Conjectures in Packing and Covering

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A $0-1$ matrix is *balanced* if it has no odd cycle submatrix.

Conjecture 1 ([4]). *Let A be a $0-1$ matrix that is balanced. Then there is a 1 that can be turned into a 0 so that the new matrix is balanced.*

A graph *perfect* if the chromatic number is equal to the clique number for every vertex induced subgraph. We saw a very brief sketch of the proof of the Strong Perfect Graph Theorem [3], stating that a graph is perfect if, and only if, it has no odd hole and no odd antihole. The proof is a major achievement in Mathematics and was the culmination of decades of progress that led to deep insight, elaborate techniques, and powerful tools. That said, the proof is quite involved, thereby making it difficult for an interested non-expert to read and follow the proof.

Question 2. *Is there a short proof of the Strong Perfect Graph Theorem?*

This theorem equivalently states that given a graph G, at least one of the following holds:

- there is a proper vertex coloring with $\omega(G)$ colors, or
- there is an odd hole or an odd antihole.

In fact, there is a polynomial algorithm that finds either the optimal vertex coloring or one of the forbidden induced subgraphs [7, 2]. However, the algorithm borrows tools from convex geometry and semidefinite programming, and is not purely "combinatorial".

Question 3. *Is there a combinatorial algorithm that given a simple graph G runs in time polynomial in $|V(G)|$ and outputs one of the following?*

- *a proper vertex coloring with $\omega(G)$ colors, or*
- *an odd hole or an odd antihole.*

Let $D = (V, A)$ be a digraph. A *dijoin* is an arc subset whose contraction results in a strongly connected digraph. A *dicut* is cut whose arcs are either all incoming or all outgoing. We saw that the minimum cardinality of a dicut is equal to the maximum value of a fractional packing of dijoins.

Conjecture 4 ([9]). *The clutter of dijoins of a digraph packs. That is, given a digraph, the minimum cardinality of a dicut is equal to the maximum number of pairwise arc-disjoint dijoins.*

Let $G = (V, E)$ be a graph, and let T be a nonempty vertex subset of even cardinality. A *T-join* is an edge subset whose odd-degree vertices are precisely T, and a *T-cut* is a cut $\delta(U)$ such that $|U \cap T|$ is odd. We saw that the minimum cardinality of a T-cut is equal to the maximum value of a fractional packing of T-joins.

Conjecture 5 ([8])**.** *Let $G = (V, E)$ be a graph, and let T be a nonempty vertex subset of even cardinality. Then the minimum cardinality of a T-cut is equal to the maximum value of a fractional packing of T-joins that is $\frac{1}{4}$-integral.*

A clutter is *ideal* if the associated set covering polyhedron is integral. We saw that testing idealness of a clutter is a co-NP-complete problem [6]. Recall that a clutter is ideal if, and only if, its blocker is ideal. So what if we were given both the clutter and its blocker as part of the input?

Conjecture 6 ([1])**.** *There is an algorithm that given clutters $\mathcal{C}, \mathcal{B}$ over ground set V outputs one of the following in time polynomial in $|V|, |\mathcal{C}|, |\mathcal{B}|$:*

(i) $\mathcal{C}, \mathcal{B}$ are not blockers,

(ii) at least one of $\mathcal{C}, \mathcal{B}$ is not ideal,

(iii) $\mathcal{C}, \mathcal{B}$ are blocking ideal clutters.

A clutter has the *packing property* if every minor of it, including the clutter itself, packs. We saw that if a clutter has the packing property, then it is ideal.

Conjecture 7 ([5])**.** *If a clutter has the packing property, then it is Mengerian.*

Here, a clutter is *Mengerian* if the corresponding set covering system is totally dual integral. Recall that if a clutter is Mengerian, then it is ideal.

References

[1] Abdi, A., Cornuéjols, G., Lee, D.: Intersecting restrictions in clutters. Submitted.

[2] Chudnovsky, M., Cornuéjols, G., Liu, X., Seymour, P., Vušković, K.: Recognizing Berge graphs. Combinatorica **25**(2), 143–186 (2005)

[3] Chudnovsky, M., Robertson, N., Seymour, P., Thomas, R.: The strong perfect graph theorem. Ann. Math. **164**(1), 51–229 (2006)

[4] Conforti, M. and Rao, M.R.: Structural properties and decomposition of linear balanced matrices. Math. Program. **55**(1-3), 129–168 (1992)

[5] Conforti, M. and Cornuéjols, G.: Clutters that pack and the max-flow min-cut property: a conjecture. (Available online at `http://www.dtic.mil/dtic/tr/fulltext/u2/a277340.pdf`) The Fourth Bellairs Workshop on Combinatorial Optimization (1993)

[6] Ding, G., Feng, L., Zang, W.: The complexity of recognizing linear systems with certain integrality properties. Math. Program. Ser. A **114**, 321–334 (2008)

[7] Grötschel, M., Lovász, L., Schrijver, A.: Polynomial algorithms for perfect graphs. Ann. Discr. Math. **21**, 325–356 (1984)

[8] Seymour, P.D.: On multi-colourings of cubic graphs, and conjectures of Fulkerson and Tutte. Proc. London Math. Soc. **38**(3), 423–460 (1979)

[9] Woodall, D.R.: Minimax theorems in graph theory, in Selected Topics in Graph Theory (eds. Beineke, L.W. and Wilson, R.J.). Academic Press, London, 237–269 (1978)