tau}e^{\eta} + 17e^{4\tau}e^{\eta} - 39e^{2\tau}e^{2\eta} + 21e^{\tau}e^{2\eta} \\ & + 18e^{3\tau}e^{2\eta} + 5e^{2\tau}e^{3\eta} + e^{\tau}e^{3\eta} + 2e^{4\eta} - 2e^{\tau}e^{4\eta}, \end{aligned}$$

and where I have used the fact that $d\tau/d\eta = -\Phi_{\eta}/\Phi_{\tau}$ (as given by [A7]) and have substituted for terms involving $(\tau + \eta)^2$ using (A6) above. As the denominator is clearly positive for all $\tau > \eta \geq 0$, it follows that the sign of $m'(\cdot)$ will be determined by the sign of $F(\tau, \eta) = (\tau + \eta)A(\tau, \eta) + B(\tau, \eta)$. It is easily shown that $F(\tau, \eta) < 0$ for all $\tau > \eta \geq 0$.[45]

[45] The most straightforward approach is to show that $F(\eta, \eta)$, $F_{\tau}(\eta, \eta)$, $F_{\tau\tau}(\eta, \eta)$, and $F_{\tau\tau\tau}(\eta, \eta)$ are all less than or equal to zero for all $\eta \geq 0$ and then to show that $F_{\tau\tau\tau\tau}(\tau, \eta) < 0$ for all $\tau > \eta \geq 0$.

D. h(N − T)

The total labor in industries in which learning continues equals

$$
\begin{aligned}
L_{LBD} &= \int_T^A \|C\| Lf(s)\, a(s)\, ds \\
&= \frac{\|C\| L\lambda}{2} \int_T^A \bar{a}^2 e^{-Z} e^{-T} e^{s-T} - \bar{a}^2 e^{-2T} e^{2s-2T} ds \\
&= \frac{\|C\| L\lambda \bar{a}^2 e^{-2T} (2e^{\tau} e^{\eta} - 2e^{\tau} - e^{2\eta} + 1)}{4} \qquad \text{(A12)} \\
&= EL \frac{2e^{\tau} e^{\eta} - 2e^{\tau} - e^{2\eta} + 1}{e^{2\tau}(\tau + \eta) + 1 - (e^{2\tau}/2) - (e^{2\eta}/2)} \\
&= L_M \frac{2e^{\tau} e^{\eta} - 2e^{\tau} - e^{2\eta} + 1}{2 + e^{2\tau} - 4e^{\tau} + 2e^{\eta} e^{\tau} - e^{2\eta}},
\end{aligned}
$$

which is the origin of (23) in the text. For $N - T > \eta^*$, $\tau = \eta = \eta^*$, and $h(\cdot)$ is invariant with respect to $N - T$. For $N - T < \eta^*$, $\eta = N - T$ and, hence,

$$
\begin{aligned}
h'(\cdot) &= \frac{\partial h}{\partial \eta} + \left(\frac{\partial h}{\partial \tau}\right)\left(\frac{d\tau}{d\eta}\right) \\
&= \frac{2e^{\eta}(e^{\tau} - e^{\eta})(e^{\tau} - 1)^2}{(2 + e^{2\tau} - 4e^{\tau} + 2e^{\eta} e^{\tau} - e^{2\eta})^2} \qquad \text{(A13)} \\
&\quad + \frac{2e^{\tau}(e^{\eta} - 1)(e^{\tau} - e^{\eta})(1 - e^{\tau})}{(2 + e^{2\tau} - 4e^{\tau} + 2e^{\eta} e^{\tau} - e^{2\eta})^2} \left(\frac{d\tau}{d\eta}\right),
\end{aligned}
$$

which, given $d\tau/d\eta < 0$, is clearly strictly positive for all $\tau > \eta \geq 0$.

Appendix B

Mathematical Details of the Steady State

A. Flow of Profits to Firm T *in the Stagnant Steady State*

From (14)–(16) we know that the flow of profits to any firm s such that $a(s) \leq a(Z)$ equals

$$\pi(s) = \frac{\|C\| L\lambda [a(Z) - a(s)]^2}{4}. \qquad \text{(B1)}$$

Substituting for $\|C\|$ and λ, using (A8), (A9), and (21) in the text, yields

$$
\begin{aligned}
\pi(s) &= \frac{L_M(e^{\tau} - e^{s-T})^2}{2 + e^{2\tau} - 4e^{\tau} + 2e^{\eta} e^{\tau} - e^{2\eta}} \quad \text{if } A \geq s \geq T \\
&= \frac{L_M(e^{\tau} - e^{T-s})^2}{2 + e^{2\tau} - 4e^{\tau} + 2e^{\eta} e^{\tau} - e^{2\eta}} \quad \text{if } Z \leq s \leq T. \qquad \text{(B2)}
\end{aligned}
$$

In the stagnant steady state, $L_M = L$ and $\eta = N - T = 0$. Consequently,

$$\pi(T) = \frac{L(e^{\tau} - 1)^2}{1 + e^{2\tau} - 2e^{\tau}} = L. \qquad \text{(B3)}$$

B. V(N(t), t) = L_MT(N − T, ρ, g)

For a firm s invented at time 0 (i.e., $s = N(0)$), $s - T = \eta - gt$ until such time as $t = \eta/g$, after which $T - s = gt - \eta$, until such time as $t = (\eta/g) + (\tau/g)$, after which the firm earns zero profits. The steady-state value of firm $N(t)$ equals the present discounted value of its future profits:

$$V(N(t), t) = V(N(0), 0) = \int_0^\infty e^{-\rho t}\pi(N(0), t)\,dt \tag{B4}$$

or (using [B2] above)

$$V(N(t), t) = \frac{L_M\left[\int_0^{\eta/g}(e^\tau - e^{\eta - gt})^2 e^{-\rho t}dt + \int_{\eta/g}^{(\eta/g)+(\tau/g)}(e^\tau - e^{gt-\eta})^2 e^{-\rho t}dt\right]}{2 + e^{2\tau} - 4e^\tau + 2e^\eta e^\tau - e^{2\eta}} \tag{B5}$$

$$= L_M T(N - T, \rho, g)$$

(I remind the reader that τ and η are determined by $N - T$ alone). In the preceding I have assumed that $N - T \leq \eta^*$. If $N - T > \eta^*$, then $\tau = \eta = \eta^*$ and[46]

$$T(N - T, \rho, g) =$$

$$\frac{e^{-\rho(N-T-\eta^*)/g}\left[\int_0^{\eta^*/g}(e^{\eta^*} - e^{\eta^* - gt})^2 e^{-\rho t}dt + \int_{\eta^*/g}^{2\eta^*/g}(e^{\eta^*} - e^{gt-\eta^*})^2 e^{-\rho t}dt\right]}{2 - 4e^{\eta^*} + 2e^{2\eta^*}}. \tag{B6}$$

References

Aghion, Philippe, and Howitt, Peter. "A Model of Growth through Creative Destruction." *Econometrica* 60 (March 1992): 323–51.

Alchian, Armen A. "Reliability of Progress Curves in Airframe Production." *Econometrica* 31 (October 1963): 679–93.

Arrow, Kenneth J. "The Economic Implications of Learning by Doing." *Rev. Econ. Studies* 29 (June 1962): 155–73.

Asher, Harold. "Cost-Quantity Relationships in the Airframe Industry." Paper no. R-291. Santa Monica, Calif.: Rand Corp., July 1, 1956.

Baloff, Nicholas. "Startups in Machine-intensive Production Systems." *J. Indus. Engineering* 17 (January 1966): 25–32.

Bardhan, Pranab K. *Economic Growth, Development, and Foreign Trade: A Study in Pure Theory*. New York: Wiley-Interscience, 1970.

Boldrin, Michele, and Scheinkman, José A. "Learning-by-Doing, International Trade and Growth: A Note." In *The Economy as an Evolving Complex System,* edited by Philip W. Anderson, Kenneth J. Arrow, and David Pines. Reading, Mass.: Addison-Wesley, 1988.

Burke, James. *The Day the Universe Changed*. Boston: Little, Brown, 1985.

Carr, Gardner W. "Peacetime Cost Estimating Requires New Learning Curves." *Aviation* 45 (April 1946): 76–77.

Cipolla, Carlo M. *Guns, Sails and Empires: Technological Innovation and the Early Phases of European Expansion, 1400–1700*. New York: Pantheon, 1965.

[46] The term $e^{-\rho(N-T-\eta^*)/g}$ represents the discount term for the time the firm must wait until it begins to earn positive profits.

Conway, R. W., and Schultz, Andrew, Jr. "The Manufacturing Progress Function." *J. Indus. Engineering* 10 (January–February 1959): 39–53.

Elvin, Mark. *The Pattern of the Chinese Past.* Stanford, Calif.: Stanford Univ. Press, 1973.

Epple, Dennis; Argote, Linda; and Devadas, Rukmini. "Organizational Learning Curves: A Method for Investigating Intra-Plant Transfer of Knowledge Acquired through Learning by Doing." *Organization Sci.* 2 (February 1991): 58–70.

Fudenberg, Drew, and Tirole, Jean. "Learning-by-Doing and Market Performance." *Bell J. Econ.* 14 (Autumn 1983): 522–30.

Graham, Gerald S. "The Ascendancy of the Sailing Ship 1850–85." *Econ. Hist. Rev.* 2d ser., 9 (August 1956): 74–88.

Grossman, Gene M., and Helpman, Elhanan. "Quality Ladders in the Theory of Growth." *Rev. Econ. Studies* 58 (January 1991): 43–61.

Harley, Charles K. "The Shift from Sailing Ships to Steamships, 1850–1890: A Study in Technological Change and Its Diffusion." In *Essays on a Mature Economy: Britain after 1840,* edited by Donald N. McCloskey. Princeton, N.J.: Princeton Univ. Press, 1971.

Head, C. Keith. "Learning by Doing and Infant Industries." Ph.D. dissertation, Massachusetts Inst. Tech., October 1991.

Hirsch, Werner Z. "Firm Progress Ratios." *Econometrica* 24 (April 1956): 136–43.

Hollingdale, S. H., and Tootill, G. C. *Electronic Computers.* Middlesex: Penguin, 1965.

Jewkes, John; Sawers, David; and Stillerman, Richard. *The Sources of Invention.* London: Macmillan, 1958.

Jones, Eric Lionel. *Growth Recurring: Economic Change in World History.* Oxford: Clarendon, 1988.

Krugman, Paul R. "The Narrow Moving Band, the Dutch Disease, and the Competitive Consequences of Mrs. Thatcher: Notes on Trade in the Presence of Dynamic Scale Economies." *J. Development Econ.* 27 (October 1987): 41–55.

———. "Endogenous Innovation, International Trade, and Growth." In *Rethinking International Trade.* Cambridge, Mass.: MIT Press, 1990.

Levhari, David, and Sheshinski, Eytan. "Experience and Productivity in the Israel Diamond Industry." *Econometrica* 41 (March 1973): 239–53.

Lieberman, Marvin B. "The Learning Curve and Pricing in the Chemical Processing Industries." *Rand J. Econ.* 15 (Summer 1984): 213–28.

Lucas, Robert E., Jr. "On the Mechanics of Economic Development." *J. Monetary Econ.* 22 (July 1988): 3–42.

Mokyr, Joel. *The Lever of Riches: Technological Creativity and Economic Progress.* New York: Oxford Univ. Press, 1990.

Musson, Alfred E., and Robinson, Eric. "The Origins of Engineering in Lancashire." *J. Econ. Hist.* 20 (June 1960): 209–33.

Romer, Paul M. "Endogenous Technological Change." *J.P.E.* 98, no. 5, pt. 2 (October 1990): S71–S102.

Rosenberg, Nathan. *Perspectives on Technology.* Cambridge: Cambridge Univ. Press, 1976.

Segerstrom, Paul S.; Anant, T. C. A.; and Dinopoulos, Elias. "A Schumpeterian Model of the Product Life Cycle." *A.E.R.* 80 (December 1990): 1077–91.

Spence, A. Michael. "The Learning Curve and Competition." *Bell J. Econ.* 12 (Spring 1981): 49–70.

Stokey, Nancy L. "The Dynamics of Industrywide Learning." In *Essays in Honor of Kenneth J. Arrow,* vol. 2, *Equilibrium Analysis,* edited by Walter P. Heller, Ross M. Starr, and David A. Starrett. Cambridge: Cambridge Univ. Press, 1986.

———. "Learning by Doing and the Introduction of New Goods." *J.P.E.* 96 (August 1988): 701–17.

———. "Human Capital, Product Quality, and Growth." *Q.J.E.* 106 (May 1991): 587–616.

Succar, Patricia. "The Need for Industrial Policy in LDC's—a Re-Statement of the Infant Industry Argument." *Internat. Econ. Rev.* 28 (June 1987): 521–34.

Temple, Robert K. G. *The Genius of China: 3000 Years of Science, Discovery, and Invention.* New York: Simon & Schuster, 1986.

Wan, Henry Y., Jr. "Trade, Development and Inventions—the Evolutionary Path." Manuscript. Ithaca, N.Y.: Cornell Univ., July 1975.

Wright, T. P. "Factors Affecting the Cost of Airplanes." *J. Aeronautical Sci.* 3 (February 1936): 122–28.

Young, Alwyn. "Learning by Doing and the Dynamic Effects of International Trade." *Q.J.E.* 106 (May 1991): 369–405.

———. "Substitution and Complementarity in Endogenous Innovation." *Q.J.E.* 108 (1993), in press.