47853 Packing and Covering: Lecture 11

Ahmad Abdi

February 21, 2019

8.2 T-joins and T-cuts

Last time we proved the following theorem:

Theorem 8.10 (Seymour 1981 [7]). *Take a bipartite graph $G = (V, E)$, and a nonempty subset $T \subseteq V$ of even cardinality. Then the minimum cardinality of a T-join is equal to the maximum number of disjoint T-cuts. That is, the clutter of minimal T-cuts of a bipartite graph packs.*

This result is actually sufficient to guarantee certificates of optimality for minimum T-joins in general graphs:

Theorem 8.11. *Take a graph $G = (V, E)$ and a nonempty subset $T \subseteq V$ of even cardinality. Denote by $\mathcal{C}$ be the clutter of minimal T-cuts over ground set E. Then the following statements hold:*

(1) (Seymour 1981 [7]) *For weights $w \in \mathbb{Z}_+^E$ where every cycle has total even weight, the minimum weight of a T-join is equal to the maximum size of a weighted packing of T-cuts:*

$$\tau(\mathcal{C}, w) = \nu(\mathcal{C}, w).$$

(2) (Lovász 1975 [4]) *For arbitrary weights $w \in \mathbb{Z}_+^E$, the minimum weight of a T-join is equal to the maximum value of a half-integral weighted packing of T-cuts:*

$$\tau(\mathcal{C}, w) = \max_{2y \in \mathbb{Z}_+^{\mathcal{C}}} \left\{ \mathbf{1}^\top y : \sum (y_C : e \in C \in \mathcal{C}) \leq w_e \ \ \forall e \in E \right\}.$$

(3) (Edmonds and Johnson 1973 [3]) *The clutter $\mathcal{C}$ of minimal T-cuts is ideal, that is, the polyhedron*

$$\left\{ x \geq \mathbf{0} : \sum (x_e : e \in B) \geq 1 \ \ \forall \, T\text{-cuts } B \right\}$$

is integral, and its vertices are the incidence vectors of the minimal T-joins.

Proof. **(1)** If there is a T-join of weight 0, then there is nothing to show. We may therefore assume that the minimum weight of a T-join is nonzero. Let (G', T') be the pair obtained from (G, T) after contracting all edges of weight 0, and for each edge e with $w_e \geq 1$, replacing e by w_e edges in series (the intermediate vertices will not be included in T'). Notice that every cycle C in G corresponds to a cycle in G' of length $w(C)$, and conversely, every cycle C' in G' corresponds to a cycle in G of weight $|C'|$. Thus, since every cycle of G has even weight, it follows that G' is a bipartite graph. Moreover, it is clear that every T-join J in G corresponds to a T'-join in G' of length $w(J)$, and conversely, every T'-join J' in G' corresponds to a T-join in G of weight $|J'|$. In particular, $T' \neq \emptyset$. It therefore follows from Theorem 8.10 that the minimum cardinality of a T'-join in G' is equal to the maximum number of disjoint T'-cuts of G'. As every packing of T'-cuts in G' corresponds to a weighted packing of T-cuts in G, it follows that $\tau(\mathcal{C}, w) = \nu(\mathcal{C}, w)$, as required. **(2)** Take arbitrary weights $w \in \mathbb{Z}_+^E$. It follows from (1) that

$$2\tau(\mathcal{C}, w) = \tau(\mathcal{C}, 2w) = \nu(\mathcal{C}, 2w) = \max_{y \in \mathbb{Z}_+^{\mathcal{C}}} \left\{ \mathbf{1}^\top y : \sum (y_C : e \in C \in \mathcal{C}) \leq 2w_e \ \ \forall e \in E \right\},$$

thereby proving (2). **(3)** follows immediately from (2). □

After applying Theorem 7.8 to part (3), we get the following:

Corollary 8.12. *Take a graph $G = (V, E)$ and a nonempty subset $T \subseteq V$ of even cardinality. Then the clutter of minimal T-joins is ideal. That is, for all weights $w \in \mathbb{Z}_+^E$, the minimum weight of a T-cut is equal to the maximum value of a fractional weighted packing of T-joins.*

Seymour 1979 [6] conjectures that in the above corollary, the minimum weight of a T-cut should be equal to the maximum value of a quarter-integral weighted packing of T-joins. In contrast to T-cuts, packing T-joins is a difficult problem. To illustrate this, we need a definition. A 3-*graph* is a connected bridgeless graph $G = (V, E)$ where every vertex has degree 3.

Proposition 8.13. *Let $G = (V, E)$ be a plane 3-graph. Then the following statements are equivalent:*

(i) G has three disjoint perfect matchings,

(ii) G has two disjoint V-joins,

(iii) G has a proper 4-face-coloring.

Proof. **(i)** ⇒ **(ii)** holds trivially. **(ii)** ⇒ **(iii)**: Suppose that G has disjoint minimal V-joins J_1, J_2. Let $G^\star = (V^\star, E)$ be the plane dual of G, and notice that every face of $G^\star$ is a triangle. Notice that the V-cuts of G are in correspondence with the cycles of $G^\star$ bounding an odd number of triangles, implying in turn that the V-cuts of G are in correspondence with the odd cycles of $G^\star$. Since each J_i is a minimal cover of the V-cuts of G, each J_i is also a minimal cover of the odd cycles of $G^\star$, implying in turn that there is a nonempty cut $\delta(U_i), U_i \subseteq V^\star$ of $G^\star$ such that $\delta(U_i) = E - J_i$. Since $J_1 \cap J_2 = \emptyset$, it follows that $U_1 \cap U_2, U_1 \cap \overline{U_2}, \overline{U_1} \cap U_2, \overline{U_1} \cap \overline{U_2}$ are stable sets of $G^\star$, thereby yielding a proper 4-vertex-coloring of $G^\star$, and hence a proper 4-face-coloring of G. **(iii)** ⇒ **(i)**: Let $h \in \{(0,0), (0,1), (1,0), (1,1)\}^{\{\text{faces}\}}$ be a proper 4-face-coloring of G. For each edge e, whose neighboring faces are F_1 and F_2, let

$$g(e) := h(F_1) + h(F_2) \pmod 2.$$

Since F_1, F_2 are adjacent faces, and therefore have different colors, it follows that $g(e) \in \{(0,1),(1,0),(1,1)\}$. Let

$$J_1 := \{e \in E : g(e) = (0,1)\}$$
$$J_2 := \{e \in E : g(e) = (1,0)\}$$
$$J_3 := \{e \in E : g(e) = (1,1)\}.$$

We claim that each J_i is a perfect matching. To see this, take an arbitrary vertex v, whose neighboring faces are F_1, F_2, F_3. Then the three edges incident with v have g-values $h(F_1) + h(F_2), h(F_2) + h(F_3), h(F_3) + h(F_1) \pmod 2$. As $h(F_1), h(F_2), h(F_3)$ are pairwise distinct, we get that the g-values of the three edges incident with v are different, so v is indicent with exactly one edge from each J_i. As this is true for each vertex, it follows that each J_i is a perfect matching, as required. □

It is widely known that properly 4-face-coloring plane 3-graphs is just as general as properly 4-face-coloring arbitrary plane graphs. Thus, the implication (ii) ⇒ (iii) implies that finding just two disjoint T-joins in a graph can be a difficult problem. Appel and Haken 1977 [1], and again Robertson, Sanders, Seymour and Thomas 1996 [5], proved that plane graphs are properly 4-face-colorable. As a consequence, the implication (iii) ⇒ (i) implies that,

Theorem 8.14. *The clutter of minimal T-joins of a planar 3-graph packs.*

This result does not extend to non-planar 3-graphs. For instance, the Petersen graph is a 3-graph whose clutter of minimal T-joins does not pack, as it is not properly 3-edge-colorable.

8.3 Testing idealness is co-NP-complete.

We saw two different classes of ideal clutters, namely the clutter of dijoins of a digraph and the clutter of T-joins of a graph. These examples demonstrate that idealness is a rich and complex property, and suggest that idealness as a property is hard to recognize. This is indeed the case. To elaborate, let A be a $0-1$ matrix. Consider the following problem:

Is A an ideal matrix?

This is a co-NP problem: to certify that A is nonideal, all we need is a fractional point $x^\star \in Q(A) = \{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$ along with a full-rank row subsystem $A'x \geq b'$ of $x \geq \mathbf{0}, Ax \geq b$ such that $A'x^\star = b'$. In fact, as the following result claims, this problem is one of the most difficult problems in the co-NP class:

Theorem 8.15 (Ding, Feng, Zang 2008 [2])**.** *Let A be a $0-1$ matrix, where every column has exactly two 1s. Then the problem*

Is A an ideal matrix?

is co-NP-complete.

In other words, given a general $0-1$ matrix that is a priori ideal, we cannot convince an adversary in polynomial time that A is indeed an ideal matrix, unless P and co-NP are equal. This means that unlike perfect clutters, ideal clutters do not admit a polynomial characterization. (The authors above proved that "Is A a Mengerian matrix?" is also a co-NP-complete problem.)

9 Minimally nonideal clutters

By Remark 7.11, we know that if a clutter is ideal, then so is any minor of it. In other words, the class of ideal clutters is minor-closed. As a result, we may indirectly study the class by characterizing the excluded minors defining the class. We say that a clutter is *minimally nonideal (mni)* if it is nonideal, and every proper minor of it is ideal. Observe that every mni clutter has at least 3 elements, and that the only mni clutter with 3 elements is $\big\{\{a,b\},\{b,c\},\{c,a\}\big\}$. It follows from Remark 7.11 and Theorem 7.8 that,

Remark 9.1. *The following statements hold:*

- *a nonideal clutter is minimally nonideal if every single deletion and contraction minor is ideal,*
- *a clutter is ideal if, and only if, it has no minimally nonideal minor,*
- *if a clutter is minimally nonideal, then so is its blocker.*

As we will see, mni clutters split into two classes that behave quite differently from one another. We will study each class separately.

9.1 The deltas

Two clutters $\mathcal{C},\mathcal{C}'$ are *isomorphic*, written as $\mathcal{C}\cong\mathcal{C}'$, if one is obtained from the other after relabeling its ground set. Take an integer $n\geq 3$. Consider the clutter over ground set $[n]:=\{1,2,3,\ldots,n\}$ whose members are

$$\Delta_n := \big\{\{1,2\},\{1,3\},\ldots,\{1,n\},\{2,3,\ldots,n\}\big\}$$

and whose incidence matrix is

$$M(\Delta_n)=\begin{pmatrix}1 & 1 & & & \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1\\ & 1 & 1 & \cdots & 1\end{pmatrix}.$$

We refer to Δ_n, and any clutter isomorphic to it, as a *delta of dimension* n. Notice that the elements and members of Δ_n correspond to the points and lines of a degenerate projective plane.[1]

Theorem 9.2. *Take an integer* $n\geq 3$. *Then,*

[1] In the literature, a delta of dimension n is called a degenerate projective plane of order $n-1$. However, as there are other degenerate projective planes, we refrain from using this terminology.

(1) $b(\Delta_n) = \Delta_n$*,*

(2) $\min\{\mathbf{1}^\top x : M(\Delta_n)x \geq \mathbf{1}, x \geq \mathbf{0}\}$ *has no integral optimal solution, and*

(3) Δ_n *is minimally nonideal.*

Proof. **(1)** As Δ_n does not have disjoint members, every member is also a cover, so every member of Δ_n contains a member of $b(\Delta_n)$. Conversely, let B be a minimal cover of Δ_n. If $1 \notin B$, then as B intersects each one of $\{1,2\}, \{1,3\}, \ldots, \{1,n\}$, it follows that $\{2,3,\ldots,n\} \subseteq B$. If $1 \in B$, then as B intersects $\{2,3,\ldots,n\}$, it follows that $\{1,i\} \subseteq B$ for some $i \in \{2,3,\ldots,n\}$. In both cases, we see that B contains a member, so every member of $b(\Delta_n)$ contains a member of Δ_n. It therefore follows from Remark 6.6 that $b(\Delta_n) = \Delta_n$. **(2)** In particular, $\tau(\mathcal{C}) = 2$. Consider now the fractional feasible solution $x^\star := \begin{pmatrix} \frac{n-2}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \end{pmatrix}$. The objective value of this solution is $1 + \frac{n-2}{n-1} < 2 = \tau(\mathcal{C})$, so (2) holds. **(3)** It follows from (2) that Δ_n is nonideal. To prove that Δ_n is mni, we need to show for each $e \in [n]$ that $\Delta_n \setminus e$ and Δ_n/e are ideal clutters. In fact, since

$$\Delta_n \setminus e = b(b(\Delta_n \setminus e)) = b(b(\Delta_n)/e) = b(\Delta_n/e)$$

by (1), it suffices by Theorem 7.8 to show that one of $\Delta_n \setminus e, \Delta_n/e$ is ideal. By the symmetry between the elements $2,3,\ldots,n$, we may assume that $e \in \{1,n\}$. Observe that

$$\Delta_n \setminus 1 = \{\{2,3,\ldots,n\}\}$$

and

$$\Delta_n/n = \{\{1\},\{2,\ldots,n-1\}\}.$$

We leave it as an exercise for the reader to see that these clutters are indeed ideal. Thus, Δ_n is mni. □

References

[1] Appel, K. and Haken, W.: Every planar map is four colorable. Part I: Discharging. Illinois J. Math. **21**(3), 429–490 (1977)

[2] Ding, G., Feng, L., Zang, W.: The complexity of recognizing linear systems with certain integrality properties. Math. Program. Ser. A **114**, 321–334 (2008)

[3] Edmonds, J. and Johnson, E.L.: Matchings, Euler tours and the Chinese postman problem. Math. Prog. **5**, 88–124 (1973)

[4] Lovász, L.: 2-matchings and 2-covers of hypergraphs. Acta Math. Acad. Sci. Hungar. **26**, 433–444 (1975)

[5] Robertson, N., Sanders, D.P., Seymour, P., Thomas, R.: A new proof of the four-colour theorem. Electronic Research Announcements of the AMS **2**(1), 17–25 (1996)

[6] Seymour, P.D.: On multi-colourings of cubic graphs, and conjectures of Fulkerson and Tutte. Proc. London Math. Soc. **38**(3), 423–460 (1979)

[7] Seymour, P.D.: On odd cuts and plane multicommodity flows. Proc. London Math. Soc. **42**(1), 178–192 (1981)