47853 Packing and Covering: Lecture 13

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February 28, 2019

9.2 Minimally nonideal clutters different from the deltas

Today we will finish Lehman's characterizations of mni clutters different from the deltas. We started the proof of the following theorem last time:

Theorem 9.9 (Lehman 1990 [4])**.** *Let $\mathcal{C}$ be a minimally nonideal clutter over ground set E that is not a delta, and let $n := |E|$. Let $x^\star$ be a fractional extreme point of $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$. Then the following statements hold:*

(1) $\mathbf{0} < x^\star < \mathbf{1}$*,*

(2) $x^\star$ *lies on exactly* n *facets, that correspond to members* $C_1, \ldots, C_n \in \mathcal{C}$ *– so* $x^\star$ *is a simple vertex,*

(3) the n *neighbors of* $x^\star$ *are integral vertices, that correspond to covers* $B_1, \ldots, B_n$ *labeled so that for distinct* $i, j \in [n]$, $|C_i \cap B_i| > 1$ *and* $|C_i \cap B_j| = 1$*,*

(4) $B_1, \ldots, B_n$ *are minimal covers,*

(5) $C_1, \ldots, C_n$ *are precisely the minimum cardinality members of* $\mathcal{C}$*,*

(6) $x^\star$ *is the unique fractional extreme point of* $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$*,*

(7) there is an integer $d \geq 1$ *such that for each* $i \in [n]$, $|C_i \cap B_i| = 1 + d$*.*

In particular, $x^\star$ *is the unique fractional extreme point of* $\{x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$*.*

Proof. We have already proved (1) and the following claim:

Claim 1. *Let* $x^\star$ *be a fractional extreme point of* P*, and let* A *be an* $n \times n$ *nonsingular submatrix of* $M(\mathcal{C})$ *such that* $Ax^\star = \mathbf{1}$*. Then* A *is cross regular.*

We'll now use this claim to prove the following two claims:

Claim 2. *Every fractional extreme point of* P *is simple, that is, it lies on exactly* n *facets. Thus* **(2)** *holds.*

Proof of Claim. Suppose for a contradiction that P has a non-simple fractional extreme point $x^\star$. Let A be an $n \times n$ nonsingular submatrix of $M(\mathcal{C})$ such that $Ax^\star = \mathbf{1}$. As $x^\star$ is non-simple, there is another row a' of $M(\mathcal{C})$ such that $a'^\top x^\star = 1$. Pick a row a of A such that the matrix A' obtained by replacing a and a' is nonsingular. (To find a, write a' as a linear combination of the rows of A, and pick a row a whose coefficient is nonzero.) Then by Claim 1, both A and A' are cross regular, a contradiction to Lemma 9.8 (2) as A and A' differ in exactly one row. $\diamondsuit$

Claim 3. *P does not have neighboring fractional extreme points. Thus* **(3)** *holds.*

Proof of Claim. Suppose for a contradiction that P has neighboring fractional extreme points $x^\star, y^\star$. Then there are $n \times n$ nonsingular submatrices A, B of $M(\mathcal{C})$ that differ in exactly one row such that $Ax^\star = \mathbf{1} = By^\star$. By Claim 1, both A and B are cross regular, a contradiction to Lemma 9.8 (2). $\diamondsuit$

Now pick a fractional extreme point $x^\star$ of P. By Claims 2 and 3, $x^\star$ lies on n facets and has precisely n neighbors, all of which are integral. Let $C_1, \ldots, C_n \in \mathcal{C}$ be the members corresponding to the facets $x^\star$ sits on, and let $B_1, \ldots, B_n$ be the covers corresponding to the neighbors of $x^\star$, where our labeling satisfies for $i, j \in [n]$ the following:

$$|C_i \cap B_j| \begin{cases} > 1 & \text{if } i = j \\ = 1 & \text{if } i \neq j. \end{cases}$$

Let A (resp. B) be the $0-1$ matrix whose columns are labeled by E and whose rows are the incidence vectors of $C_1, \ldots, C_n$ (resp. $B_1, \ldots, B_n$). Then the equalities above imply that

$$AB^\top = J + \text{Diag}\left(|C_1 \cap B_1| - 1, \ldots, |C_n \cap B_n| - 1\right).$$

In particular, $AB^\top$ is nonsingular, implying in turn that B is nonsingular. Moreover, by Claim 1, A is cross regular. Let G be the bipartite representation of A, where column e and row C are adjacent if $e \notin C$. Since A is cross regular, it follows that adjacent vertices of G have the same degree. In particular, every connected component of G is regular and so it has the same number of vertices in the two parts of the bipartition.

Claim 4. *G is connected.*

Proof of Claim. Suppose for a contradiction that G is not connected. Then there exist a partition of the rows of A into nonempty parts X_1, X_2 and a partition of the columns of A into nonempty parts $Y_1, Y_2 \subseteq E$ such that $|X_1| = |Y_1|$, $|X_2| = |Y_2|$, and the (X_2, Y_1) and (X_1, Y_2) blocks of A are submatrices of all ones. If $|Y_1| = 1$ or $|Y_2| = 1$, then A has a row with $n-1$ ones, so $\mathcal{C}$ has a delta minor by Theorem 9.3, implying in turn by minimality that $\mathcal{C}$ is a delta, a contradiction as $\mathcal{C}$ is not a delta. Otherwise, $|X_1| = |Y_1| \geq 2$ and $|X_2| = |Y_2| \geq 2$. As a result, for each $i \in [n]$, $|B_i \cap Y_1| = |B_i \cap Y_2| = 1$, implying in turn that the columns of the matrix B corresponding to Y_1 have the same sum as the columns of B corresponding to Y_2, a contradiction as B is nonsingular. $\diamondsuit$

In particular, G is a regular graph, implying in turn that for some integer $r \geq 2$, every row and every column of A has exactly r ones – this has two consequences. Firstly, each B_i is a minimal cover. For if not, then $B_i - \{e\}$ is a cover for some $e \in B_i$, implying in turn that column e of A has at least $n-1$ zero entries, implying in turn that $r \leq 1$, which is not the case. Thus **(4)** holds. Secondly, since A is nonsingular, it follows that $x^\star = \left(\frac{1}{r}\ \frac{1}{r}\ \cdots \frac{1}{r}\right)$. As a result, as $x^\star \in P$, every row of $M(\mathcal{C})$ has at least r ones, and as $x^\star$ is simple, every row of $M(\mathcal{C})$ not in A has at least $r+1$ ones, so **(5)** holds. In particular, we cannot run this argument for another fractional extreme point, so $x^\star$ is the unique fractional extreme point of P, so **(6)** holds. Finally, for each $i \in [n]$, let $d_i := |C_i \cap B_i| - 1 \in \{1, \ldots, r-1\}$, and let $D := \mathrm{Diag}(d_1, \ldots, d_n)$. Then

$$(n+d_1, n+d_2, \ldots, n+d_n) = \mathbf{1}^\top (J+D) = \mathbf{1}^\top (AB^\top) = (\mathbf{1}^\top A) B^\top = r \cdot (B\mathbf{1})^\top.$$

Since there is at most one multiple of r in $\{n+1, \ldots, n+r-1\}$, it follows that $d := d_1 = d_2 = \cdots = d_n$, implying in turn that **(7)** holds, thereby finishing the proof. □

For an integer $k \geq 1$, a square $0-1$ matrix is *k-regular* if every row and every column has exactly k ones. We will need the following tool:

Theorem 9.10 (Bridges and Ryser 1969 [2]). *Take an integer $n \geq 3$, and let A, B be $n \times n$ matrices with $0-1$ entries such that*

$$AB = J + dI$$

for some integer $d \geq 1$. Then A, B are nonsingular matrices that commute

$$BA = J + dI,$$

and for some integers $r, s \geq 2$ such that $rs = n + d$, A is r-regular and B is s-regular.

Proof. As $J + dI$ is nonsingular, it follows that both A, B are nonsingular matrices. In particular, neither A nor B has a zero row or a zero column. We have

$$I = (J+dI)\left(\frac{1}{d}I - \frac{1}{d(n+d)}J\right) = (AB)\left(\frac{1}{d}I - \frac{1}{d(n+d)}J\right) = A\left(\frac{1}{d}B - \frac{1}{d(n+d)}BJ\right),$$

so A and $\frac{1}{d}B - \frac{1}{d(n+d)}BJ$ are inverses of one another. Thus,

$$I = \left(\frac{1}{d}B - \frac{1}{d(n+d)}BJ\right)A = \frac{1}{d}BA - \frac{1}{d(n+d)}(B\mathbf{1})(A^\top\mathbf{1})^\top,$$

so

$$BA = \frac{1}{n+d}(B\mathbf{1})(A^\top\mathbf{1})^\top + dI.$$

For each $i \in [n]$, denote by $s_i \in \{1, 2, \ldots, n\}$ the number of ones in row i of B, and by $r_i \in \{1, 2, \ldots, n\}$ the number of ones in column i of A. Then the previous equation implies that

(1) for all $i, j \in [n]$, $n + d \mid s_i r_j$.

As $\mathrm{trace}(AB) = \mathrm{trace}(BA)$, it follows that

$$n + nd = \frac{1}{n+d}\sum_{i=1}^{n} s_i r_i + nd,$$

so

$$n(n+d) = \sum_{i=1}^{n} s_i r_i \geq n(n+d),$$

implying in turn that

(2) for each $i \in [n]$, $n + d = s_i r_i$.

(1) and (2) imply that $r := r_1 = r_2 = \cdots = r_n$ and $s := s_1 = s_2 = \cdots = s_n$. As a consequence,

$$BA = \frac{1}{n+d}(B\mathbf{1})(A^\top \mathbf{1})^\top + dI = J + dI = AB.$$

Analyzing the equation $AB = J + dI$, we proved that every row of B has the same s number of ones, and every column of A has the same r number of ones. The same argument on the equation $BA = J + dI$ implies that every row of A has the same number of ones, and the number inevitably has to be r, while every column of B has the same number of ones, and the number inevitably has to be s. In particular, A is r-regular and B is s-regular. As $rs = n + d$ and $r, s < n + d$, it follows that $r, s \geq 2$, thereby finishing the proof. □

We are now ready for Lehman's combinatorial characterization of the mni clutters different from the deltas:

Theorem 9.11 (Lehman 1990 [4])**.** *Suppose $\mathcal{C}$ is a minimally nonideal clutter over ground set E that is not a delta, and let $\mathcal{B} := b(\mathcal{C})$. Denote by $\overline{\mathcal{C}}, \overline{\mathcal{B}}$ the clutters over ground set E of the minimum cardinality members of $\mathcal{C}, \mathcal{B}$, respectively. Then*

(1) $M(\overline{\mathcal{C}})$ and $M(\overline{\mathcal{B}})$ are square and nonsingular matrices,

(2) for some integers $r \geq 2$ and $s \geq 2$, $M(\overline{\mathcal{C}})$ is r-regular and $M(\overline{\mathcal{B}})$ is s-regular,

(3) for $n := |E|$, $rs \geq n + 1$,

(4) after possibly permuting the rows of $M(\overline{\mathcal{B}})$, we have

$$M(\overline{\mathcal{C}})M(\overline{\mathcal{B}})^\top = J + (rs-n)I = M(\overline{\mathcal{B}})^\top M(\overline{\mathcal{C}}),$$

that is, there is a labeling $C_1, \ldots, C_n$ of the members of $\overline{\mathcal{C}}$ and a labeling $B_1, \ldots, B_n$ of the members of $\overline{\mathcal{B}}$ such that for all $i, j \in [n]$,

$$|C_i \cap B_j| = \begin{cases} rs - n + 1 & \text{if } i = j \\ 1 & \text{if } i \neq j, \end{cases}$$

and for all elements $g, h \in E$,

$$\left|\left\{i \in [n] : g \in C_i, h \in B_i\right\}\right| = \begin{cases} rs - n + 1 & \text{if } g = h \\ 1 & \text{if } g \neq h. \end{cases}$$

Proof. Let $x^\star \in [0,1]^E$ be a fractional extreme point of $P(\mathcal{C})$. After applying Theorem 9.9 to the mni clutter $\mathcal{C}$, we get the following implications. The point $x^\star \in [0,1]^E$ is the unique fractional extreme point of $P(\mathcal{C})$, $\mathbf{1} > x^\star > \mathbf{0}$ and $x^\star$ is simple. Let A be the submatrix of $M(\mathcal{C})$ such that $Ax^\star = \mathbf{1}$. We have that $A = M(\overline{\mathcal{C}})$. Let $B_1, \ldots, B_n$ be the minimal covers that correspond to the neighbors of $x^\star$, and let B be the matrix whose rows are the incidence vectors of $B_1, \ldots, B_n$. Then after possibly permuting the rows of B, $AB^\top = J + dI$ for some integer $d \geq 1$.

It now follows from Theorem 9.10 that A, B are nonsingular matrices such that $AB^\top = J + dI = B^\top A$, and for some integers $r, s \geq 2$ such that $rs = n + d$, A is r-regular and B is s-regular. To finish the proof, it remains to show that $B = M(\overline{\mathcal{B}})$. To this end, notice that $x^\star$ is equal to $(\frac{1}{r} \ \cdots \ \frac{1}{r})$, and the neighbors of $x^\star$ lie on the hyperplane $\sum_{i=1}^n x_i = s$. Therefore, the inequality $\sum_{i=1}^n x_i \geq s$ is valid for all the integer extreme points of P, implying in turn that every member of $\mathcal{B}$ has cardinality at least s. As a result, $(\frac{1}{s} \ \cdots \ \frac{1}{s})$ is a fractional extreme point of $P(\mathcal{B})$. Applying Theorem 9.9 to the mni clutter $\mathcal{B}$, we see that $(\frac{1}{s} \ \cdots \ \frac{1}{s})$ must be the unique fractional extreme point of $P(\mathcal{B})$ and $B = M(\overline{\mathcal{B}})$, as required. □

9.3 Immediate applications

The first application of Theorem 9.11 is that $\{\Delta_n : n \geq 4\}$ are the only mni clutters requiring unequal weights to violate the width-length inequality. The following application is the true analogue of the max-max inequality, Theorem 5.5:

Theorem 9.12. *A clutter without a $\{\Delta_n : n \geq 4\}$ minor is ideal if, and only if, for each minor $\mathcal{C}$ over ground set E,*

$$\min\{|C| : C \in \mathcal{C}\} \cdot \min\{|B| : B \in b(\mathcal{C})\} \leq |E|.$$

Proof. If the clutter is ideal, then the inequality follows from the width-length inequality of Theorem 7.8. Conversely, it suffices to prove that for an mni clutter $\mathcal{C}$ over ground set E that is not one of $\Delta_n, n \geq 4$,

$$\min\{|C| : C \in \mathcal{C}\} \cdot \min\{|B| : B \in b(\mathcal{C})\} > |E|.$$

This is obviously true if $\mathcal{C} \cong \Delta_3$. Otherwise, $\mathcal{C}$ is not a delta, and let n, r, s be the parameters as in Theorem 9.11. Then the inequality $rs \geq n + 1$ implies the inequality above, as required. □

A second application of Theorem 9.11 is the following truly remarkable result that, to test the integrality of an n-dimensional set covering polyhedron, it is sufficient to test just 3^n directions:

Theorem 9.13. *If $\mathcal{C}$ is a minimally nonideal clutter, then*

$$\min\{\mathbf{1}^\top x : M(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}\}$$

has no integral optimal solution. As a consequence, if $\mathcal{C}$ is a nonideal clutter over ground set E, then there exists a $w \in \{0, 1, +\infty\}^E$ such that

$$\min\{w^\top x : M(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}\}$$

has no integral optimal solution.

Proof. If $\mathcal{C}$ is a delta, then the result follows from Theorem 9.2 (2). Otherwise, $\mathcal{C}$ is not a delta, and let n, r, s be as in Theorem 9.11. As every member has cardinality at least r, it follows that $x^\star := \left(\frac{1}{r}\ \frac{1}{r}\ \cdots \frac{1}{r}\right)$ is a feasible solution, and its objective value is $\frac{n}{r} \leq \frac{rs-1}{r} < s$. However, the minimum cardinality of a cover is s, so $\min\{\mathbf{1}^\top x : M(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}\}$ has no integral optimal solution. The second part follows from the first part after applying Remark 7.10. □

A clutter $\mathcal{C}$ *fractionally packs* if it has a fractional packing of value $\tau(\mathcal{C})$. It follows from the preceding theorem that an mni clutter does not fractionally pack. Thus,

Theorem 9.14. *A clutter is ideal if, and only if, every minor fractionally packs.*

We say that a clutter has the *packing property* if every minor packs. An immediate consequence of the preceding theorem is that,

Corollary 9.15. *If a clutter has the packing property, then it is ideal.*

Conforti and Cornuéjols 1993 [3] conjecture that if a clutter has the packing property, then it must be Mengerian!

References

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