CO750 Packing and Covering

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1 What is packing and covering?

1.1 A packing example: Menger's theorem

Let $G = (V, E)$ be a loopless graph, and take distinct vertices $s, t \in V$. An *st-path* is a minimal edge subset connecting s and t. What is the maximum number of (pairwise) disjoint st-paths? In other words, how many st-paths can we *pack*? Denote by ν the maximum number of disjoint st-paths.

An *st-cut* is an edge subset of the form

$$\delta(U) := \{e \in E : |e \cap U| = 1\}$$

where $U \subseteq V$ satisfies $U \cap \{s, t\} = \{s\}$. We will refer to U and $V - U$ as the *shores* of G. Notice that every st-path intersects an st-cut. Thus, ν is at most the cardinality of any st-cut. Let τ be the minimum cardinality of an st-cut. Then

$$\tau \geq \nu.$$

Theorem 1.1 (Menger 1927)**.** *Let $G = (V, E)$ be a loopless graph, and take distinct vertices $s, t \in V$. Then the maximum number of disjoint st-paths is equal to the minimum cardinality of an st-cut, that is, $\tau = \nu$.*

Proof. We prove this by induction on $|V| + |E| \geq 3$. The result is obvious for $|V| + |E| = 3$. For the induction step, assume that $|V| + |E| \geq 4$. Let τ be the minimum cardinality of an st-cut. We may assume that $\tau \geq 1$. We will find τ disjoint st-paths.

Claim 1. *If an edge e does not appear in a minimum st-cut, then G has τ disjoint paths.*

Proof of Claim. Notice that the cardinality of a minimum st-cut in $G \setminus e$ is still τ. As a result, the induction hypothesis implies the existence of τ disjoint st-paths in $G \setminus e$, and therefore in G. $\Diamond$

We may therefore assume that every edge appears in a minimum st-cut. An st-cut $\delta(U)$ is *trivial* if either $|U| = 1$ or $|V - U| = 1$.

Claim 2. *If there is a minimum st-cut which is not trivial, then G has τ disjoint paths.*

Proof of Claim. Let $\delta(U), s \in U \subseteq V - \{t\}$ be a minimum st-cut which is non-trivial. Let G_1 be the graph obtained from G by shrinking U to a single vertex s', and let G_2 be the graph obtained from G after shrinking $V - U$ to a single vertex t'. Since $\delta(U)$ is non-trivial, it follows that $|V(G_i)| + |E(G_i)| < |V| + |E|$, for each $i \in [2]$. We may therefore apply the induction hypothesis to G_1 and G_2. Notice that τ is still the minimum cardinality of an $s't$-cut in G_1 and of an st'-cut in G_2. Thus, by the induction hypothesis, G_1 has τ disjoint $s't$-paths and G_2 has disjoint st'-paths. Gluing these paths along the edges of $\delta(U)$ gives us τ disjoint st-paths in G. $\Diamond$

We may therefore assume that every minimum st-cut is trivial. Since every edge appears in a minimum st-cut, it follows that every edge has either s or t as an end. In this case, G has a special form and it is clear that $\tau = \nu$ for this graph, thereby completing the induction step. $\square$

This result was the first of many packing theorems. Just to mention a few, we will see some of these packing results:

- Given a connected loopless graph G and distinct vertices s, t, the maximum number of disjoint st-cuts is equal to the minimum cardinality of an st-path.
- Ford and Fulkerson 1956: given a directed graph G and distinct vertices s, t, the maximum number of disjoint directed (s, t)-paths is equal to the minimum cardinality of an (s, t)-cut.
- Edmonds 1972: given a directed graph G and a root r, the maximum number of disjoint spanning r-arborescences is equal to the minimum cardinality of an r-cut.
- Edmonds and Johnson 1973: given a graph G and even subset T of vertices, the maximum value of a fractional packing of T-joins is equal to the minimum cardinality of a T-cut.
- Lucchesi and Younger 1976: given a directed graph G, the maximum number of disjoint dicuts is equal to the minimum cardinality of a dijoin.
- Conjecture (Woodall 1978): given a directed graph G, the maximum number of disjoint dijoins is equal to the minimum cardinality of a dicut.
- Guenin 2001: in a signed graph without an odd-K_5 minor, the maximum value of a fractional packing of odd circuits is equal to the minimum cardinality of a signature.

1.2 A covering example: Dilworth's theorem

Take a partially ordered set $(E, \leq)$, that is, the following statements hold for all $a, b, c \in E$:

- $a \leq a$,
- if $a \leq b$ and $b \leq a$, then $a = b$,
- if $a \leq b$ and $b \leq c$, then $a \leq c$.

We say that a, b are *comparable* if $a \geq b$ or $b \geq a$; otherwise they are *incomparable*. A *chain* is a set of pairwise comparable elements. What is the minimum number of (not necessarily disjoint) chains whose union is E? That is, what is the least number of chains needed to *cover* the ground set? Let θ be the minimum size of a covering.

An *antichain* is a set of pairwise incomparable elements. Given an antichain A, every chain picks at most one element from A. Thus, θ is at least the cardinality of an antichain. Let α be the maximum cardinality of an antichain. Then

$$\theta \geq \alpha.$$

Theorem 1.2 (Dilworth 1950)**.** *Let $(E, \leq)$ be a partially ordered set. Then the minimum number of chains needed to cover E is equal to the maximum cardinality of an antichain. That is, $\theta = \alpha$.*

Proof. We prove this by induction on $|E|$. The base case $|E| = 1$ is obvious. For the induction step, assume that $|E| \geq 2$. Let α be the maximum cardinality of an antichain. We will find α chains covering E. If $\alpha = |E|$, then $\theta = \alpha = |E|$ and we are done. Otherwise, $\alpha < |E|$, implying in turn that there is a chain $\{a, b\}$ where a is a minimal element and b is a maximal element. Let $E' := E - \{a, b\}$.

Claim. *If the maximum cardinality of an antichain of* $(E', \leq)$ *is* $\alpha - 1$*, then there are* α *chains covering* E.

Proof of Claim. By the induction hypothesis, there are $\alpha - 1$ chains of E' covering $E - \{a, b\}$. Together with $\{a, b\}$, we get a covering of E using α chains. $\Diamond$

We may therefore assume that E' has an antichain A such that $|A| = \alpha$. Let

$$E^+ := A \cup \{x \in E - A : x \geq z \text{ for some } z \in A\}$$
$$E^- := A \cup \{y \in E - A : y \leq z \text{ for some } z \in A\}.$$

Since A is an antichain, $E^+ \cap E^- = A$, and since it is a maximum antichain, $E^+ \cup E^- = E$. As a is minimal and $a \notin A$, it follows that $a \notin E^+$. As b is maximal and $b \notin A$, we get that $b \notin E^-$. In particular, $|E^+|, |E^-| < |E|$. Thus, by the induction hypothesis, E^+ has α chains covering it, and E^- has α chains covering it. Gluing these chains together, we get α chains covering $E^+ \cup E^- = E$, thereby completing the induction step. $\square$

This result was the first of many covering results. To name a few:

- In a partially ordered set, the minimum number of antichains needed to cover the ground set is equal to the maximum cardinality of a chain.
- Kőnig 1931: In a bipartite graph, the minimum number of colors needed for an edge-coloring is equal to the maximum degree of a vertex.
- Kőnig 1931: In a bipartite graph, the minimum number of vertices needed to cover the edges is equal to the maximum cardinality of a matching.
- Gallai 1962, Suranyi 1968: In a chordal graph, the minimum number of cliques needed to cover the vertices is equal to the maximum cardinality of a stable set.
- Sachs 1970: In a chordal graph, the minimum number of colors needed for a vertex-coloring is equal to the maximum cardinality of a clique.
- Chudnovski, Robertson, Thomas and Seymour 2006: In a graph without an odd hole or an odd hole complement, the minimum number of cliques needed to cover the vertices is equal to the maximum cardinality of a stable set.

2 A review of integral polyhedra and totally dual integral linear systems

Take integers $m, n \geq 1$, a rational $m \times n$ matrix M, and a rational m-dimensional (column) vector b. The set

$$P := \left\{ x \in \mathbb{R}^n : Mx \geq b, x \geq \mathbf{0} \right\}$$

is called a *polyhedron*. Hereinafter, $\mathbf{0}$ is the all-zeros vector of appropriate dimension. If P is a bounded set, then it is called a *polytope*. We will always be working with non-empty and full-dimensional polyhedra. A *vertex*, or an *extreme point*, of P is a point $x^\star \in P$ satisfying any of the following equivalent conditions:

- if for $x_1, x_2 \in P$ we have $x^\star = \frac{1}{2}x_1 + \frac{1}{2}x_2$, then $x_1 = x_2 = x^\star$,
- there is a row subsystem $M'x \geq b'$ of $\begin{pmatrix} M \\ I \end{pmatrix} x \geq \begin{pmatrix} b \\ \mathbf{0} \end{pmatrix}$, where $\operatorname{rank}(M') = n$ and $M'x^\star = b'$,
- there exists an integral cost vector $w \in \mathbb{Z}^n$ such that $x^\star$ is the unique optimal solution to the linear program

$$\min \left\{ w^\top x : x \in P \right\}.$$

We say that P is *integral* if all its vertices are integral.

For a variable cost vector $w \in \mathbb{Z}^n$, consider the primal linear program

$$(P) \qquad \begin{array}{ll} \min & w^\top x \\ \text{s.t.} & Mx \geq b \\ & x \geq \mathbf{0} \end{array}$$

and the dual linear program

$$(D) \qquad \begin{array}{ll} \max & b^\top y \\ \text{s.t.} & M^\top y \leq w \\ & y \geq \mathbf{0}. \end{array}$$

By LP Strong Duality, the optimal values of these two programs are equal, whenever the primal (P) is feasible and has a finite optimum. We say that the linear system $Mx \geq b, x \geq \mathbf{0}$ is *totally dual integral (TDI)* if, for all $w \in \mathbb{Z}^n$ for which the primal (P) is feasible and has a finite optimum, the primal (P) and the dual (D) have integral optimal solutions. (Warning: this definition is not standard!) By definition, if $Mx \geq b, x \geq \mathbf{0}$ is TDI, then the polyhedron $\{x \geq \mathbf{0} : Mx \geq b\}$ is integral.

Theorem 2.1 (Hoffman 1974, Edmonds and Giles 1977)**.** *The following statements are equivalent:*

- $Mx \geq b, x \geq \mathbf{0}$ *is totally dual integral,*
- *for all* $w \in \mathbb{Z}^n$ *for which the primal (P) is feasible and has a finite optimum, the dual (D) has an integral optimal solution.*

3 Packing and covering models

There are only two polyhedra that we are interested in. Let A, B be $0-1$ matrices, where B has no column of all zeros. We will call

$$\{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$$

the *set covering polyhedron*, and

$$\{x \geq \mathbf{0} : Bx \leq \mathbf{1}\}$$

the *set packing polytope*. Here, $\mathbf{1}$ is the all-ones vectors of appropriate dimension. When are these polyhedra integral? When are the associated linear systems TDI? These questions will form the underlying theme of the entire course. The short answers are, the questions have been answered for the set packing case, and they are widely open for the set covering case. But first, why are we even interested?

3.1 The set covering polyhedron

Let A be a $0-1$ matrix. Consider the *set covering program*

$$(P) \qquad \begin{array}{ll} \min & w^\top x \\ \text{s.t.} & Ax \geq \mathbf{1} \\ & x \geq \mathbf{0} \end{array}$$

and its dual

$$(D) \qquad \begin{array}{ll} \max & \mathbf{1}^\top y \\ \text{s.t.} & A^\top y \leq w \\ & y \geq \mathbf{0} \end{array}$$

for an integral cost vector w.[1] Notice that if w has a negative entry, then (P) does not have a finite optimum. We may therefore focus on non-negative cost vectors w.

Packing st-paths. Let $G = (V, E)$ be a graph and take distinct vertices s, t. Let A be the $0-1$ matrix whose columns are labeled by E and whose rows are the incidence vectors of st-paths. Let $w \in \mathbb{Z}^E_+$. Then the set covering program (P) can be rewritten as

$$\begin{array}{ll} \min & \sum (w_e x_e : e \in E) \\ \text{s.t.} & \sum (x_e : e \in P) \geq 1 \quad \forall\ st\text{-paths } P \\ & x_e \geq 0 \quad \forall e \in E. \end{array}$$

Note that every st-cut gives a feasible solution to (P). In particular, the minimum *weight* of an st-cut is an upper-bound on the optimal value of (P). Let G_w be the graph obtained from G after replacing each edge e by w_e parallel edges. Then the minimum weight of an st-cut in G is simply the minimum cardinality of an st-cut in

[1] Believe it or not, Fulkerson (1970) called this dual LP the "packing program" for reasons that will become clear soon. Why are we then calling (P) the set covering program? That will become clear in the next section.

G_w. Consider now the dual program (D), which may be rewritten as

$$\begin{array}{ll} \max & \sum (y_P : P \text{ is an } st\text{-path}) \\ \text{s.t.} & \sum (y_P : P \text{ is an } st\text{-path such that } e \in P) \leq w_e \quad \forall e \in E \\ & y_P \geq 0 \quad \forall\, st\text{-paths } P. \end{array}$$

Then a packing of st-paths in G_w gives a feasible solution to (D). We will think of a packing of st-paths in G_w as a *weighted* packing of st-paths in G (where each edge e appears in at most w_e many st-paths, and where an st-path may be packed more than once). Hence, the maximum value of a weighted packing of st-paths in G is a lower-bound on the optimal value of (D). It therefore follows from Theorem 1.1 that,

Corollary 3.1. *Let G be a graph and take distinct vertices s, t. Then the set covering system corresponding to the st-paths of G is totally dual integral. In particular, the set covering polyhedron*

$$\left\{x \in \mathbb{R}^E_+ : \sum (x_e : e \in P) \geq 1 \quad \forall\, st\text{-paths } P\right\}$$

is integral.

3.2 The set packing polytope

Let B be a $0-1$ matrix without a column of all zeros. Consider the *set packing program*

$$(P) \qquad \begin{array}{ll} \max & w^\top x \\ \text{s.t.} & Bx \leq \mathbf{1} \\ & x \geq \mathbf{0} \end{array}$$

and its dual

$$(D) \qquad \begin{array}{ll} \min & \mathbf{1}^\top y \\ \text{s.t.} & B^\top y \geq w \\ & y \geq \mathbf{0} \end{array}$$

for an integral cost vector w.[2] Notice that if w has a negative entry, then the corresponding variable in an optimal solution will always be set to 0. We may therefore focus on non-negative cost vectors w.

Covering with chains. Let $(E, \leq)$ be a partially ordered set. Let B be the $0-1$ matrix whose columns are labeled by E and whose rows are the incidence vectors of chains. Then the set packing program (P) can be rewritten as

$$\begin{array}{ll} \max & \sum (w_e x_e : e \in E) \\ \text{s.t.} & \sum (x_e : e \in C) \leq 1 \quad \forall \text{ chains } C \\ & x_e \geq 0 \quad \forall e \in E. \end{array}$$

Observe that an antichain gives a feasible solution to (P). In particular, the maximum *weight* of an antichain is a lower-bound on the optimal value of (P). Let $(E_w, \leq)$ be the partially ordered set obtained from $(E, \leq)$ after replacing each element e by w_e pairwise incomparable copies. Then the maximum weight of an antichain of

[2] Fulkerson (1970) called this dual LP the "covering program".

$(E, \leq)$ is simply the maximum cardinality of an antichain of $(E_w, \leq)$. Consider now the dual program (D), rewritten as

$$\begin{array}{ll} \min & \sum (y_C : C \text{ is a chain}) \\ \text{s.t.} & \sum (y_C : C \text{ is a chain such that } e \in C) \geq w_e \quad \forall e \in E \\ & y_C \geq 0 \quad \forall \text{ chains } C. \end{array}$$

Then a covering of E_w with chains gives a feasible solution to (D). We will think of a covering of E_w with chains as a *weighted* covering of E with chains (where each element e is covered at least w_e times, and chains can be used in a covering more than once). Thus, the minimum value of a weighted covering of E with chains is an upper-bound on the optimal value of (D). It therefore follows from Theorem 1.2 that,

Corollary 3.2. *Let $(E, \leq)$ be a partially ordered set. Then the set packing system corresponding to the chains of $(E, \leq)$ is totally dual integral. In particular, the set packing polytope*

$$\left\{ x \in \mathbb{R}_+^E : \sum (x_e : e \in C) \leq 1 \quad \forall \textit{ chains } C \right\}$$

is integral.

4 Balanced matrices

Let A, B be $0-1$ matrices, where B has no column of all zeros. Why is

$$\{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$$

called the set covering polyhedron and

$$\{x \geq \mathbf{0} : Bx \leq \mathbf{1}\}$$

the set packing polytope? There is a neat way to look at these polyhedra that explains the terminology and gives us good intuition about what is coming. Take a loopless graph $G = (V, E)$. Let A be the edge-vertex incidence matrix of G, that is, the columns are labeled by V and the rows are the incidence vectors of the edges. Then the $0-1$ points of

$$\{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$$

correspond to the vertex covers of G, hence the "set covering polyhedron". (A *vertex cover* of a graph is a set of vertices whose deletion makes the graph stable.) Let B be the vertex-edge incidence matrix of G, i.e. $B = A^\top$. Then the $0-1$ points of

$$\{x \geq \mathbf{0} : Bx \leq \mathbf{1}\}$$

correspond to the matchings of G, hence the "set packing polytope".

It follows from various well-known theorems of Kőnig (1931) that if G is bipartite, then the set covering and the set packing systems associated to the (edge-vertex or vertex-edge) incidence matrix are totally dual integral. Well, in general, we can think of any $0-1$ matrix as the (vertex-edge or edge-vertex) incidence matrix of a "hypergraph". How can we generalize the notion of bipartite-ness to hypergraphs? However way we do this, we want the definition to be invariant of taking matrix transpose.

An odd square matrix of the form

$$\begin{pmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & 1 & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ 1 & & & & & 1 \end{pmatrix}$$

is called an *odd cycle matrix*. A $0-1$ matrix is *balanced* if it has no odd cycle submatrix (even after rearranging its rows and columns). Observe that if a matrix is balanced, then so is its transpose. Notice that an odd cycle matrix is the incidence matrix of a graph odd cycle. As a result, the incidence matrix of a bipartite graph is always balanced. We may therefore think of balanced matrices as generalizations of bipartite graphs.

4.1 A bicoloring characterization of balanced matrices

A *bicoloring* of a $0-1$ matrix is a partition of the columns into two color classes, where every row with at least two ones gets both colors. For instance, $R = \{1, 2\}$ and $B = \{3, 4\}$ yields a bicoloring of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

whose columns are labeled $1, 2, 3, 4$ from left to right.

Theorem 4.1 (Berge 1970)**.** *A $0-1$ matrix is balanced if, and only if, every submatrix has a bicoloring.*

Proof. Let A be a $0-1$ matrix. ($\Leftarrow$) Since an odd cycle is not bipartite, an odd cycle matrix is not bicolorable. So, if every submatrix of A is bicolorable, A must be balanced. ($\Rightarrow$) Suppose otherwise. We may assume that A is a balanced matrix that is not bicolorable, but every proper submatrix is bicolorable. In particular, every row of A has at least two ones. Let V collect the column labels of A.

Claim. *For every $v \in V$, there exist rows of the form $\{v, u\}, \{v, w\}$ for some distinct $u, w \in V - \{v\}$.*

Proof of Claim. For if not, bicolor the column submatrix of A corresponding to the columns $V - \{v\}$. Our contrary assumption allows us to extend this bicoloring to a bicoloring of A, a contradiction. $\diamondsuit$

Let G be the graph on vertices V whose edges correspond to the rows in A with exactly two ones. Since A is balanced, and the edge-vertex incidence matrix of G is a submatrix of A, it follows that G is bipartite. By Claim 1, every vertex of G has degree at least 2. In particular, G has a vertex v_0 that is not a cut-vertex. Now bicolor the column submatrix of A corresponding to the columns $V - \{v_0\}$, and extend this bicoloring uniquely to a bicoloring of A, determined by the path in $G \setminus v_0$ between two neighbors of v_0, a contradiction. This finishes the proof of Theorem 4.1. $\square$

A *hypergraph* is a pair $G = (V, E)$ where V is a finite set of *vertices*, and each element of E is a non-empty subset of V, called an *edge*. A hypergraph is *balanced* if its incidence matrix is balanced.

Corollary 4.2 (Berge 1972)**.** *Let $G = (V, E)$ be a balanced hypergraph, and let $k \geq 2$ be the minimum cardinality of an edge. Then there exists a partition of V into k color classes where every edge gets at least one vertex of each color.*

Proof. For $k = 2$, the result follows immediately from Theorem 4.1. We may therefore assume that $k \geq 3$. Let $(S_1, \ldots, S_k)$ be an arbitrary partition of V. For each edge e, let

$$k_e := |\{i \in [k] : e \cap S_i \neq \emptyset\}| \in \{1, \ldots, k\}.$$

If each k_e is k, then we have a k-coloring. Otherwise, assume that $k_g < k$ for some edge g. Since $|g| \geq k$, we may assume that

$$|g \cap S_{k-1}| \geq 2 \quad \text{and} \quad g \cap S_k = \emptyset.$$

Let A be the edge-vertex incidence matrix of G. Since A is balanced, by Theorem 4.1, we may bicolor the column submatrix of A corresponding to $S_{k-1} \cup S_k$ and get a bicoloring $S'_{k-1} \cup S'_k$. Consider now the partition $(S_1, \cdots, S_{k-2}, S'_{k-1}, S'_k)$. Notice that g intersects k_g+1 many of these parts, and every other edge e intersects at least k_e many of these parts. By applying this argument recursively, we will achieve the desired k-coloring. $\square$

For an integer $k \geq 2$, a hypergraph is *k-partite* if its vertices can be partitioned into k parts such that every edge intersects each part at most once. As an immediate consequence of the preceding result, we have the following:

Corollary 4.3. *Take an integer $k \geq 2$ and a hypergraph where every edge has cardinality k. If G is balanced, then it is k-partite.*

4.2 Integral polyhedra associated with balanced matrices

Take a $0-1$ matrix A with column labels E, and consider the polytope

$$P(A) := \{\mathbf{1} \geq x \geq \mathbf{0} : Ax = \mathbf{1}\}.$$

Notice that for each $e \in E$,

$$P(A) \cap \{x : x_e = 0\} = P(A') \quad \text{and} \quad P(A) \cap \{x : x_e = 1\} = P(A'')$$

where A', A'' are appropriate submatrices of A. (Equality holds above after extending $P(A'), P(A'')$ to $\mathbb{R}^E$ by setting new coordinates to either 0 or 1.)

Proposition 4.4. *Let A be a balanced matrix. Then the polytope $P(A)$ is integral.*

Proof. Suppose otherwise. Let E be the column labels of A. We may assume that $P(A)$ is not integral, but for every proper submatrix A' of A, $P(A')$ is integral. In particular, for every $e \in E$, the two polytopes

$$P(A) \cap \{x : x_e = 0\} \quad \text{and} \quad P(A) \cap \{x : x_e = 1\}$$

are integral. Let $x^\star$ be a fractional extreme point of $P(A)$. Since the polytopes above are integral, it follows that $\mathbf{1} > x^\star > \mathbf{0}$. Our minimality assumption implies that A is a square non-singular matrix.

Claim. *Every row of A has exactly two ones.*

Proof of Claim. By our minimal choice, every row of A has at least two ones. Let A' be the matrix obtained from A after removing the first row. Since $P(A')$ is integral and $x^\star \in P(A')$, it follows that $x^\star$ lies on an edge of $P(A')$. So for some vertices $\chi_S, \chi_T \in P(A')$ and $\lambda \in (0,1)$,

$$x^\star = \lambda \chi_S + (1-\lambda)\chi_T.$$

Since $\mathbf{1} > x^\star > \mathbf{0}$, it follows that $S \cap T = \emptyset$ and $S \cup T = E$. Since $A'\chi_S = \mathbf{1} = A'\chi_T$, every row of A other than the first row has exactly two ones. A similar argument applied to the second row implies that even the first row has exactly two ones. $\Diamond$

Since A is balanced, it is the incidence edge-vertex incidence matrix of a bipartite graph G. As A is a square matrix, G has an even cycle, which in turn contradicts the non-singularity of A. This finishes the proof of Proposition 4.4. $\square$

Theorem 4.5 (Fulkerson, Hoffman, Oppenheim 1974)**.** *Let* $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ *be a balanced matrix. Then the polyhedron*

$$P = \{x \geq \mathbf{0} : Ax \geq \mathbf{1}, Bx \leq \mathbf{1}, Cx = \mathbf{1}\}$$

is integral. In particular, the set packing polytope and the set covering polyhedron corresponding to a balanced matrix are both integral.

Proof. Let $x^\star$ be an extreme point of P. Observe that $x^\star \leq \mathbf{1}$, and that $x^\star$ is also an extreme point of the polytope $\{\mathbf{1} \geq x \geq \mathbf{0} : Dx = \mathbf{1}\}$, where D is the row submatrix of $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ corresponding to the constraints of $Ax \geq \mathbf{1}, Bx \leq \mathbf{1}, Cx = \mathbf{1}$ that are tight at $x^\star$. Since $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ is balanced, so is D, so by Proposition 4.4, $x^\star$ is integral, as required. $\square$

In fact, the linear system above is totally dual integral. We will prove a similar result in the next section.

4.3 Hall's theorem for balanced hypergraphs

Let $G = (V, E)$ be a hypergraph. A *matching* is a packing of pairwise disjoint edges. A *perfect matching* is a matching that uses every vertex. Recall Hall's condition for the existence of perfect matchings in bipartite graphs:

Theorem 4.6 (Hall 1935)**.** *Let G be a bipartite graph. Then the following statements are equivalent:*

- *G has no perfect matching,*
- *there exist disjoint vertex sets R, B such that $|R| > |B|$ and every edge with an end in R has an end in B.*

We will see a generalization of this to balanced hypergraphs. We will need two lemmas.

Lemma 4.7. *Let A be an $m \times n$ balanced matrix. Then the polyhedron*

$$P = \{x, s, t \geq \mathbf{0} : Ax + Is - It = \mathbf{1}\}$$

is integral.

Proof. Denote by a_i the ith row of A, for each $i \in [m]$. Take an extreme point $(x^\star, s^\star, t^\star)$ of P. Since the corresponding columns are linearly dependent, we see that $s_i^\star t_i^\star = 0$ for each $i \in [m]$. As a result, $x^\star$ is also an extreme point of the polyhedron

$$\left\{ x \geq \mathbf{0} : \begin{array}{lll} a_i^\top x & \leq 1 & \forall i \in [m] \text{ s.t. } s_i^\star > 0 \\ a_i^\top x & \geq 1 & \forall i \in [m] \text{ s.t. } t_i^\star > 0 \\ a_i^\top x & = 1 & \text{otherwise.} \end{array} \right\}$$

By Theorem 4.5, this polyhedron is integral, implying in turn that $x^\star$ is integral. This easily implies that $(x^\star, s^\star, t^\star)$ is also integral, thereby finishing the proof. $\square$

Lemma 4.8. *Let A be a balanced matrix. Then the linear system $x, s, t \geq \mathbf{0}, Ax + Is - It = \mathbf{1}$ is totally dual integral.*

Proof. We prove this by induction on the number of rows of A. The base case is obvious. For the induction step, consider for integral weights b, c, d the primal program

$$(P) \qquad \begin{array}{ll} \max & b^\top x + c^\top s + d^\top t \\ \text{s.t.} & Ax + Is - It = \mathbf{1} \\ & x, s, t \geq \mathbf{0} \end{array}$$

and the dual

$$(D) \qquad \begin{array}{lll} \min & \mathbf{1}^\top y & \\ \text{s.t.} & A^\top y & \geq b \\ & y & \geq c \\ & -y & \geq d. \end{array}$$

We will construct an integral optimal solution to (D). To this end, take an optimal solution $\bar{y}$ to (D). If $\bar{y}$ is integral, we are done. Otherwise, we may assume that $\bar{y}_1$ is fractional. Write $\bar{y} = (\bar{y}_1, \bar{z})$. Let a be the first row of A, and let A' (resp. c', d') be the matrix (resp. vector) obtained from A (resp. c, d) after removing the first row. Consider the program

$$(D') \qquad \begin{array}{lll} \min & \mathbf{1}^\top z & \\ \text{s.t.} & A'^\top z & \geq b - \lceil \bar{y}_1 \rceil a \\ & z & \geq c' \\ & -z & \geq d'. \end{array}$$

Since $\bar{y} = (\bar{y}_1, \bar{z})$ is feasible for (D), we get that $\bar{z}$ is feasible for (D'). Our induction hypothesis implies that (D') has an integral optimal solution $z^\star$. In particular,

$$\mathbf{1}^\top \bar{z} \geq \mathbf{1}^\top z^\star.$$

As $z^\star$ is feasible for (D'), and c, d are integral, it follows that $(\lceil \bar{y}_1 \rceil, z^\star)$ is feasible for (D), so

$$\lceil \bar{y}_1 \rceil + \mathbf{1}^\top z^\star \geq \mathbf{1}^\top \bar{y} = \bar{y}_1 + \mathbf{1}^\top \bar{z}.$$

Combining the preceding two inequalities yields

$$\lceil \bar{y}_1 \rceil + \mathbf{1}^\top z^\star \geq \mathbf{1}^\top \bar{y} \geq \bar{y}_1 + \mathbf{1}^\top z^\star.$$

By Lemma 4.7, (P) has an integral optimal solution, so as b, c, d are integral, (P) has an integer optimal value. Thus, by LP Strong Duality, $\mathbf{1}^\top \bar{y}$ is an integer. Hence, the inequalities above imply that $\lceil \bar{y}_1 \rceil + \mathbf{1}^\top z^\star = \mathbf{1}^\top \bar{y}$, so $(\lceil \bar{y}_1 \rceil, z^\star)$ is an integral optimal solution for (D), as required. This completes the induction step. □

We are now ready to prove the following generalization of Theorem 4.6:

Theorem 4.9 (Conforti, Cornuéjols, Kapoor, Vušković 1996)**.** *Let $G = (V, E)$ be a balanced hypergraph. Then the following statements are equivalent:*

- *G has no perfect matching,*
- *there are disjoint vertex sets R, B such that $|R| > |B|$ and for every edge e, $|e \cap B| \geq |e \cap R|$.*

Proof. ($\Leftarrow$) Suppose for a contradiction that G has a perfect matching $e_1, \ldots, e_k$. Then

$$|R| = \sum_{i=1}^{k} |e_i \cap R| \leq \sum_{i=1}^{k} |e_i \cap B| = |B| < |R|,$$

a contradiction. ($\Rightarrow$) Suppose G has no perfect matching. Let A be the vertex-edge incidence matrix of G. Notice that A is a balanced matrix. Consider the linear program

$$(P) \qquad \begin{array}{ll} \max & \mathbf{0}^\top x - \mathbf{1}^\top s - \mathbf{1}^\top t \\ \text{s.t.} & Ax + Is - It = \mathbf{1} \\ & x, s, t \geq \mathbf{0} \end{array}$$

Since G has no perfect matching, (P) has no integer feasible solution of value ≥ 0. It therefore follows from Lemma 4.7 that the optimal value of (P) is < 0. As a result, by Lemma 4.8, the dual program has an integral feasible solution of negative value, that is, there is an integral point $\bar{y}$ such that

$$\begin{aligned} \mathbf{1}^\top y &< 0 \\ A^\top y &\geq \mathbf{0} \\ y &\leq \mathbf{1} \\ y &\geq -\mathbf{1} \end{aligned}$$

Let $B := \{v \in V : \bar{y}_v = 1\}$ and $R := \{v \in V : \bar{y}_v = -1\}$. Clearly, $B \cap R = \emptyset$. The first inequality implies that $|R| > |B|$ while the second inequality implies that, for each edge e, $|e \cap B| \geq |e \cap R|$, as required. □

This result has a nice Kőnig-type consequence. Given a hypergraph, the *degree* of a vertex is the number of edges containing that vertex. For an integer $d \geq 1$, a hypergraph is *d-regular* if every vertex has degree d.

Corollary 4.10. *The edges of a balanced hypergraph with maximum degree d can be partitioned into d matchings.*

Proof. Let $G = (V, E)$ be a balanced hypergraph with maximum degree $d \geq 1$. Let us first prove the result for d-regular hypergraphs:

Claim 1. *If G is d-regular, then its edges can be partitioned into d perfect matchings.*

Proof of Claim. We prove this by induction on $d \geq 1$. The base case $d = 1$ is obvious. Assume that $d \geq 2$. Let us use Theorem 4.9 to find a perfect matching in G. Take disjoint vertex subsets R, B of V such that for every edge e, $|e \cap B| \geq |e \cap R|$. Then

$$d \cdot |B| = \sum_{e \in E} |e \cap B| \geq \sum_{e \in E} |e \cap R| = d \cdot |R|,$$

implying in turn that $|B| \geq |R|$. It therefore follows from Theorem 4.9 that G has a perfect matching $M_d \subseteq E$. Notice that $G \setminus M_d$ is $(d-1)$-regular, so by the induction hypothesis, the edges of $G \setminus M_d$ can be partitioned into $d-1$ perfect matchings $M_1, \ldots, M_{d-1}$. Together with M_d, we get a partition of the edges of G into d perfect matchings, thereby completing the induction step. $\Diamond$

Claim 2. *There is a d-regular balanced hypergraph $H = (V, E')$ such that $E \subseteq E'$.*

Proof of Claim. To obtain H, for every vertex v of G, add $d - \deg(v)$ edges of the form $\{v\}$. It is clear that H is a d-regular hypergraph. It is easy to see that H is a balanced hypergraph. $\Diamond$

By Claim 1, the edges of H can be partitioned into d perfect matchings. It is easy to see that this corresponds to a partition of the edges of G into d matchings, thereby finishing the proof. $\square$

In particular,

Theorem 4.11 (Kőnig 1931)**.** *Let G be a loopless bipartite graph of maximum degree d. Then the edges of G can be partitioned into d matchings, that is, G can be d-edge-colored.*

5 Perfect graphs

Let $G = (V, E)$ be a simple graph. Denote by $\chi(G)$ the minimum number of stable sets needed to cover V. Notice that $\chi(G)$ records the *chromatic number of* G, i.e. the minimum number of colors needed for a vertex-coloring. Denote by $\omega(G)$ the maximum cardinality of a clique. Since the vertices of a clique get different colors in any vertex-coloring, it follows that

$$\chi(G) \geq \omega(G).$$

Denote by $\overline{G}$ the *complement* of G, that is, $\overline{G}$ has vertex set V where distinct vertices u, v are adjacent in $\overline{G}$ if they are non-adjacent in G. Notice that the cliques and stable sets of $\overline{G}$ are precisely the stable sets and cliques of $\overline{G}$.

Remark 5.1. *Let $G = (V, E)$ be a simple graph. Then*

$$\theta(G) := \chi(\overline{G})$$

is the minimum number of cliques of G needed to cover V, and

$$\alpha(G) := \omega(\overline{G})$$

is the maximum cardinality of a stable set. In particular, $\theta(G) \geq \alpha(G)$.

Recall the following two theorems from Assignment 1:

Theorem 5.2 (Kőnig 1931)**.** *In a loopless bipartite graph, the minimum cardinality of a vertex cover is equal to the maximum cardinality of a matching.*

Theorem 5.3. *In a partially ordered set, the minimum number of antichains needed to cover the ground set is equal to the maximum cardinality of a chain.*

We will need this result moving forward, as well as a few notions. The *line graph* of a simple graph G is the graph on vertex set $E(G)$ where distinct $e, f \in E(G)$ are adjacent if e, f share a vertex of G. Given a partially ordered set $(V, \leq)$, its *comparability graph* is the graph on vertex set V where distinct $u, v \in V$ are adjacent if they are comparable.

The main theme of this section is, when does equal hold in $\chi \geq \omega$?

Theorem 5.4. *$\chi(G) = \omega(G)$ if G is any of the following graphs:*

(1) G or $\overline{G}$ is bipartite,

(2) G or $\overline{G}$ is the line graph of a bipartite graph,

(3) G or $\overline{G}$ is a comparability graph.

Proof. **(1)** Let G be a bipartite graph. Then $\chi(G) = 2 = \omega(G)$. We need to show that $\theta(G) = \alpha(G)$. Clearly,

$$\alpha(G) = |V| - k$$

where k is the minimum cardinality of a vertex cover. Since G is bipartite,

$$\theta(G) = |V| - m$$

where m is the maximum cardinality of a matching. By Theorem 5.2, $m = k$, implying in turn that $\theta(G) = \alpha(G)$, as required. **(2)** Let G be the line graph of a bipartite graph H. Observe that the stable sets and cliques of G are in correspondence with the matchings and stars of H, respectively. Thus $\chi(G)$ is equal to the minimum number of colors needed in an edge-coloring of H, while $\omega(G)$ is equal to the maximum degree of a vertex of H. It therefore follows from Theorem 4.11 that $\chi(G) = \omega(G)$. Moreover, $\theta(G)$ is equal to the minimum cardinality of a vertex cover, while $\alpha(G)$ is equal to the maximum cardinality of a matching. So by Theorem 5.2, $\theta(G) = \alpha(G)$. **(3)** Let $G = (V, E)$ be the comparability graph of a partially ordered set $(V, \leq)$. Then the cliques and stable sets of G are in correspondence with the chains and antichains of $(V, \leq)$. It therefore follows from Theorem 1.2 that $\theta(G) = \alpha(G)$, and it follows from Theorem 5.3 that $\chi(G) = \omega(G)$. $\square$

Equality does not always hold in $\chi \geq \omega$. For instance, for the odd cycle C_5 on five vertices, $\chi(C_5) = 3 > 2 = \omega(C_5)$. Can we characterize when equality does hold? Is this even a well-posed question? Let H be an arbitrary graph, and let $k := \chi(H) - \omega(H) \geq 0$. Let $C \subseteq V(H)$ be a maximum clique of H. Let G be the graph obtained from H after adding k vertices and just enough edges so as to grow C into a clique of cardinality $\omega(H) + k$. Notice now that $\chi(G) = \chi(H) = \omega(H) + k = \omega(G)$. Starting from an arbitrary graph, we just constructed a graph for which equality holds in $\chi \geq \omega$. This construction tells us that asking when equality holds in

$$\chi \geq \omega$$

is an ill-posed question. To make sure this construction is ruled out, we will come up with a stronger notion.

Let $G = (V, E)$ be a simple graph. For $X \subseteq V$, the subgraph of G induced on vertices X is called an *induced subgraph* and is denoted $G[X]$. We say that G is *perfect* if, for every induced subgraph G' of G, $\chi(G') = \omega(G')$. (Notice that G' may be G.) In words, a simple graph is perfect if in each induced subgraph, the maximum cardinality of a clique is equal to the chromatic number. It follows from the preceding theorem that,

Corollary 5.5. *The following graphs are perfect:*

(1) bipartite graphs, and their complements,

(2) line graphs of bipartite graphs, and their complements,

(3) comparability graphs, and their complements.

The obvious question this corollary leads to is, does complementation preserve perfection? Claude Berge asked the same question in 1961. Although this may seem too good to be true, the answer is yes!

5.1 The max-max inequality and the weak perfect graph theorem

As a tribute to Manfred Padberg, we follow Gasparyan (1996) for the proof of the following result:

Theorem 5.6 (Lovász 1972)**.** *Let G be a simple graph. The following statements are equivalent:*

(i) G is perfect,

(ii) $\omega(H) \cdot \alpha(H) \geq |V(H)|$ for every induced subgraph H.

Proof. **(i) ⇒ (ii)**: Let H be an induced subgraph. By definition, $\chi(H) = \omega(H)$, that is, $V(H)$ can be covered by $\omega(H)$ stable sets. Since each stable set has cardinality at most $\alpha(H)$, it follows that

$$|V(H)| \leq \omega(H) \cdot \alpha(H).$$

(ii) ⇒ (i): Suppose for a contradiction that G is not perfect. Let H be an induced subgraph of G that is not perfect, but every proper induced subgraph of H is perfect. Let $\omega := \omega(H)$, $\alpha := \alpha(H)$ and $n := |V(H)|$. Note that $n > 1$. Clearly,

$$\omega \geq \omega(H \setminus S) \geq \omega - 1 \qquad \text{for every non-empty stable set } S \subseteq V(H);$$

since $H \setminus S$ is perfect and H is not, it follows that

$$\omega(H \setminus S) = \omega \qquad \text{for every non-empty stable set } S \subseteq V(H).$$

Let S_0 be a maximum stable set of H. Then for every vertex $v \in S_0$, $H \setminus v$ is perfect, so its vertices can be partitioned into $\omega(H \setminus v) = \omega$ non-empty stable sets. As S_0 has α vertices, we get $\alpha\omega$ stable sets $S_1, \ldots, S_{\alpha\omega}$.

Claim. *Every maximum clique of H intersects all but one of $S_0, S_1, \ldots, S_{\alpha\omega}$ exactly once.*

Proof of Claim. Let C be a maximum clique of H. Clearly C intersects each one of $S_0, S_1, \ldots, S_{\alpha\omega}$ at most once. For a vertex $v \in S_0$, if

- $v \in C$: then C intersects all but one stable set in every partition of $V(H \setminus v)$ into ω stable sets,
- $v \notin C$: then C intersects all stable sets in every partition of $V(H \setminus v)$ into ω stable sets.

This observation immediately implies the claim. ◊

For each $i \in \{0, 1, \ldots, \alpha\omega\}$, let C_i be a maximum clique of $H \setminus S_i$; notice that $|C_i| = \omega$. Let A be the $0-1$ matrix whose columns are labeled by $V(H)$, and whose rows are the incidence vectors of $S_0, S_1, \ldots, S_{\alpha\omega}$. Let B be the $0 - 1$ matrix whose columns are labeled by $V(H)$, and whose rows are the incidence vectors of $C_0, C_1, \ldots, C_{\alpha\omega}$. It then follows from the claim above that $AB^\top = J - I$, where J is the all-ones matrix and I the identity matrix of appropriate dimensions. Since $J - I$ is a non-singular $(\alpha\omega + 1) \times (\alpha\omega + 1)$ matrix, it follows that both A and B have full row rank, implying in turn that

$$|V(H)| = n \geq \alpha\omega + 1 = \alpha(H) \cdot \omega(H) + 1 > |V(H)|,$$

a contradiction. □

As a consequence, we get the so-called weak perfect graph theorem:

Theorem 5.7 (Lovász 1972)**.** *If a graph is perfect, then so is its complement.*

Proof. Suppose that G is perfect. Then by Theorem 5.6, for every induced subgraph H of G,

$$\omega(H) \cdot \alpha(H) \geq |V(H)|,$$

implying in turn that for every induced subgraph $\overline{H}$ of $\overline{G}$,

$$\alpha(\overline{H}) \cdot \omega(\overline{H}) \geq |V(\overline{H})|,$$

so by Theorem 5.6, $\overline{G}$ is perfect, as required. $\square$

5.2 Odd holes and odd antiholes

We say that a simple graph is *minimally imperfect* if it is not perfect, but every proper induced subgraph is perfect. Equivalently, a simple graph G is minimally imperfect if $\chi(G) > \omega(G)$, but for every proper induced subgraph G', $\chi(G') = \omega(G')$. The latter implies that a minimally imperfect graph is always connected.

Remark 5.8. *A graph is perfect if, and only if, it has no minimally imperfect induced subgraph.*

Let H be an odd circuit with at least 5 vertices. Then $3 = \chi(H) > \omega(H) = 2$, so G is imperfect. Since every proper induced subgraph of H is bipartite, and therefore perfect, it follows that H is minimally imperfect. Notice that Theorem 5.7 equivalently states that,

Corollary 5.9. *The complement of a minimally imperfect graph is also minimally imperfect.*

Thus, the complement of an odd circuit with at least 5 vertices is also minimally imperfect. Let G be a simple graph. We say that G has an *odd hole* if it has as an induced subgraph an odd circuit with at least 5 vertices, and we say that G has an *odd antihole* if $\overline{G}$ has an odd hole. It follows from the preceding remark that,

Remark 5.10. *A perfect graph has no odd hole and no odd antihole.*

In 1961, Claude Berge conjectured that the converse of this statement is also true. In 2006, this conjecture was proved by Chudnovsky, Robertson, Seymour and Thomas, and their theorem is referred to as the strong perfect graph theorem. We will see some of the milestones and highlights leading to the proof, as well as a sketch of the proof.

5.3 Star cutsets and antitwins

Let $G = (V, E)$ be a simple graph. A *star cutset* is a non-empty $X \subseteq V$ such that

- a vertex of X is adjacent to all the other vertices in X, and
- $G \setminus X$ is not connected.

Lemma 5.11 (Chvátal 1985). *A minimally imperfect graph does not have a star cutset.*

Proof. Let $G = (V, E)$ be a minimally imperfect graph, and let $\omega := \omega(G)$. Then

$$\omega(G \setminus S) = \omega \quad \text{for every stable set } S \subseteq V.$$

Suppose for a contradiction that G has a star cutset $X \subseteq V$. Then the vertices of $G \setminus X$ can be partitioned into non-empty parts V_1, V_2 such that G has no edge between V_1 and V_2. Since every proper induced subgraph of G is perfect, for each $i \in [2]$, there is a vertex-coloring $f_i : X \cup V_i \to [\omega]$ of the induced subgraph $G[X \cup V_i]$. Since X is a star cutset, it has a vertex v that is adjacent to all other vertices of X. For $i \in [2]$, let $S_i := \{w \in X \cup V_i : f_i(w) = f_i(v)\}$. Clearly, each S_i is stable and $S_i \cap X = \{v\}$. Moreover, since there are no edges between V_1 and V_2, it follows that $S := S_1 \cup S_2$ is also stable. In particular, $\omega(G \setminus S) = \omega$, so $G \setminus S$ has a clique C of cardinality ω. However, either $C \subseteq X \cup V_1$ or $C \subseteq X \cup V_2$, implying in turn that C is an ω-clique of some $G[X \cup V_i] \setminus S_i$, which has an $(\omega - 1)$-vertex-coloring, a contradiction. $\square$

This lemma was a key milestone for what led to the proof of the strong perfect graph theorem. To demonstrate the power of this lemma, let us see some applications of it. Let G_1 be a perfect graph, and take a vertex $v \in V(G_1)$. To *duplicate* v is to introduce a new vertex $\bar{v}$, join it to all the neighbors of v, and then join it to $\bar{v}$. More generally, given another perfect graph G_2 over a disjoint vertex set, to *substitute* G_2 *for* v is to remove v, and join every vertex of G_2 to all the neighbors of v in $G_1 \setminus v$.

Theorem 5.12 (Lovász 1972). *Let G_1, G_2 be perfect graphs over disjoint vertex sets. If G is obtained by substituting G_2 for a vertex v of G_1, then G is perfect. In particular, duplication preserves perfection.*

Proof. Suppose otherwise. Since every induced subgraph of G is either an induced subgraph of G_1, or of G_2, or arises from induced subgraphs of G_1, G_2 by substitution, we may assume that G is minimally imperfect. Clearly, G_2 has at least two vertices, and $G_1 \setminus v$ has at least one vertex. Take an arbitrary vertex u of G_2, and denote by N its neighbors of G in $V(G_1 \setminus v)$. Notice that for each vertex in $V(G_2)$, its neighbors of G in $V(G_1 \setminus v)$ is precisely N. As G is minimally imperfect, $\overline{G}$ is minimally imperfect by Corollary 5.9, so $\overline{G}$ is connected, implying in turn that $V(G_1 \setminus v) - N \neq \emptyset$. Let $X := \{u\} \cup N$. Then X is a star cutset as u is adjacent to all the vertices in N, and in $G \setminus X$, there are no edges between $V(G_2) - \{u\}$ and $V(G_1 \setminus v) - N$. This contradicts the Star Cutset Lemma 5.11. $\square$

Let $G = (V, E)$ be a simple graph. A *skew partition* is a partition of V into a pair (A, B) such that $G[A]$ is not connected and $\overline{G}[B]$ is not connected. Notice that if (A, B) is a skew partition for G, then it is a skew partition for $\overline{G}$. Notice further that if X is a star cutset and $|X| \geq 2$, then $(V - X, X)$ is a skew partition. In an attempt to generalize Lemma 5.11, Chvátal (1985) conjectured that a minimally imperfect graph does not have a skew partition. The *length* of a path is the number of edges in it. A path of $\overline{G}$ is called an *antipath of* G. We say that a skew partition (A, B) is *balanced* if

- there is no induced odd path between non-adjacent vertices in B with interior in A,

- there is no induced odd antipath between adjacent vertices in A with interior in B.

Theorem 5.13 (Chudnovsky, Robertson, Seymour, Thomas 2006)**.** *A minimally imperfect graph does not have a balanced skew partition.*

Let $G = (V, E)$ be a simple graph. Distinct vertices u, v are *antitwins* if every other vertex is adjacent to precisely one of u, v. Notice that if u, v are antitwins in G, then they are also antitwins in $\overline{G}$. The proof of the following lemma highlights the special role odd holes and odd antiholes have as minimally imperfect graphs.

Lemma 5.14 (Oraliu 1988)**.** *A minimally imperfect graph does not have antitwins.*

Proof. Let $G = (V, E)$ be a minimally imperfect graph, and let $\omega := \omega(G)$. Suppose for a contradiction that G has antitwins u, v. Let $A \subseteq V - \{u, v\}$ be the neighbors of u other than possibly v, and let $B \subseteq V - \{u, v\}$ be the neighbors of v other than possibly u. Since u, v are antitwins, it follows that A, B partition $V - \{u, v\}$.

Claim 1. *B contains a clique of cardinality $\omega - 1$ that does not extend to a clique of cardinality ω in $A \cup B$.*

Proof of Claim. Let $f : V - \{v\} \to [\omega]$ be an ω-vertex-coloring of $G \setminus v$, and let $S := \{w \in V - \{v\} : f(w) = f(u)\}$. Notice that $u \in S \subseteq \{u\} \cup B$. Recall that $G \setminus S$ has a clique K of cardinality ω. As the vertices of $G \setminus v \setminus S$ are $(\omega - 1)$-vertex-colored, it follows that

- $v \in K$, implying in turn that $K - \{v\} \subseteq B$,
- $K - \{v\}$ does not extend to a clique of cardinality ω in $A \cup B$.

$K - \{v\}$ is the desired clique. $\Diamond$

Let $\alpha := \alpha(G)$. By Corollary 5.9, $\overline{G}$ is also minimally imperfect. Thus, since u, v are also antitwins in $\overline{G}$, Claim 1 applied to $\overline{G}$ implies that,

Claim 2. *A contains a stable set of cardinality $\alpha - 1$ that does not extend to a stable set of cardinality α in $A \cup B$.*

Let $C \subseteq B$ be the clique from Claim 1, and let $S \subseteq A$ be the stable set from Claim 2. Among all the vertices in C, pick one x with the least number of neighbors in S. Since S does not extend to a stable set in $A \cup B$, it follows that x has a neighbor $y \in S$. Since C does not extend to a clique in $A \cup B$, y has a non-neighbor $z \in C$. As z has at least as many neighbors in S as x does, there is a vertex $t \in S$ that is a neighbor of z but is not a neighbor of x. Observe now that $\{u, y, x, z, t\}$ induces an odd hole (and an odd antihole), which is imperfect, thereby contradicting the minimality of G. $\square$

Let $G = (V, E)$ be a simple graph. Take disjoint non-empty subsets $A, B \subseteq V$ such that $|A| + |B| \geq 3$ and $|V - (A \cup B)| \geq 2$. The pair (A, B) is *homogeneous* if for each $v \in V - (A \cup B)$,

- if v is adjacent to a vertex of A, then it is adjacent to all of A, and

- if v is adjacent to a vertex of B, then it is adjacent to all of B.

Note that if (A, B) is homogeneous for G, then it is homogeneous for $\overline{G}$. Observe that if $|V(G)| \geq 5$ and u, v are antitwins both of which have a neighbor in $V(G) - \{u, v\}$, then $(N(u) - \{u\}, N(v) - \{v\})$ is homogeneous, where $N(u), N(v)$ denote the neighbors of u, v, respectively. The following theorem generalizes the Antitwin Lemma 5.14:

Theorem 5.15 (Chvátal and Sbihi 1987)**.** *A minimally imperfect graph does not have a homogeneous pair.*

Let $G = (V, E)$ be a simple graph. A 2*-join* is a partition of V into parts V_1, V_2 and non-empty disjoint subsets $A_1, B_1 \subseteq V_1$ and $A_2, B_2 \subseteq V_2$ such that

- $|V_1| \geq 3$ and $|V_2| \geq 3$,
- all the vertices in A_1 are adjacent to all the vertices in A_2, and all the vertices in B_1 are adjacent to all the vertices in B_2,
- there are no other adjacencies between V_1 and V_2.

Notice that an odd circuit of length at least 7 has a 2-join.

Theorem 5.16 (Cornuéjols and Cunningham 1985)**.** *Let G be a minimally imperfect graph. If G has a* 2*-join, then it is an odd hole, and if $\overline{G}$ has a* 2*-join, then G is an odd antihole.*

5.4 The strong perfect graph theorem

Let $G = (V, E)$ be a simple graph. We say that G is *Berge* if it has no odd hole and no odd antihole. Clearly, the complement of a Berge graph, as well as its induced subgraphs, are also Berge. By Remark 5.10, a perfect graph is always Berge. Conversely, the strong perfect graph theorem proves that a Berge graph is always perfect. The main idea behind the proof is that Berge graphs are a very small (yet rich) class of graphs, and a lot more than just perfection can be said about them. It is shown that apart from a few basic classes of graphs that happen to be perfect, Berge graphs enjoy properties that we saw in the preceding section do not hold for minimally imperfect graphs.

As for the basic classes of Berge graphs, we need a definition. We say that a simple graph G is a *double split graph* if $V(G)$ can be partitioned into four parts $\{a_1, \ldots, a_m\}, \{b_1, \ldots, b_m\}, \{c_1, \ldots, c_n\}$ and $\{d_1, \ldots, d_n\}$ for some $m, n \geq 2$ such that

- for each $i \in [m]$, a_i and b_i are adjacent, and for each $j \in [n]$, c_j and d_j are not adjacent,
- for $1 \leq i < i' \leq m$, there are no edges between $\{a_i, b_i\}, \{a_{i'}, b_{i'}\}$, and for $1 \leq j < j' \leq n$, the four edges betwen $\{c_j, d_j\}, \{c_{j'}, d_{j'}\}$ are present,
- for $i \in [m]$ and $j \in [n]$, there are precisely two edges between $\{a_i, b_i\}, \{c_j, d_j\}$, and these two edges have no vertex in common.

Notice that if a graph is a double split graph, then so is its complement. We leave the following as an exercise:

Proposition 5.17. *Double split graphs are perfect.*

Let us say that a simple graph G is *basic* if either

- G or $\overline{G}$ is bipartite,
- G or $\overline{G}$ is the line graph of a bipartite graph, or
- G is a double split graph.

Clearly, if a graph is basic, then so is its complement. Notice that by Corollary 5.5 and Proposition 5.17, basic graphs are perfect, and so they are Berge. The following theorem is the main piece to proving that Berge graphs are perfect:

Theorem 5.18 (Chudnovsky, Robertson, Seymour, Thomas 2006)**.** *Let G be a Berge graph that is not basic. Then either G has a balanced skew partition, or G has a homogeneous pair, or one of $G, \overline{G}$ has a 2-join.*

Combining this result with the results from the previous section, we get the strong perfect graph theorem:

Theorem 5.19. *A graph is perfect if, and only if, it has no odd hole and no odd antihole.*

Proof. Let G be a simple graph. ($\Rightarrow$) If G is perfect, then by Remark 5.10, G has no odd hole and no odd antihole. ($\Leftarrow$) Suppose conversely that G has no odd hole and no odd antihole, that is, G is Berge. Suppose for a contradiction that G is not perfect. We may assume that G is minimally imperfect. Since G is imperfect, it follows that G is not basic. Thus, by Theorem 5.18, either G has a balanced skew partition, or G has a homogeneous pair, or one of $G, \overline{G}$ has a 2-join. It follows from Theorems 5.13 and 5.15 that one of $G, \overline{G}$ has a 2-join. But then Theorem 5.16 implies that G is either an odd hole or an odd antihole, a contradiction as G is Berge. Thus G is perfect. $\square$

As a consequence,

Corollary 5.20. *Every simple graph G satisfies at least one of the following statements:*

- $\chi(G) = \omega(G)$*, or*
- *G has an odd hole or an odd antihole.*

6 Perfect matrices

Let $G = (V, E)$ be a perfect graph. Let A be the $0-1$ matrix whose columns are labeled by V and whose rows are the incidence vectors of the stable sets of G. Take weights $c \in \mathbb{Z}_+^V$. Consider the set packing primal-dual pair

$$(P) \quad \begin{array}{ll} \max & c^\top x \\ \text{s.t.} & Ax \leq \mathbf{1} \\ & x \geq \mathbf{0} \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{ll} \min & \mathbf{1}^\top y \\ \text{s.t.} & A^\top y \geq c \\ & y \geq \mathbf{0}. \end{array}$$

We can rewrite the primal as

$$(P) \quad \begin{array}{ll} \max & \sum (c_v x_v : v \in V) \\ \text{s.t.} & \sum (x_v : v \in S) \leq 1 \quad \forall \text{ stable sets } S \\ & x_v \geq 0 \quad \forall v \in V. \end{array}$$

Observe that a clique gives a feasible solution to this program. So the maximum weight of a clique is a lower-bound on the optimal value of (P). To make this precise, let G_c be the graph obtained from G after replacing each vertex v by c_v duplicates. (If $c_v = 0$ then delete v.) Notice that by Theorem 5.12, G_c is also a perfect graph. Observe that the maximum weight of a clique of G is equal to the maximum cardinality $\omega(G_c)$ of a clique of G_c. Thus, $\omega(G_c)$ is a lower-bound on the optimal value of (P). Let us next rewrite the dual as

$$(D) \quad \begin{array}{ll} \min & \sum (y_S : \text{ stable sets } S) \\ \text{s.t.} & \sum (y_S : \text{ stable sets } S \text{ such that } v \in S) \geq c_v \quad \forall v \in V \\ & y_S \geq 0 \quad \forall \text{ stable sets } S. \end{array}$$

Observe that a covering of $V(G_c)$ using stable sets gives a feasible solution to (D). Thus, the minimum number of stable sets needed to cover $V(G_c)$, which is $\chi(G_c)$, is an upper-bound on the optimal value of (D). Since G_c is perfect, we have $\chi(G_c) = \omega(G_c)$, implying in turn that,

Corollary 6.1. *Let G be a perfect graph. Then the set packing system corresponding to the stable sets of G is totally dual integral. In particular, the set packing polytope*

$$\left\{ x \in \mathbb{R}_+^V : \sum (x_v : v \in S) \leq 1 \quad \forall \textit{ stable sets } S \right\}$$

is integral.

In fact, we will see that these are essentially the *only* examples of integral set packing polytopes and totally dual integral set packing systems! To this end, let A be a $0-1$ matrix without a column of all zeros. We say that A is *perfect* if the set packing polytope $\{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$ is integral.

6.1 Perfection implies total dual integrality

From the discussion in the previous section, it seems more natural to call a matrix perfect when the corresponding set packing system is totally dual integral. The following amazing result justifies our choice of terminology:

Theorem 6.2 (Fulkerson 1972)**.** *Let A be a perfect matrix. Then the linear system $x \geq \mathbf{0}, Ax \leq \mathbf{1}$ is totally dual integral.*

Proof. Denote by E the column labels of A. Consider the set packing primal-dual pair

$$(P) \quad \begin{array}{ll} \max & c^\top x \\ \text{s.t.} & Ax \leq \mathbf{1} \\ & x \geq \mathbf{0} \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{ll} \min & \mathbf{1}^\top y \\ \text{s.t.} & A^\top y \geq c \\ & y \geq \mathbf{0} \end{array} \qquad c \in \mathbb{Z}^E.$$

As A is perfect, (P) has an integral optimal solution for all $c \in \mathbb{Z}^E$. We will prove by induction on the optimal value $\omega \in \mathbb{Z}_+$ of (P) that (D) has an integral dual solution for all $c \in \mathbb{Z}^E$. If $\omega = 0$ for some $c \in \mathbb{Z}^E$, then as A has no column of all zeros, it follows that $c \leq \mathbf{0}$, implying in turn that $\mathbf{0}$ is an optimal solution for (D). For the induction step, assume that $\omega \geq 1$ for some $c \in \mathbb{Z}^E$. Take an arbitrary row a of A such that

$$a^\top x^\star = 1 \qquad \text{for all optimal solutions } x^\star \text{ of (P).}$$

(To find this row, take an optimal dual solution $y^\star$, and pick a so that $y^\star_a > 0$; apply the complementary slackness conditions.) We may assume that a is the first row of A. Consider the set packing primal-dual pair

$$(P') \quad \begin{array}{ll} \max & (c-a)^\top x \\ \text{s.t.} & Ax \leq \mathbf{1} \\ & x \geq \mathbf{0} \end{array} \qquad \text{and} \qquad (D') \quad \begin{array}{ll} \min & \mathbf{1}^\top y \\ \text{s.t.} & A^\top y \geq c - a \\ & y \geq \mathbf{0} \end{array}$$

Clearly, the optimal value of (P') is at most ω, and our choice of a implies that it is exactly $\omega - 1$. Thus, by the induction hypothesis, (D') has an integral optimal solution $\bar{y} = (\bar{y}_1, \bar{z})$ of value $\omega - 1$. Let $y^\star := (\bar{y}_1 + 1, \bar{z})$. Then $y^\star$ is an integral feasible solution for (D) and has value ω, so it is optimal. This completes the induction step. $\square$

6.2 The pluperfect graph theorem

In an attempt to prove it, Ray Fulkerson proposed and proved a polyhedral analogue of the weak perfect graph theorem, and he called it the *pluperfect graph theorem*. To prove his theorem, we will need two ingredients. Let A be a non-negative matrix without a column of all zeros. Let

$$P := \{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}.$$

The *antiblocker of P* is the set

$$a(P) := \{y \geq \mathbf{0} : x^\top y \leq 1 \ \forall x \in P\}.$$

Proposition 6.3. *Let A be a non-negative matrix without a column of all zeros. Let B be the matrix whose rows are the extreme points of $P := \{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$. Then B is non-negative, has no column of all zeros, and*

$$a(P) = \{y \geq \mathbf{0} : By \leq \mathbf{1}\}$$
$$a(a(P)) = P.$$

Proof. Clearly, B is a non-negative matrix. Since A has no column of all zeros, P is a polytope, so every point of P can be written as a convex combination of the rows of B – this has two consequences. Firstly, as $\epsilon\mathbf{1} \in P$ for a sufficiently small $\epsilon > 0$, B cannot have a column of all zeros. Secondly, $\{y \geq \mathbf{0} : By \leq \mathbf{1}\} \subseteq a(P)$. As the reverse inclusion holds trivially, we see that $a(P) = \{y \geq \mathbf{0} : By \leq \mathbf{1}\}$. For the next equation, by definition

$$a(a(P)) = \{x \geq \mathbf{0} : y^\top x \leq 1 \ \ \forall y \in a(P)\}.$$

So clearly, $P \subseteq a(a(P))$. To prove the reverse inclusion, it suffices to show that every row a of A belongs to $a(P)$. Since $a \geq \mathbf{0}$ and $Ba \leq \mathbf{1}$, the result follows. $\square$

Next we study the extreme points of the antiblocker. Let's see an example first. Consider the matrix

$$A := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then the extreme points of $P := \{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$ are the rows of the matrix

$$B := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

By Proposition 6.3, the antiblocker of P is the polytope $a(P) = \{x \geq \mathbf{0} : Bx \leq \mathbf{1}\}$. Aside from the three rows of A, the extreme points of $a(P)$ are $(1\,0\,0), (0\,1\,0), (0\,0\,1), (0\,0\,0)$, which are all orthogonal projections of the rows of A. We will show that this is true in general. Given vectors x, y of the same dimension, if x is obtained from y after setting some of the coordinates to 0, then we say that x is a *projection* of y.

Proposition 6.4. *Let A be a non-negative matrix and let $P := \{x \in \mathbb{R}^n_+ : Ax \leq \mathbf{1}\}$. Then the following statements hold:*

(1) Let $\bar{x}$ be an extreme point of P for which

$$\bar{x} \leq \sum_{i=1}^{k} \lambda_i x^i$$

for some points $x^1, \ldots, x^k \in P$ and scalars $\lambda_1, \ldots, \lambda_k > 0$ with $\sum_{i=1}^k \lambda_i = 1$. Then $\bar{x}$ is a projection of each x^i.

(2) Suppose that A has no column of all zeros. Then every extreme point of $a(P)$ is a (possibly trivial) projection of a row of A.

Proof. **(1)** If $\bar{x} = \mathbf{0}$, then we are done. Otherwise, after possibly rearranging the coordinates, we have $\bar{x} = (\bar{z}, \mathbf{0})$ for some $\ell \geq 1$ and $\bar{z} \in \mathbb{R}^\ell$ such that $\bar{z} > \mathbf{0}$. For each $i \in [k]$, denote by z^i the vector consisting of the first ℓ coordinates of x^i. Then

$$\bar{z} \leq \sum_{i=1}^{k} \lambda_i z^i =: z.$$

Notice that z consists of the first ℓ coordinates of $\sum_{i=1}^{k} \lambda_i x^i$. As $\bar{x}$ is an extreme point of P, there is an $\ell \times \ell$ non-singular submatrix E of A such that $E\bar{z} = \mathbf{1}$. On the one hand, as E is non-negative and $z \geq \bar{z}$, it follows that $Ez \geq E\bar{z} = \mathbf{1}$. On the other hand, as $Ax \leq \mathbf{1}$, it follows that $Ez \leq \mathbf{1}$. Thus, $Ez = E\bar{z} = \mathbf{1}$, implying in turn that $z = \bar{z}$. As a result,

$$\bar{x} = (\bar{z}, \mathbf{0}) = (z, \mathbf{0}) = \sum_{i=1}^{k} \lambda_i (z^i, \mathbf{0}).$$

Since $\bar{x}$ is an extreme point, and each $(z^i, \mathbf{0})$ belongs to P, it follows that $\bar{x} = (z^1, \mathbf{0}) = \cdots = (z^k, \mathbf{0})$, as required.

(2) Denote by B the matrix whose rows are the extreme points of the polytope P. Then by Proposition 6.3, B is a non-negative matrix without a column of all zeros, and $a(P) = \{y \geq \mathbf{0} : By \leq \mathbf{1}\}$. Denote by A' the matrix whose rows are the extreme points of the polytope $a(P)$. Then by Proposition 6.3,

$$\{x \geq \mathbf{0} : Ax \leq \mathbf{1}\} = a(a(P)) = \{x \geq \mathbf{0} : A'x \leq \mathbf{1}\}.$$

Take an extreme point a' of $a(P)$, which is also a row of A'. Since $a'^{\top} x \leq \mathbf{1}$ is valid for $\{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$, it follows that a' is bounded above by a convex combination of the rows of A. Applying (1) to $a(P)$, we see that a' must be a projection of a row of A, as required. $\square$

We are now ready for the pluperfect graph theorem:

Theorem 6.5 (Fulkerson 1972)**.** *Let A be a non-negative matrix without a column of all zeros, and let B be the matrix whose rows are the extreme points of $\{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$. If A is perfect, then so is B.*

Proof. Suppose that A is perfect, that is, A is a $0-1$ matrix whose associated set packing polytope $P := \{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$ is integral. So B is a $0-1$ matrix. By Proposition 6.3, B has no column of all zeros and $a(P) = \{y \geq \mathbf{0} : By \leq \mathbf{1}\}$. Therefore, by Proposition 6.4 (2), every extreme point of $\{y \geq \mathbf{0} : By \leq \mathbf{1}\}$ is a projection of a row of A. In particular, $\{y \geq \mathbf{0} : By \leq \mathbf{1}\}$ is integral, that is, B is perfect. $\square$

6.3 Clutters and antiblockers

Let V be a finite set of *elements*, and let $\mathcal{A}$ be a family of subsets of V, called *members*. We say that $\mathcal{A}$ is a *clutter* over *ground set* V if no member is contained in another one.[3] The *incidence matrix* of $\mathcal{A}$, denoted $M(\mathcal{A})$, is the $0-1$ matrix whose columns are labeled by V and whose rows are the incidence vectors of the members.

Remark 6.6. *Let $\mathcal{A}_1, \mathcal{A}_2$ be clutters over the same ground set, where every member of $\mathcal{A}_1$ contains a member of $\mathcal{A}_2$, and every member of $\mathcal{A}_2$ contains a member of $\mathcal{A}_1$. Then $\mathcal{A}_1 = \mathcal{A}_2$.*

Proof. Take $A_1 \in \mathcal{A}_1$. Then A_1 contains a member A of $\mathcal{A}_2$, and A contains a member of $\mathcal{A}_1$. As $\mathcal{A}_1$ is a clutter, it must be that $A_1 \subseteq A \subseteq A_1$, implying in turn that $A = A_1$. Thus, $\mathcal{A}_1 \subseteq \mathcal{A}_2$. Similarly, $\mathcal{A}_2 \subseteq \mathcal{A}_1$, so $\mathcal{A}_1 = \mathcal{A}_2$. $\square$

[3] Clutters are also referred to as *Sperner families*.

Let $\mathcal{A}$ be a clutter over ground set V, where every element is contained in a member. Consider the set packing polytope associated with $\mathcal{A}$:

$$\left\{x \in \mathbb{R}_+^V : \sum (x_v : v \in A) \leq 1 \ \ \forall A \in \mathcal{A}\right\} = \{x \geq \mathbf{0} : M(\mathcal{A})x \leq \mathbf{1}\}.$$

Notice that the $0-1$ points of $P(\mathcal{A})$ correspond to the sets in

$$\{B \subseteq V : |B \cap A| \leq 1 \ \ \forall A \in \mathcal{A}\},$$

and that every $0-1$ point of the polytope is in fact an extreme point. We say that $\mathcal{A}$ is a *perfect clutter* if the associated set packing polytope is integral, that is, when the associated incidence matrix $M(\mathcal{A})$ is perfect. Notice that an arbitrary $0-1$ matrix A is perfect if, and only if, the clutter corresponding to the maximal rows of A is perfect. As a consequence, studying perfect clutters is just as general as studying perfect matrices.

Let $\mathcal{A}$ be a clutter over ground set V. The maximal sets of $\{B \subseteq V : |B \cap A| \leq 1 \ \ \forall A \in \mathcal{A}\}$ form another clutter over the same ground set, called the *antiblocker of* $\mathcal{A}$ and denoted $a(\mathcal{A})$. If every element is used in a member of $\mathcal{A}$, then the members of $a(\mathcal{A})$ are precisely the maximal integral points contained in the set packing polytope. For instance,

$$\begin{aligned}\text{the antiblocker of } \{\{1,2\},\{2,3\},\{3,1\}\} &= \{\{1\},\{2\},\{3\}\}\\ \text{the antiblocker of } \{\{1\},\{2\},\{3\}\} &= \{\{1,2,3\}\}\\ \text{the antiblocker of } \{\{1,2,3\}\} &= \{\{1\},\{2\},\{3\}\}.\end{aligned}$$

One natural question to ask is, when do we have $a(a(\mathcal{A})) = \mathcal{A}$? Perhaps surprisingly, the answer is very simple:

Proposition 6.7 (Fulkerson 1971)**.** *Let $\mathcal{A}$ be a clutter over ground set V. Then the following statements are equivalent:*

(i) $a(a(\mathcal{A})) = \mathcal{A}$*,*

(ii) $\mathcal{A}$ consists of the maximal stable sets of a graph over vertex set V.

Proof. **(ii)** $\Rightarrow$ **(i)**: Suppose $\mathcal{A}$ consists of the maximal stable sets of $G = (V, E)$. Then a vertex set intersects every stable set at most once if, and only if, it is a clique. This implies that $a(\mathcal{A})$ consists of the maximal cliques of G. Applying the same argument to $\overline{G}$ implies that $a(a(\mathcal{A}))$ consists of the maximal stable sets of G, so $a(a(\mathcal{A})) = \mathcal{A}$. **(i)** $\Rightarrow$ **(ii)**: Suppose $a(a(\mathcal{A})) = \mathcal{A}$. Let G be the graph over vertex set V, where distinct vertices u, v are non-adjacent if there is a member containing both u, v. Clearly, every member of $\mathcal{A}$ is a stable set of G. Conversely, let $S \subseteq V$ be a stable set of G. We claim that

$$(\star) \qquad |S \cap B| \leq 1 \quad \forall B \in a(\mathcal{A}).$$

Suppose otherwise. Then for distinct vertices u, v of G, $\{u, v\} \subseteq S \cap B$. However, as u and v are non-adjacent, $\{u, v\} \subseteq A$ for some member $A \in \mathcal{A}$, but then $\{u, v\} \subseteq A \cap B$, a contradiction as $B \in a(\mathcal{A})$. This proves $(\star)$, implying in turn that S is contained in a member of $a(a(\mathcal{A})) = \mathcal{A}$. Remark 6.6 implies that $\mathcal{A}$ consists of the maximal stable sets of G, as required. $\square$

As a consequence,

Theorem 6.8 (Padberg 1973)**.** *If a clutter is perfect, then its members are the maximal stable sets of a simple graph.*

Proof. Let $\mathcal{A}$ be a perfect clutter over ground set V, and let A be the corresponding incidence matrix. Let B be the matrix whose rows are the extreme points of $P := \{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$, and let $Q := \{y \geq \mathbf{0} : By \leq \mathbf{1}\}$. Then by Proposition 6.3, $a(P) = Q$ and $a(Q) = P$. Moreover, since the clutter $\mathcal{A}$ is perfect, the matrix A is perfect, so by Theorem 6.5, B is a perfect matrix. Let $\mathcal{B}$ be the clutter over ground set V whose members correspond to the maximal rows of B. Notice that $a(\mathcal{A})$ corresponds to the maximal integral extreme points of P, so $a(\mathcal{A}) = \mathcal{B}$. Similarly, $a(\mathcal{B})$ corresponds to the maximal integral extreme points of Q, so $a(\mathcal{B}) = \mathcal{A}$. It therefore follows from Proposition 6.7 that $\mathcal{A}$ consists of the maximal stable sets of a graph, as required. $\square$

In fact, as we will see on Assignment 2, the simple graph above is perfect:

Theorem 6.9 (Chvátal 1975)**.** *Let $G = (V, E)$ be a simple graph. If the clutter of the maximal stable sets of G is perfect, then G is a perfect graph.*

Summarizing the results of this section and the previous one, we get the following characterization of when the set packing polytope is integral:

Corollary 6.10. *The following statements hold:*

(1) Let A be a $0-1$ matrix without a column of all zeros whose set packing polytope $\{x \geq \mathbf{0} : Ax \leq \mathbf{1}\}$ is integral. Then the linear system $x \geq \mathbf{0}, Ax \leq \mathbf{1}$ is totally dual integral, the maximal rows of A correspond to the maximal stable sets of a simple graph, and the graph is perfect.

(2) Let G be a simple graph. Then G is perfect if, and only if, it has no odd hole and no odd antihole.

7 Integral and totally dual integral set covering programs

Let $\mathcal{C}$ be a clutter over ground set E. Consider the set covering polyhedron associated with $\mathcal{C}$:

$$\left\{x \in \mathbb{R}^E_+ : \sum (x_e : e \in C) \geq 1 \quad \forall C \in \mathcal{C}\right\} = \{x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}.$$

A *cover* is a subset of E that intersects every member.[4] Notice that the covers of $\mathcal{C}$ correspond precisely to the $0-1$ points of the associated set covering polyhedron. If a set is a cover then so is every superset of it, so not all covers are interesting.

7.1 Clutters and blockers

Let $\mathcal{C}$ be a clutter over ground set E. The *blocker of* $\mathcal{C}$, denoted $b(\mathcal{C})$, is the clutter over ground set E whose members are the minimal covers of $\mathcal{C}$.[5] Unlike antiblockers,

Theorem 7.1 (Isbell 1958, Edmonds and Fulkerson 1970)**.** *Given a clutter $\mathcal{C}$, we have $b(b(\mathcal{C})) = \mathcal{C}$.*

Proof. Denote by E the ground set of $\mathcal{C}$. We need to show that the minimal covers of $b(\mathcal{C})$ are precisely the members of $\mathcal{C}$. By Remark 6.6, it suffices to show that (a) every member of $\mathcal{C}$ is a cover of $b(\mathcal{C})$, and (b) every minimal cover of $b(\mathcal{C})$ contains a member of $\mathcal{C}$.

(a) Take $C \in \mathcal{C}$. Since $C \cap B \neq \emptyset$ for every $B \in b(\mathcal{C})$, we get that C is a cover of $b(\mathcal{C})$.

(b) Take a minimal cover C' of $b(\mathcal{C})$. Then $E - C'$ cannot contain a member of $b(\mathcal{C})$, so $E - C'$ is not a cover of $\mathcal{C}$, implying in turn that $E - C'$ is disjoint from a member of $\mathcal{C}$. Consequently, C' contains a member of $\mathcal{C}$.

Thus, $b(b(\mathcal{C})) = \mathcal{C}$. $\square$

That is, if $\mathcal{B}$ is the blocker of $\mathcal{C}$, then $\mathcal{C}$ is the blocker of $\mathcal{B}$. Let us see some examples of blocking pairs of clutters:

Remark 7.2. *The following statements hold:*

(1) Let G be a graph and take distinct vertices s, t. Over ground set $E(G)$, the clutter of st-paths and the clutter of minimal st-cuts are blockers.

(2) Let G be a simple graph. Over ground set $V(G)$, the clutter of edges and the clutter of minimal vertex covers are blockers.

(3) Consider the clutter of the triangles of K_4 over ground set $E(K_4)$:

$$Q_6 := \big\{\{1,3,5\}, \{1,4,6\}, \{2,3,6\}, \{2,4,5\}\big\}.$$

Its blocker consists of the triangles, as well as the perfect matchings:

$$b(Q_6) = \big\{\{1,3,5\}, \{1,4,6\}, \{2,3,6\}, \{2,4,5\}, \{1,2\}, \{3,4\}, \{5,6\}\big\}.$$

[4] In the literature, a cover is also referred to as a *hitting set*, a *blocking set*, a *transversal*, etc.

[5] Berge (1989) referred to $b(\mathcal{C})$ as the *transversal of* $\mathcal{C}$ and denoted it $\mathrm{Tr}(\mathcal{C})$.

Proof. **(1)** Let $\mathcal{C}$ be the clutter of st-paths over ground set $E(G)$. Clearly, every st-cut is a cover for $\mathcal{C}$. Let B be a minimal cover of $\mathcal{C}$. By definition, $E(G) - B$ does not contain an st-path of G, implying in turn that in $G \setminus B$ the vertices s, t are disconnected, so $G \setminus B$ has an empty st-cut, implying in turn that B contains an st-cut of G. Thus, $b(\mathcal{C})$ consists of the minimal st-cuts, as required. **(2)** follows from the definition of a vertex cover. **(3)** We leave this as an easy exercise. □

7.2 Packing and covering parameters

To each clutter $\mathcal{C}$, we can associate two dual parameters. A *packing* is a collection of pairwise disjoint members.[6] The *packing number*, denoted $\nu(\mathcal{C})$, is the maximum size of a packing. The *covering number*, denoted $\tau(\mathcal{C})$, is the minimum cardinality of a cover. Since a cover picks up a different element from each member of a packing, we see that

$$\tau(\mathcal{C}) \geq \nu(\mathcal{C}).$$

For instance, for the clutter $\big\{\{1,2\},\{2,3\},\{3,1\}\big\}$, the packing number is 1, while the covering number is 2 – so the two parameters are not always equal. We say that $\mathcal{C}$ *packs* if $\tau(\mathcal{C}) = \nu(\mathcal{C})$.

Proposition 7.3. *The following statements hold:*

(1) Given a graph G with distinct vertices s, t, the clutter of st-paths packs, and the clutter of minimal st-cuts packs.

(2) Given a bipartite simple graph G, the clutter of edges packs, and the clutter of minimal vertex covers packs.

(3) Q_6 does not pack, and $b(Q_6)$ packs.

Proof. **(1)** By Theorem 1.1 (Menger), the maximum number of edge-disjoint st-paths is equal to the minimum cardinality of an st-cut, so the clutter of st-paths packs. Denote by $\mathcal{C}$ the clutter of minimal st-cuts of G. We may assume that G has no empty st-cut, so G has at least one st-path. Notice that $\tau(\mathcal{C})$, the minimum cardinality of an st-path, is simply the distance between s, t. To prove that $\mathcal{C}$ packs, it suffices to exhibit $\tau(\mathcal{C})$ disjoint st-cuts. To this end, for each $i \in [\tau(\mathcal{C})]$, denote by $U_i \subseteq V(G)$ the set of vertices within distance $i-1$ from s. Notice that $s = U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_{\tau(\mathcal{C})} \subseteq V(G) - \{t\}$, and that $\delta(U_1), \delta(U_2), \ldots, \delta(U_{\tau(\mathcal{C})})$ are disjoint st-cuts, as required.[7] **(2)** It follows from Theorem 5.2 (Kőnig) that the maximum cardinality of a matching in G is equal to the minimum cardinality of a vertex cover of G, so the clutter of edges of G packs. We leave it as an easy exercise to prove that the clutter of minimal vertex covers of G packs. **(3)** Q_6 does not pack as $\tau(Q_6) = 2 > 1 = \nu(Q_6)$. On the other hand, $b(Q_6)$ packs as $\tau(b(Q_6)) = 3$ and $b(Q_6)$ has disjoint members $\{1,2\}, \{3,4\}, \{5,6\}$. □

Let $\mathcal{C}$ be a clutter over ground set E. Take non-negative weights $w \in \mathbb{R}^E_+$. A *weighted packing* is a collection of members such that every element e is contained in at most w_e of the members. (Notice that a member may be taken more than once.) Denote by $\nu(\mathcal{C}, w)$ the maximum size of a weighted packing. Given a cover B, its

[6] A packing is also referred to as a *matching*.

[7] We just proved the min-work max-potential theorem of Duffin (1962).

weight is $w(B) := \sum_{e\in B} w_e$. Denote by $\tau(\mathcal{C}, w)$ the minimum weight of a cover. Notice that for weights $\mathbf{1}$, weighted packings are precisely packings and cover weights are precisely cover cardinalities, so $\nu(\mathcal{C}, \mathbf{1}) = \nu(\mathcal{C})$ and $\tau(\mathcal{C}, \mathbf{1}) = \tau(\mathcal{C})$.

Remark 7.4. *Given a clutter $\mathcal{C}$ over ground set E and weights $w \in \mathbb{R}^E_+$,*

$$\tau(\mathcal{C}, w) \geq \nu(\mathcal{C}, w).$$

Proof. Take a cover B and a weighted packing $C_1, \dots, C_k$. Then

$$w(B) = \sum_{e\in B} w_e \geq \sum_{e\in B} |\{i \in [k] : e \in C_i\}| = \sum_{i\in[k]} |\{e \in B : e \in C_i\}| = \sum_{i\in[k]} |B \cap C_i| \geq k.$$

Since this is true for all covers and weighted packings, the inequality $\tau(\mathcal{C}, w) \geq \nu(\mathcal{C}, w)$ follows. $\square$

Consider the associated set covering program

$$\begin{array}{llll} & \min & w^\top x & \\ (P) & \text{s.t.} & \sum (x_e : e \in C) \geq 1 & \forall C \in \mathcal{C} \\ & & x \geq \mathbf{0} & \end{array}$$

As the $0-1$ solutions of (P) are precisely the covers, it follows that $\tau(\mathcal{C}, w)$ computes the optimal value of a $0-1$ solution, and hence an integral solution, to (P). Consider the dual program

$$\begin{array}{llll} & \max & \sum (y_C : C \in \mathcal{C}) & \\ (D) & \text{s.t.} & \sum (y_C : C \in \mathcal{C}, e \in C) \leq w_e & \forall e \in E \\ & & y \geq \mathbf{0} & \end{array}$$

As the integral solutions of (D) are precisely the weighted packings, we get that $\nu(\mathcal{C}, w)$ computes the optimal value of an integral solution to (D). In particular, linear program duality offers an alternate proof of the inequality $\tau(\mathcal{C}, w) \geq \nu(\mathcal{C}, w)$. We will refer to each solution of (D) as a *fractional* weighted packing, and its *value* is the objective value of the solution.

We say that $\mathcal{C}$ is *Mengerian* if for all weights $w \in \mathbb{Z}^E_+$, the minimum weight of a cover is equal to the maximum size of a weighted packing:

$$\tau(\mathcal{C}, w) = \nu(\mathcal{C}, w).$$

The discussion we just had implies that $\mathcal{C}$ is Mengerian if, and only if, the corresponding set covering program (P) is totally dual integral.[8] As we know, total dual integrality is a notion stronger than primal integrality. We say that $\mathcal{C}$ is *ideal* if for all weights $w \in \mathbb{Z}^E_+$, the minimum weight of a cover is equal to the maximum value of a fractional weighted packing. Equivalently, by LP Strong Duality, $\mathcal{C}$ is ideal if for all weights $w \in \mathbb{Z}^E_+$, the set covering program (P) has an integral optimal solution, i.e. the optimal value of (P) is $\tau(\mathcal{C}, w)$. Recall that $\mathcal{C}$ is ideal if, and only if, the set covering polyhedron $\left\{x \in \mathbb{R}^E_+ : M(\mathcal{C})x \geq \mathbf{1}\right\}$ is integral. Studying Mengerian and ideal clutters is just as general as studying integral and totally dual integral set covering systems:

[8] In the literature, the Mengerian property is also referred to as the *max-flow min-cut* property.

Remark 7.5. *Take a $0-1$ matrix A with column labels E. Let $\mathcal{C}$ be the clutter over ground set E whose members correspond to the minimal rows of A. Then the following statements hold:*

- $x \geq \mathbf{0}, Ax \geq \mathbf{1}$ *is totally dual integral if, and only if, $\mathcal{C}$ is Mengerian,*
- $\{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$ *is integral if, and only if, $\mathcal{C}$ is ideal.*

Notice that a Mengerian clutter is always ideal. In contrast to Theorem 6.2 in the set packing case, an ideal clutter is not necessarily Mengerian:

Remark 7.6. *The following statements hold:*

(1) Q_6 is an ideal clutter that is not Mengerian,

(2) $b(Q_6)$ is a Mengerian clutter.

Proof. **(1)** We saw in Assignment 1 that Q_6 is ideal. On the other hand, as Q_6 does not pack, it is not Mengerian. **(2)** We leave this as an exercise. □

This remark also shows that being Mengerian is not closed under taking blockers. However, much like the pluperfect graph theorem – Theorem 6.5 – in the set packing case, being ideal *is* closed under taking blockers.

7.3 The width-length inequality

The following "width-length" inequality is the analogue of the max-max inequality, Theorem 5.6, for set covering polyhedra. Alfred Lehman proved this inequality and wrote it up in 1963, taught it to Ray Fulkerson in 1965 at RAND Corporation, but the result was not published until much later in 1979:

Theorem 7.7 (Lehman 1979)**.** *Let $\mathcal{C}$ be a clutter over ground set E. Then $\mathcal{C}$ is ideal if, and only if, for all $w, \ell \in \mathbb{R}^E_+$,*

$$\min\{w(C) : C \in \mathcal{C}\} \cdot \min\{\ell(B) : B \in b(\mathcal{C})\} \leq w^\top \ell.$$

Proof. Suppose first that $\mathcal{C}$ is ideal. Take $w, \ell \in \mathbb{R}^E_+$. Let $\tau := \tau(\mathcal{C}, \ell) = \min\big\{\ell(B) : B \in b(\mathcal{C})\big\}$. Since $\mathcal{C}$ is ideal, there is a fractional ℓ-weighted packing $y \in \mathbb{R}^{\mathcal{C}}_+$ of value τ:

$$\sum \big(y_C : C \in \mathcal{C}\big) = \tau$$
$$\sum \big(y_C : e \in C \in \mathcal{C}\big) \leq \ell_e \quad \forall e \in E.$$

Now we have

$$\begin{aligned} w^\top \ell = \sum_{e \in E} w_e \ell_e \geq \sum_{e \in E} w_e \Big[\sum \big(y_C : e \in C \in \mathcal{C}\big)\Big] &= \sum_{C \in \mathcal{C}} y_C \cdot w(C) \\ &\geq \min\big\{w(C) : C \in \mathcal{C}\big\} \cdot \sum_{C \in \mathcal{C}} y_C \\ &= \min\big\{w(C) : C \in \mathcal{C}\big\} \cdot \tau \\ &= \min\big\{w(C) : C \in \mathcal{C}\big\} \cdot \min\big\{\ell(B) : B \in b(\mathcal{C})\big\}, \end{aligned}$$

as required. Suppose conversely that the width-length inequality holds for all $w, \ell \in \mathbb{R}^E_+$. We will show that $\mathcal{C}$ is ideal. To this end, take an arbitrary $\ell \in \mathbb{R}^E_+$, and let $x^\star$ be an optimal solution to

$$\begin{array}{lll} \min & \ell^\top x & \\ \text{s.t.} & x(C) \geq 1 & \forall\, C \in \mathcal{C} \\ & x \geq \mathbf{0}. & \end{array}$$

We will show that

$$\ell^\top x^\star = \min\big\{\ell(B) : B \in b(\mathcal{C})\big\},$$

thereby finishing the proof. Well, it is clear that $\leq$ holds above. We will prove that $\geq$ holds as well. By the width-length inequality,

$$\begin{aligned} \ell^\top x^\star &\geq \min\big\{\ell(B) : B \in b(\mathcal{C})\big\} \cdot \min\big\{x^\star(C) : C \in \mathcal{C}\big\} \\ &\geq \min\big\{\ell(B) : B \in b(\mathcal{C})\big\}. \end{aligned}$$

as required. □

As an immediate consequence, we get the following analogue of the pluperfect graph theorem, Theorem 6.5:

Theorem 7.8 (Lehman 1979)**.** *If a clutter is ideal, then so is its blocker.*

7.4 Deletions, contractions and minors

Let $\mathcal{C}$ be a clutter over ground set E, and take an element $e \in E$. We will define two clutters over ground set $E - \{e\}$. The *deletion* is the clutter

$$\mathcal{C} \setminus e := \{C \in \mathcal{C} : e \notin C\}$$

while the *contraction* is the clutter

$$\mathcal{C}/e := \text{ the minimal sets of } \{C - \{e\} : C \in \mathcal{C}\}.$$

Notice that deletion and contraction are blocking operations:

Proposition 7.9. *Let $\mathcal{C}$ be a clutter over ground set E. Then for $e \in E$, $b(\mathcal{C} \setminus e) = b(\mathcal{C})/e$ and $b(\mathcal{C}/e) = b(\mathcal{C}) \setminus e$.*

Proof. Let us first prove that $b(\mathcal{C} \setminus e) = b(\mathcal{C})/e$. If B' is a cover of $\mathcal{C} \setminus e$ then $B' \cup \{e\}$ is a cover of $\mathcal{C}$. So every member of $b(\mathcal{C} \setminus e)$ contains a member of $b(\mathcal{C})/e$. For the reverse inclusion, if B is a cover of $\mathcal{C}$ then $B - \{e\}$ is a cover of $\mathcal{C} \setminus e$. So every member of $b(\mathcal{C})/e$ contains a member of $b(\mathcal{C} \setminus e)$. Remark 6.6 implies that $b(\mathcal{C} \setminus e) = b(\mathcal{C})/e$. To prove the second equation, let us apply the first equation to $b(\mathcal{C})$:

$$b(b(\mathcal{C}) \setminus e) = b(b(\mathcal{C}))/e = \mathcal{C}/e.$$

Taking blockers yields $b(\mathcal{C}) \setminus e = b(\mathcal{C}/e)$, thereby proving the second equation. □

For disjoint subsets $I, J \subseteq E$, the following clutter over ground set $E - (I \cup J)$,

$$\mathcal{C} \setminus I/J := \text{ the minimal sets of } \{C - J : C \in \mathcal{C}, C \cap I = \emptyset\}$$

is a *minor* of $\mathcal{C}$ obtained after deleting I and contracting J. If $I \cup J \neq \emptyset$, then $\mathcal{C} \setminus I/J$ is a *proper* minor. By the proposition above, $b(\mathcal{C} \setminus I/J) = b(\mathcal{C})/I \setminus J$. From an optimization point of view, minors operations are quite natural:

Remark 7.10. *Take a clutter $\mathcal{C}$ over ground set E, and disjoint subsets $I, J \subseteq E$. Then the linear programs*

$$\min\{w^\top x : M(\mathcal{C} \setminus I/J)x \geq \mathbf{1}, x \geq \mathbf{0}\} = \max\{\mathbf{1}^\top y : M(\mathcal{C} \setminus I/J)^\top y \leq w, y \geq \mathbf{0}\}$$

for $w \in \mathbb{R}_+^{E-(I\cup J)}$, are equivalent to the linear programs

$$\min\{w^\top x : M(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}\} = \max\{\mathbf{1}^\top y : M(\mathcal{C})^\top y \leq w, y \geq \mathbf{0}\}$$

for $w \in \mathbb{R}_+^E$ such that $w_e = 0$ for all $e \in I$ and $w_f = +\infty$ for all $f \in J$.

As an immediate consequence,

Remark 7.11 (Seymour 1977)**.** *If a clutter is ideal (resp. Mengerian), then so is any minor of it.*

8 Ideal clutters

We will see two rich classes of ideal clutters that are quite different in nature, suggesting that ideal clutters form a much richer class than perfect clutters. Unfortunately for us, it also suggests that studying general ideal clutters is more complicated than perfect clutters. Indeed, this is confirmed by a negative complexity result on detecting idealness that we will mention at the end of this section.

8.1 Dicuts and dijoins

Let $D = (V, A)$ be a digraph. We say that D is *strongly connected* if for all distinct vertices $s, t \in V$, there is an (s, t)-dipath. Take a non-empty and proper subset U of V. We say that the cut $\delta^+(U)$ is a *dicut* if $\delta^-(U) = \emptyset$; that is, $\delta^+(U)$ is a dicut if it has no in-coming arc; we will refer to U as an *out-shore* of $\delta^+(U)$.

Remark 8.1. *A digraph is strongly connected if, and only if, it has no dicut.*

Proof. Take a digraph $D = (V, A)$. Suppose first that D is strongly connected. Let $\delta^+(U)$ be a cut, and take vertices $t \in U$ and $s \in V - U$. Since there is an (s, t)-dipath, it follows that $\delta^-(U) \neq \emptyset$, implying in turn that $\delta^+(U)$ is not a dicut. Suppose conversely that D is not strongly connected. Then there are distinct vertices s, t without an (s, t)-dipath. Let $\overline{U}$ be the set of all vertices that can be reached from s. Clearly, $s \in \overline{U}$ and $t \notin \overline{U}$, and by construction, $\delta^-(U) = \delta^+(\overline{U}) = \emptyset$, so $\delta^+(U)$ is a dicut. $\square$

Given a digraph, what is the minimum number of arcs whose contraction makes the digraph strongly connected? By the remark above, we can rephrase the question as, what is the covering number of the clutter of dicuts of a digraph? In this section, we will answer this question by showing that in a digraph, the clutter of dicuts packs. To prove this, we will need a coloring lemma.

Let V be a finite set, and let $\mathcal{S}$ be a family of subsets of V (some subsets may be equal). We say that two sets $S, S' \in \mathcal{S}$ are *crossing* if the four sets $S_1 \cap S_2, S_1 - S_2, S_2 - S_1, V - (S_1 \cup S_2)$ are non-empty. Notice that if S_1, S_2 are crossing, then so are $S_1, \overline{S_2}$. We say that $\mathcal{S}$ is *cross-free* if it has no crossing sets, that is, for all $S_1, S_2 \in \mathcal{S}$, either $S_1 \cap S_2 = \emptyset, S_1 \subseteq S_2, S_2 \subseteq S_1$ or $S_1 \cup S_2 = V$. Observe that if $\mathcal{S}$ is cross-free, then so is any family obtained from $\mathcal{S}$ after complementing some sets. We will need the following *dicut coloring lemma*: [9]

Lemma 8.2 (Lucchesi and Younger 1976)**.** *Let $D = (V, A)$ be a digraph, and $\mathcal{F}$ a family of (possibly equal) dicuts whose out-shores form a cross-free family. Take an integer $k \geq 1$. If every arc appears in at most k dicuts of $\mathcal{F}$, then the dicuts of $\mathcal{F}$ can be k-colored so that dicuts of the same color are arc-disjoint.*

Proof. Denote by $\mathcal{S}$ the family of the out-shores of $\mathcal{F}$. By definition, $\mathcal{S}$ is a cross-free family. In particular, if an arc belongs to dicuts $\delta^+(U_1), \delta^+(U_2) \in \mathcal{F}$, then either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$. As a result,

($\star$) given the dicuts of $\mathcal{F}$ containing a fixed arc, their out-shores are nested.

[9] Lucchesi and Younger (1976) called this the *disjunctive partition property*.

This observation is crucial to the proof. Take an arbitrary vertex $r \in V$, and let $\mathcal{S}'$ be obtained from $\mathcal{S}$ after complementing each out-shore containing r. Clearly, $\mathcal{S}'$ is a cross-free family, and as no set contains r, it follows that for all $S_1, S_2 \in \mathcal{S}'$, either $S_1 \cap S_2 = \emptyset$, $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$. That is, $\mathcal{S}'$ is a laminar family. We may therefore represent $\mathcal{S}'$ by an r-arborescence T' whose arcs are in a one-to-one correspondence with the sets of $\mathcal{S}'$. Let T be the directed tree obtained from T' as follows: for every set $S' \in \mathcal{S}'$ obtained by complementing an out-shore of $\mathcal{S}$, flip the arc of T' corresponding to S'. Notice the one-to-one correspondence between the arcs of T and the out-shores of $\mathcal{S}$. Notice further that by $(\star)$, the dicuts of $\mathcal{F}$ containing a fixed arc correspond to a directed path in T of length at most k. Thus, to prove the lemma, it suffices to k-color the arcs of T so that in every directed path of length at most k, the arcs get different colors. To this end, partition the vertices of T into layers $L_0, L_1, L_2, \ldots$ so that each arc of T goes from some layer L_{i+1} to the layer L_i. Color the arcs going from layer L_{i+1} to layer L_i with color $i \pmod{k}$, for each $i \geq 0$. It is then easy to see that the arcs of a directed path of length at most k get different colors, as required. $\square$

Let $D = (V, A)$ be a digraph. A *dijoin of* D is an arc subset B such that D/B is strongly connected. Notice that by Remark 8.1, an arc subset is a dijoin if and only if it intersects every dicut. In other words, the dijoins of D are precisely the covers of the clutter of dicuts. The proof of the following theorem is due to Lovász (1976).

Theorem 8.3 (Lucchesi and Younger 1976)**.** *In a digraph, the maximum number of disjoint dicuts is equal to the minimum cardinality of a dijoin. That is, the clutter of dicuts of a digraph packs.*

Proof. Let $D = (V, A)$ be a digraph. We will prove by induction on $|A| \geq 1$ that the clutter of dicuts packs. The base case $|A| = 1$ is trivial. For the induction step, assume that $|A| \geq 2$. We may assume that the underlying undirected graph of D is connected, and that D is not strongly connected. Let ν be the maximum size of a packing of dicuts. Let us say that an arc is *essential* if it is used in every maximum packing of dicuts.

Claim. *D has an an essential arc.*

Proof of Claim. Suppose otherwise. Then for each arc, we have a packing of ν disjoint dicuts of D excluding the arc. Doing this for every arc of D, we get a family $\mathcal{F}$ such that

> $(\star)$ $\mathcal{F}$ is a family of dicuts of D such that $|\mathcal{F}| = |A| \cdot \nu$, and every arc of D is used in at most $|A| - 1$ dicuts of $\mathcal{F}$.

We will recursively update the family $\mathcal{F}$ so that each intermediate family satisfies $(\star)$, and at the end, the out-shores form a cross-free family. If the out-shores of $\mathcal{F}$ form a cross-free family, then we are done. Otherwise, take dicuts $\delta^+(U_1), \delta^+(U_2) \in \mathcal{F}$ where U_1, U_2 are crossing. Then $\delta^+(U_1 \cap U_2), \delta^+(U_1 \cup U_2)$ are also dicuts such that

$$\delta^+(U_1 \cap U_2) \cap \delta^+(U_1 \cup U_2) \subseteq \delta^+(U_1) \cap \delta^+(U_2) \quad \text{and} \quad \delta^+(U_1 \cap U_2) \cup \delta^+(U_1 \cup U_2) \subseteq \delta^+(U_1) \cup \delta^+(U_2).$$

We update $\mathcal{F}$ by replacing the dicuts $\delta^+(U_1), \delta^+(U_2)$ by the dicuts $\delta^+(U_1 \cap U_2), \delta^+(U_1 \cup U_2)$. The inclusions above imply that $\mathcal{F}$ still satisfies $(\star)$. Since at each iteration, the potential $\sum_{\delta^+(U) \in \mathcal{F}} |U|^2$ strictly increases, we will eventually reach a family $\mathcal{F}$ satisfying $(\star)$ whose out-shores form a cross-free family. Therefore, by the Dicut Coloring Lemma 8.2, we may $(|A|-1)$-color the dicuts of $\mathcal{F}$ so that each color class is a packing of dicuts. One of the color classes has cardinality at least $\frac{|A|\cdot\nu}{|A|-1} > \nu$, implying in turn that D has a packing of $\nu+1$ dicuts, a contradiction. Thus, D has an essential arc. $\Diamond$

Let e be an essential arc of D, and let $C_1, \ldots, C_\nu$ be a maximum packing of dicuts such that $e \in C_\nu$. To complete the induction step, it suffices to exhibit a dijoin of cardinality ν. As e is essential, the dicuts $C_1, \ldots, C_{\nu-1}$ give a maximum packing of dicuts of D/e. Thus, by the induction hypothesis, D/e has a dijoin B' of cardinality $\nu - 1$. Notice that $B' \cup \{e\}$ is a dijoin of D of cardinality ν, as required. This finishes the proof. $\square$

Using this result, we can prove the following:

Corollary 8.4. *The clutter of dicuts of a digraph is Mengerian, and therefore ideal.*

Proof. Let $\mathcal{C}$ be the clutter of dicuts of digraph $D = (V, A)$. To prove that $\mathcal{C}$ is Mengerian, take weights $w \in \mathbb{Z}_+^A$. We need to show that $\tau(\mathcal{C}, w) = \nu(\mathcal{C}, w)$, that is, the minimum weight of a dijoin is equal to the maximum size of a weighted packing of dicuts. Construct a digraph D' starting from D as follows: for each arc e with $w_e = 0$ contract arc e, and for each arc w with $w_e \geq 1$ replace arc e by w_e arcs in series (forming a directed path). Then $\tau(\mathcal{C}, w)$ is equal to the minimum cardinality of a dijoin of D', while $\nu(\mathcal{C}, w)$ is equal to the maximum number of disjoint dicuts of D'. Therefore, Theorem 8.3 implies that $\tau(\mathcal{C}, w) = \nu(\mathcal{C}, w)$, as required. $\square$

Together with Theorem 7.8, this result implies that,

Corollary 8.5. *The clutter of dijoins of a digraph is ideal.*

Schrijver (1980) showed that in contrast to dicuts, the clutter of dijoins is *not* necessarily Mengerian. Nevertheless, Woodall (1978) conjectures that the clutter of dijoins always packs. (Why would Woodall's conjecture not imply that the clutter of dijoins is Mengerian?)

8.2 T-joins and T-cuts

Let $G = (V, E)$ be a graph where loops and parallel edges are allowed; however, loops are thought of as vertexless edges. For an edge subset $J \subseteq E$, denote by $\mathrm{odd}(J) \subseteq V$ the set of vertices incident with an odd number of edges of J – clearly $\mathrm{odd}(J)$ has even cardinality. Notice that

$$\mathrm{odd}(J_1) \triangle \mathrm{odd}(J_2) = \mathrm{odd}(J_1 \triangle J_2) \qquad J_1, J_2 \subseteq E,$$

where $\triangle$ is the symmetric difference operation. A subset $C \subseteq E$ is a *cycle* if $\mathrm{odd}(C) = \emptyset$. Observe that $\emptyset$ and loops are cycles. A *circuit* is a non-empty cycle that does not properly contain another non-empty cycle. We leave the following as an exercise:

Remark 8.6. *Let $G = (V, E)$ be a graph, and take a non-empty subset $C \subseteq E$. The C is a cycle if, and only if, C is a disjoint union of circuits.*

We will use this basic observation without reference. Take a subset $T \subseteq V$ of even cardinality. A *T-join* is an edge subset $J \subseteq E$ such that $\mathrm{odd}(J) = T$. For instance, the $\emptyset$-joins are precisely the cycles, and for distinct vertices $s, t \in V$, every st-path is an $\{s, t\}$-join.

Remark 8.7. *Take a graph $G = (V, E)$, a subset $T \subseteq V$ of even cardinality, and a T-join J. Then*

$$\{J' \subseteq E : J' \text{ is a } T\text{-join}\} = \{J \triangle C : C \text{ is a cycle}\}.$$

Proof. Suppose first that $J' \subseteq E$ is a T-join. Then $\mathrm{odd}(J' \triangle J) = \mathrm{odd}(J') \triangle \mathrm{odd}(J) = T \triangle T = \emptyset$, so $J' \triangle J$ is a cycle, and as $J' = J \triangle (J' \triangle J)$, we are done. Conversely, take a cycle C. Then $\mathrm{odd}(J \triangle C) = \mathrm{odd}(J) \triangle \mathrm{odd}(C) = T \triangle \emptyset = T$, so $J \triangle C$ is a T-join and we are done. $\square$

Given a graph and a vertex subset T of even cardinality, what is the minimum cardinality of a T-join? When $T = \emptyset$, the answer is zero as $\emptyset$ is a T-join. We may therefore focus on non-empty T. The two remarks above provide the following partial answer to this question:

Remark 8.8 (Sebő 1987)**.** *Take a graph $G = (V, E)$, a non-empty subset $T \subseteq V$ of even cardinality, and a T-join J. Define weights $w \in \{-1, 1\}^E$ as follows: for each $e \in J$ set $w_e := -1$, and for each $e \in E - J$ set $w_e := 1$. Then the following statements are equivalent:*

- *J is a minimum T-join,*
- *there is no cycle of total negative weight,*
- *there is no circuit of total negative weight.*

The reason we are not satisfied with this answer is the lack of an optimality certificate. How can we certify that a minimum T-join is truly optimal? Well, if we treat minimal T-joins as the minimal covers of a clutter, and the clutter happened to pack, then any maximum packing would give an optimality certificate.

Take a graph $G = (V, E)$ and a *non-empty* subset $T \subseteq V$ of even cardinality. A *T-cut* is a cut of the form $\delta(U) \subseteq E$ where $|U \cap T|$ is odd. For instance, for distinct vertices s, t of G, an st-cut is an $\{s, t\}$-cut.

Proposition 8.9. *Take a graph $G = (V, E)$ and a non-empty subset $T \subseteq V$ of even cardinality. Let $\mathcal{C}$ be the clutter of minimal T-joins over ground set E. Then $b(\mathcal{C})$ is the clutter of minimal T-cuts.*

Proof. We need to show that (a) every T-cut is a cover of $\mathcal{C}$, and (b) every cover of $\mathcal{C}$ contains a T-cut. **(a)** Take a T-cut $\delta(U)$. We need to show that $\delta(U)$ intersects every T-join. Suppose otherwise. Take a T-join J such that $J \cap \delta(U) = \emptyset$. Then the odd-degree vertices of $J \cap E(G[U])$ are precisely $T \cap U$, a contradiction as $|T \cap U|$ is odd. **(b)** Conversely, let $B \subseteq E$ be a cover of $\mathcal{C}$. Then the graph $H := G \setminus B$ does not contain a T-join. To prove that B contains a T-cut of G, it suffices to argue why H has an empty T-cut. To this end, let A be the vertex-edge incidence matrix of H, and let $b \in \{0,1\}^V$ be the incidence vector of $T \subseteq V$. (So the loops of H are the zero columns of A.) Since H has no T-join, it follows that the system

$$Ax \equiv b \pmod 2$$

has no $0-1$ solution. By Farkas' lemma for binary spaces, there is a certificate $c \in \{0,1\}^V$ such that

$$c^\top A \equiv \mathbf{0} \quad \text{and} \quad c^\top b \equiv 1 \pmod 2.$$

Pick $U \subseteq V$ such that $c = \chi_U$. The second equation implies that $|U \cap T|$ is odd, while the first equation implies that $\delta(U)$ is an empty cut of H, so $\delta(U)$ is an empty T-cut of H, as required. $\square$

Let's see what minors of the clutter of minimal T-joins correspond to in terms of the graph. Let $G = (V, E)$ be a graph and take a possibly empty subset $T \subseteq V$ of even cardinality. Let $\mathcal{C}$ be the clutter of minimal T-joins over ground set E. Take an edge $e \in E$. The *deletion* $(G,T) \setminus e$ is the pair $(G \setminus e, T)$. It is clear that the minimal T-joins of $(G,T) \setminus e$ are the members of $\mathcal{C} \setminus e$. The *contraction* $(G,T)/e$ is the pair $(G/e, T')$ where [10]

$$T' = \begin{cases} T - e & \text{if } |e \cap T| \text{ is even} \\ (T - e) \cup \{\text{shrunk vertex}\} & \text{if } |e \cap T| \text{ is odd.} \end{cases}$$

Observe that T' is a set of even cardinality. Notice that if J is a T-join of G, then $J - \{e\}$ is a T'-join of G/e. Conversely, if J' is a T'-join of G/e, then $J' \cup \{e\}$ contains a T-join of G. Hence, the minimal T'-joins of $(G,T)/e$ are the members of $\mathcal{C}/e$. For disjoint subsets $I, J \subseteq E$, the *minor* $(G,T) \setminus I/J$ is what is obtained after deleting I and contracting J. Notice that the minimal T'-joins of $(G \setminus I/J, T') := (G,T) \setminus I/J$ are the members of $\mathcal{C} \setminus I/J$.

Let's get back to our question regarding minimum T-joins and certifying their optimality by looking at the blocker of minimal T-joins: does the clutter of minimal T-cuts necessarily pack? Consider the complete graph K_4 on 4 vertices, let $T := V(K_4)$, and let $\mathcal{C}$ be its clutter of minimal T-cuts. Then $\mathcal{C}$ consists of the claws of K_4, and the blocker $b(\mathcal{C})$ – the minimal T-joins – consists of the claws as well as the perfect matchings. So $\tau(\mathcal{C}) = 2$, and as there are no disjoint claws, it follows that $\nu(\mathcal{C}) = 1$, so $\mathcal{C}$ does not pack. Despite this shortcoming, we can prove the following result. Our proof is due to Sebő (1987).

Theorem 8.10 (Seymour 1981)**.** *Take a bipartite graph $G = (V, E)$, and a non-empty subset $T \subseteq V$ of even cardinality. Then the minimum cardinality of a T-join is equal to the maximum number of disjoint T-cuts. That is, the clutter of minimal T-cuts of a bipartite graph packs.*

Proof. We proceed by induction on the number of vertices of G. The base case $|V| = 2$ holds trivially. For the induction step, assume that $|V| \geq 3$. Denote by τ the minimum cardinality of a T-join. We will construct τ disjoint T-cuts. If $\tau = 1$, then we are done. We may therefore assume that $\tau \geq 2$. Among all minimum T-joins, pick the one J whose longest path is the longest compared to the other ones. Define weights $w \in \{-1, 1\}^E$ as

[10] In this setting, to contract a loop is to delete it.

follows: for each $e \in J$ set $w_e := -1$, and for each $e \in E - J$ set $w_e := 1$. By Remark 8.8, G has no negative cycle, and as G is bipartite, every cycle has even weight.

Let Q be the longest path contained in J and let u, v be its ends. As Q is the longest path in J, and as G has no negative cycle, it follows that u, v each have degree 1 in J. In particular, $u, v \in \mathrm{odd}(J) = T$. Let $e^\star$ be the edge of Q incident with u. Then $J \cap \delta(u) = \{e^\star\}$.

Claim 1. *If C is a circuit such that $C \cap \delta(u) \neq \emptyset$ and $e^\star \notin C$, then $w(C) \geq 2$.*

Proof of Claim. Suppose otherwise. Since $w(C) \geq 0$ and $w(C)$ is even, it follows that $w(C) = 0$. So $J \triangle C$ is another minimum T-join, and as Q cannot be extended to a longer path in $J \triangle C$, Q and C must share a vertex other than u. Among all the vertices in $V(Q) - \{u\}$ that also belong to $V(C)$, pick the one w that is closest to u on Q. Let Q' be the uw-path in Q; as $e^\star \notin C$, it follows that $Q' \neq \emptyset$ and $Q' \cap C = \emptyset$. Let P_1, P_2 be the two uw-paths partitioning C. Since $w(P_1) + w(P_2) = w(C) = 0$ and $w(Q') < 0$, it follows that one of $P_1 \cup Q', P_2 \cup Q'$ is a negative circuit, a contradiction. ◊

Claim 2. *u cannot be adjacent to all the other vertices in T.*

Proof of Claim. Suppose otherwise. In particular, u and v are adjacent, and as G has no negative cycle, Q has length 1. Since Q is the longest path in J, it follows that J is a matching, and as $\tau \geq 2$, the matching has an edge other than the edge of Q. Since u is adjacent to the other matched vertices, G has a triangle, a contradiction as G is bipartite. ◊

Let $(G', T') := (G, T)/\delta(u)$. Notice that G' is still a bipartite graph, and by Claim 2, $T' \neq \emptyset$. Let $J' := J - \delta(u)$. Then J' is a T'-join of G' of length $\tau - 1$. In fact,

Claim 3. *J' is a minimum T'-join of G'.*

Proof of Claim. Define weights $w' \in \{-1, 1\}^{E(G')}$ on the edges of G' as follows: for each $e \in J'$ set $w'(e) := -1$, and for each $e \in E(G') - J'$ set $w'(e) := 1$. Notice that w' is simply the restriction of w to $E - \delta(u) = E(G')$. To prove that J' is a minimum T'-join of G', it suffices by Remark 8.8 to show that G' does not have a negative circuit. To this end, let C' be a circuit of G', and let C be a circuit of G such that $C' \subseteq C \subseteq C' \cup \delta(u)$. If $C = C'$ or $e^\star \in C$, then $w'(C') = w(C) \geq 0$. Otherwise, $C \cap \delta(u) \neq \emptyset$ and $e^\star \notin C$. It therefore follows from Claim 1 that

$$w'(C') = w(C) - 2 \geq 0,$$

as required. ◊

Thus, by the induction hypothesis, G' has $\tau - 1$ disjoint T-cuts; these are also disjoint T-cuts of G, and together with $\delta(u)$, they give τ disjoint T-cuts in G, thereby completing the induction step. This finishes the proof. □

This result is actually sufficient to guarantee certificates of optimality for minimum T-joins in general graphs:

Theorem 8.11 (Edmonds and Johnson 1970, 1973)**.** *Take a graph $G = (V, E)$ and a non-empty subset $T \subseteq V$ of even cardinality. Denote by $\mathcal{C}$ be the clutter of minimal T-cuts over ground set E. Then the following statements hold:*

(1) For weights $w \in \mathbb{Z}_+^E$ where every cycle has total even weight, the minimum weight of a T-join is equal to the maximum size of a weighted packing of T-cuts:

$$\tau(\mathcal{C}, w) = \nu(\mathcal{C}, w).$$

(2) (Lovász 1975) For arbitrary weights $w \in \mathbb{Z}_+^E$, the minimum weight of a T-join is equal to the maximum value of a half-integral weighted packing of T-cuts:

$$\tau(\mathcal{C}, w) = \max_{2y \in \mathbb{Z}_+^{\mathcal{C}}} \left\{ \mathbf{1}^\top y : \sum (y_C : e \in C \in \mathcal{C}) \leq w_e \ \ \forall e \in E \right\}.$$

(3) The clutter $\mathcal{C}$ of minimal T-cuts is ideal, that is, the polyhedron

$$\left\{ x \geq \mathbf{0} : \sum (x_e : e \in B) \geq 1 \ \ \forall\, T\text{-cuts } B \right\}$$

is integral, and its vertices are the incidence vectors of the minimal T-joins.

Proof. **(1)** If there is a T-join of weight 0, then there is nothing to show. We may therefore assume that the minimum weight of a T-join is non-zero. Let (G', T') be the pair obtained from (G, T) after contracting all edges of weight 0, and for each edge e with $w_e \geq 1$, replacing e by w_e edges in series (the intermediate vertices will not be included in T'). Notice that every cycle C in G corresponds to a cycle in G' of length $w(C)$, and conversely, every cycle C' in G' corresponds to a cycle in G of weight $|C'|$. Thus, since every cycle of G has even weight, it follows that G' is a bipartite graph. Moreover, it is clear that every T-join J in G corresponds to a T'-join in G' of length $w(J)$, and conversely, every T'-join J' in G' corresponds to a T-join in G of weight $|J'|$. In particular, $T' \neq \emptyset$. It therefore follows from Theorem 8.10 that the minimum cardinality of a T'-join in G' is equal to the maximum number of disjoint T'-cuts of G'. As every packing of T'-cuts in G' corresponds to a weighted packing of T-cuts in G, it follows that $\tau(\mathcal{C}, w) = \nu(\mathcal{C}, w)$, as required. **(2)** Take arbitrary weights $w \in \mathbb{Z}_+^E$. It follows from (1) that

$$2\tau(\mathcal{C}, w) = \tau(\mathcal{C}, 2w) = \nu(\mathcal{C}, 2w) = \max_{y \in \mathbb{Z}_+^{\mathcal{C}}} \left\{ \mathbf{1}^\top y : \sum (y_C : e \in C \in \mathcal{C}) \leq 2w_e \ \ \forall e \in E \right\},$$

thereby proving (2). **(3)** follows immediately from (2). □

After applying Theorem 7.8 to part (3), we get the following:

Corollary 8.12. *Take a graph $G = (V, E)$ and a non-empty subset $T \subseteq V$ of even cardinality. Then the clutter of minimal T-joins is ideal. That is, for all weights $w \in \mathbb{Z}_+^E$, the minimum weight of a T-cut is equal to the maximum value of a fractional weighted packing of T-joins.*

Cornuéjols (2001) conjectures that in the above corollary, the minimum weight of T-cut should be equal to the maximum value of a quarter-integral weighted packing of T-joins. In contrast to T-cuts, packing T-joins is a difficult problem. To illustrate this, we need a definition. A 3-*graph* is a connected bridgeless graph $G = (V, E)$ where every vertex has degree 3.

Proposition 8.13. *Let $G = (V, E)$ be a plane 3-graph. Then the following statements are equivalent:*

(i) G has three disjoint perfect matchings, so the clutter of minimal V-joins packs,

(ii) G has two disjoint V-joins,

(iii) G has a 4-face-coloring.

Proof. **(i)** $\Rightarrow$ **(ii)** holds trivially. **(ii)** $\Rightarrow$ **(iii)**: Suppose that G has disjoint minimal V-joins J_1, J_2. Let $G^\star = (V^\star, E)$ be the plane dual of G, and notice that every face of $G^\star$ is a triangle. Notice that the V-cuts of G are in correspondence with the cycles of $G^\star$ bounding an odd number of triangles, implying in turn that the V-cuts of G are in correspondence with the odd cycles of $G^\star$. Since each J_i is a minimal cover of the V-cuts of G, each J_i is also a minimal cover of the odd cycles of $G^\star$, implying in turn that there is a non-empty cut $\delta(U_i), U_i \subseteq V^\star$ of $G^\star$ such that $\delta(U_i) = E - J_i$. Since $J_1 \cap J_2 = \emptyset$, it follows that $U_1 \cap U_2, U_1 \cap \overline{U_2}, \overline{U_1} \cap U_2, \overline{U_1} \cap \overline{U_2}$ are stable sets of $G^\star$, thereby yielding a 4-vertex-coloring of $G^\star$, and hence a 4-face-coloring of G. **(iii)** $\Rightarrow$ **(i)**: Let $h \in \{(0,0),(0,1),(1,0),(1,1)\}^{\{\text{faces}\}}$ be a 4-face-coloring of G. For each edge e, whose neighboring faces are F_1 and F_2, let

$$g(e) := h(F_1) + h(F_2) \pmod 2.$$

Since F_1, F_2 are adjacent faces, and therefore have different colors, it follows that $g(e) \in \{(0,1),(1,0),(1,1)\}$. Let

$$\begin{aligned} J_1 &:= \{e \in E : g(e) = (0,1)\} \\ J_2 &:= \{e \in E : g(e) = (1,0)\} \\ J_3 &:= \{e \in E : g(e) = (1,1)\}. \end{aligned}$$

We claim that each J_i is a perfect matching. To see this, take an arbitrary vertex v, whose neighboring faces are F_1, F_2, F_3. Then the three edges incident with v have g-values $h(F_1)+h(F_2), h(F_2)+h(F_3), h(F_3)+h(F_1) \pmod 2$. As $h(F_1), h(F_2), h(F_3)$ are pairwise distinct, we get that the g-values of the three edges incident with v are different, so v is indicent with exactly one edge from each J_i. As this is true for each vertex, it follows that each J_i is a perfect matching, as required. $\square$

It is widely known that 4-face-coloring plane 3-graphs is just as general as 4-face-coloring arbitrary plane graphs. Thus, the implication (ii) $\Rightarrow$ (iii) implies that finding just two disjoint T-joins in a graph can be a difficult problem. Appel and Haken (1977), and again Robertson, Sanders, Seymour and Thomas (1996), proved that plane graphs are 4-face-colorable. As a consequence, the implication (iii) $\Rightarrow$ (i) implies that,

Theorem 8.14. *The clutter of minimal T-joins of a planar 3-graph packs.*

This result does not extend to non-planar 3-graphs. For instance, the Petersen graph is a 3-graph whose clutter of minimal T-joins does not pack, as it is not 3-edge-colorable.

8.3 Testing idealness is co-NP-complete.

Let A be a $0-1$ matrix. Consider the following problem:

Is A an ideal matrix?

This is a co-NP problem: to certify that A is non-ideal, all we need is a fractional point $x^\star \in Q(A) = \{x \geq \mathbf{0} : Ax \geq \mathbf{1}\}$ along with a full-rank row subsystem $A'x \geq b'$ of $\begin{pmatrix} A \\ I \end{pmatrix} x \geq \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}$ such that $A'x^\star = b'$. In fact, as the following result claims, this problem is one of the most difficut problems in the co-NP class:

Theorem 8.15 (Ding, Feng, Zang 2008)**.** *Let A be a $0-1$ matrix, where every column has exactly two 1s. Then the problem*

Is A an ideal matrix?

is co-NP-complete.

In other words, given a general $0-1$ matrix that is a priori ideal, we cannot convince an adversary in polynomial time that A is indeed an ideal matrix, unless P and co-NP are equal. This means that unlike perfect clutters, ideal clutters do not admit a polynomial characterization in this model. (The authors above proved that "Is A a Mengerian matrix?" is a also co-NP-complete problem.)

9 Minimally non-ideal clutters

By Remark 7.11, we know that if a clutter is ideal, then so is any minor of it. In other words, the class of ideal clutters is minor-closed. As a result, we may indirectly study the class by characterizing the excluded minors defining the class. We say that a clutter is *minimally non-ideal (mni)* if it is non-ideal, and every proper minor of it is ideal. Observe that every mni clutter has at least 3 elements, and that up to isomorphism, the only mni clutter with 3 elements is $\big\{\{1,2\},\{2,3\},\{3,1\}\big\}$.[11] It follows from Remark 7.11 and Theorem 7.8 that,

Remark 9.1. *The following statements hold:*

- *a non-ideal clutter is minimally non-ideal if every single deletion and contraction minor is ideal,*
- *a clutter is ideal if, and only if, it has no minimally non-ideal minor,*
- *if a clutter is minimally non-ideal, then so is its blocker.*

As we will see, mni clutters split into two classes that behave quite differently from one another. We will study each class independently.

9.1 The deltas

Take an integer $n \geq 3$. Consider the clutter over ground set $[n] := \{1,2,3,\dots,n\}$ whose members are

$$\Delta_n := \big\{\{1,2\},\{1,3\},\dots,\{1,n\},\{2,3,\dots,n\}\big\}$$

and whose incidence matrix is

$$M\big(\Delta_n\big) = \begin{pmatrix} 1 & 1 & & & \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \\ & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

We refer to Δ_n, and any clutter isomorphic to it, as a *delta of dimension* n. Notice that the elements and members of Δ_n correspond to the points and lines of a degenerate projective plane.[12]

Theorem 9.2. *Take an integer $n \geq 3$. Then,*

(1) $b(\Delta_n) = \Delta_n$*,*

(2) $\min\{\mathbf{1}^\top x : M(\Delta_n)x \geq \mathbf{1}\}$ *has no integral optimal solution, and*

(3) Δ_n *is minimally non-ideal.*

[11] Given a clutter $\mathcal{C}$, we may obtain another clutter $\mathcal{C}'$ by relabeling the elements of $\mathcal{C}$. We will say that $\mathcal{C},\mathcal{C}'$ are *isomorphic* and write $\mathcal{C} \cong \mathcal{C}'$.

[12] In the literature, a delta of dimension n is called a degenerate projective plane of order $n-1$. However, as there are other degenerate projective planes, we refrain from using this terminology.

Proof. **(1)** As Δ_n does not have disjoint members, every member is also a cover, so every member of Δ_n contains a member of $b(\Delta_n)$. Conversely, let B be a minimal cover of Δ_n. If $1 \notin B$, then as B intersects each one of $\{1,2\},\{1,3\},\dots,\{1,n\}$, it follows that $\{2,3,\dots,n\} \subseteq B$. If $1 \in B$, then as B intersects $\{2,3,\dots,n\}$, it follows that $\{1,i\} \subseteq B$ for some $i \in \{2,3,\dots,n\}$. In both cases, we see that B contains a member, so every member of $b(\Delta_n)$ contains a member of Δ_n. It therefore follows from Remark 6.6 that $b(\Delta_n) = \Delta_n$. **(2)** In particular, $\tau(\mathcal{C}) = 2$. Consider now the fractional feasible solution $x^\star := \begin{pmatrix} \frac{n-2}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \end{pmatrix}$. The objective value of this solution is $1 + \frac{n-2}{n-1} < 2 = \tau(\mathcal{C})$, so (2) holds. **(3)** It follows from (2) that Δ_n is non-ideal. To prove that Δ_n is mni, we need to show for each $e \in [n]$ that $\Delta_n \setminus e$ and Δ_n / e are ideal clutters. In fact, since

$$\Delta_n \setminus e = b(b(\Delta_n \setminus e)) = b(b(\Delta_n)/e) = b(\Delta_n/e)$$

by (1), it suffices by Theorem 7.8 to show that one of $\Delta_n \setminus e, \Delta_n/e$ is ideal. By the symmetry between the elements $2,3,\dots,n$, we may assume that $e \in \{1,n\}$. Observe that

$$\Delta_n \setminus 1 = \{\{2,3,\dots,n\}\}$$

and

$$\Delta_n/n = \{\{1\},\{2,\dots,n-1\}\}.$$

We leave it as an exercise for the reader to see that these clutters are indeed ideal. Thus, Δ_n is mni. □

The deltas form an important class of mni clutters that is tractable in the sense that it is easy to see whether a clutter has a delta minor or not. To see why, we need the following result:

Theorem 9.3 (Abdi, Cornuéjols, Pashkovich 2017)**.** *Take a clutter $\mathcal{C}$ over ground set E and an element $e \in E$. If there are distinct members C_1, C_2, C such that $e \in C_1 \cap C_2$, $e \notin C$ and $(C_1 \cup C_2) - \{e\} \subseteq C$, then $\mathcal{C}$ has a delta minor that can be found in time $O(|E||\mathcal{C}|)$.*

Proof. Let us call (C_1, C_2, C) a *bad triple through* e. We may assume that in every proper minor of $\mathcal{C}$ where e is present, no bad triple through e exists. We will prove that $\mathcal{C}$ itself is a delta. The minimality assumption implies that

(1) $C_1 \cap C_2 = \{e\}$,

because for $I := (C_1 \cap C_2) - \{e\}$, the minor $\mathcal{C}/I$ has the bad triple $(C_1 - I, C_2 - I, C - I)$ through e.

The minimality assumption also implies that

(2) $\{e\} \cup C = E$,

because for $J := E - (\{e\} \cup C)$, $\mathcal{C} \setminus J$ has the same bad triple (C_1, C_2, C) through e.

Next we claim that

(3) $|C_1| = |C_2| = 2$.

To see this, suppose for a contradiction that one of C_1, C_2, say C_1, has cardinality at least 3. Pick an element $h \in C_1 - \{e\}$, and note that by (1), $h \notin C_2$. Consider the minor $\mathcal{C}' := \mathcal{C}/h$, for which $C_1' := C_1 - \{h\}$ and $C' := C - \{h\}$ are still members. Notice that C_2 contains a member C_2' of $\mathcal{C}'$, for which it is easy to see that $e \in C_2'$ and $C_2' \neq \{e\}$. But now $\mathcal{C}'$ has a bad triple (C_1', C_2', C') through e, a contradiction to our minimality assumption. This proves (3).

Now let $X := \{f \in E : \{e, f\} \text{ is a member}\}$. So $|X| \geq 2$ by (3), and $X \subseteq C$ by (2). Our last claim is that

(4) $X = C$.

For if not, pick an element $h \in C - X$, and note that $\mathcal{C}/h$ has a bad triple $(C_1, C_2, C - \{h\})$ through e, contradicting the minimality assumption. Thus, $X = C$. Hence,

$$\mathcal{C} \supseteq \{\{e, f\} : f \in C\} \cup \{C\}.$$

Since $\{e\} \cup C = E$ by (2), and $\mathcal{C}$ is a clutter, equality must hold above, implying in turn that $\mathcal{C}$ indeed is a delta, as required. $\square$

Two elements of a clutter are *exclusive* if they are never used together in a member. Notice that exclusive elements remain exclusive in every minor that they are present in. The preceding result has the following immediate consequence:

Corollary 9.4. *Let $\mathcal{C}$ be a clutter without a delta minor, and take distinct elements e, f, g. If $\{e, f\}, \{e, g\}$ are members, then f, g are exclusive.*

We are now ready to prove the following:

Theorem 9.5 (Abdi, Cornuéjols, Pashkovich 2017)**.** *Let $\mathcal{C}$ be a clutter over ground set E. Then in time $O(|E|^3|\mathcal{C}|^3)$, one can find a delta minor or certify that none exists.*

Proof. We claim that the following statements are equivalent:

(i) $\mathcal{C}$ does not have a delta minor,

(ii) for all distinct members C_1, C_2 with $C_1 \cap C_2 \neq \emptyset$ and for all elements e, f, g with $e \in C_1 \cap C_2, f \in C_1 - C_2, g \in C_2 - C_1$, the following holds: for $X := (C_1 \cup C_2) - \{e, f, g\}$ and $\mathcal{C}' := \mathcal{C}/X$, either $\{e, f\} \notin \mathcal{C}'$ or $\{e, g\} \notin \mathcal{C}'$ or f, g are exclusive elements of $\mathcal{C}'$.

(ii) $\Rightarrow$ **(i)**: Assume that (i) does not hold. Suppose $\mathcal{C}$ has a delta minor obtained after deleting $I \subseteq E$ and contracting $J \subseteq E$. Pick elements $e, f, g \in E - (I \cup J)$ such that $\{e, f\}, \{e, g\}$ are members of the delta minor. Notice that f, g are not exclusive elements in the delta minor, and so they are not exclusive in $\mathcal{C}$. Let C_1, C_2 be members of $\mathcal{C}$ such that $\{e, f\} \subseteq C_1 \subseteq \{e, f\} \cup J$ and $\{e, g\} \subseteq C_2 \subseteq \{e, g\} \cup J$. It can be readily checked that C_1, C_2 and e, f, g do not satisfy (ii). Thus, (ii) does not hold. **(i)** $\Rightarrow$ **(ii)**: Assume that (i) holds. Take $C_1, C_2, e, f, g, X, \mathcal{C}'$ as in (ii) where $\{e, f\} \in \mathcal{C}'$ and $\{e, g\} \in \mathcal{C}'$. Since $\mathcal{C}$ has no delta minor, neither does $\mathcal{C}'$, so by Corollary 9.4, f and g are exclusive elements of $\mathcal{C}'$, so (ii) holds. Hence, (i) and (ii) are equivalent. Since (ii) may be verified in time $O(|E|^3|\mathcal{C}|^3)$, and if (ii) does not hold, a delta minor can be found in time $O(|E||\mathcal{C}|)$ using Theorem 9.3, we can find a delta minor or certify that none exists in time $O(|E|^3|\mathcal{C}|^3)$. $\square$

9.2 The other minimally non-ideal clutters

We now move on to the mni clutters different from the deltas. Take an odd integer $n \geq 5$. Consider the clutter over ground set $[n]$ whose members are

$$\mathcal{C}_n^2 := \{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\}\,.$$

The clutter $\mathcal{C}_n^2$, and any clutter isomorphic to it, is called an *odd hole of dimension* n. It may be readily checked that odd holes are mni. In contrast to Theorem 9.5,

Theorem 9.6 (Ding, Feng, Zang 2008)**.** *Finding an odd hole minor in a clutter is an NP-complete problem.*

That is, unless P and NP are equal, there is no algorithm for finding an odd hole minor in a clutter $\mathcal{C}$ over ground set E, whose running time is polynomial in $|E|$ and $|\mathcal{C}|$. Theorems 9.5 and 9.6 highlight the difference between the deltas and the other mni clutters. There are many mni clutters: other than the two infinite classes $\{\mathcal{C}_{2n-1}^2 : n \geq 3\}$ and $\{b(\mathcal{C}_{2n-1}^2) : n \geq 3\}$, there are at least two other infinite classes of mni clutters different from the deltas, as well as many sporadic examples. For instance, the clutter of the lines of the Fano plane

$$\mathbb{L}_7 = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,7\},\{2,5,6\},\{3,4,6\},\{3,5,7\}\} = b(\mathbb{L}_7),$$

as well as $\mathcal{C}_9^2 \cup \{\{3,6,9\}\}$ are mni. It may now seem that there is no good characterization of the mni clutters different from the deltas, but this is not the case – Alfred Lehman (1990) provided powerful geometric and combinatorial characterizations of these clutters. Before getting to his characterizations, let us briefly study the geometric aspects of ideal clutters and of minor operations. First off, it is easier to work with polytopes rather than polyhedra:

Proposition 9.7. *Take a clutter $\mathcal{C}$ over ground set E. Then $\mathcal{C}$ is ideal if, and only if, $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$ is an integral polytope.*

Proof. Let $Q := \{x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$ and $P := \{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$. If Q is not integral, it has a fractional extreme point $x^\star$, and as $x^\star \leq \mathbf{1}$, it follows that $x^\star$ is also an extreme point of P, so P is not integral. Conversely, assume that P is not integral, and let $x^\star$ be a fractional extreme point. Let

$$I_{x^\star} := \{e \in E : x^\star_e = 1\}.$$

We prove by induction on $|I_{x^\star}| \geq 0$ that Q has a fractional extreme point. If $I_{x^\star} = \emptyset$, then $x^\star$ is also an extreme point of Q, so we are done. For the induction step, we assume that $|I_{x^\star}| \geq 1$. If for each $e \in I$, there is a member C such that $C \cap I_{x^\star} = \{e\}$, then $x^\star$ is an extreme point of Q also, so we are done. Otherwise, for some $f \in I_{x^\star}$ there is no member C such that $C \cap I_{x^\star} = \{f\}$. That is, there is no member C such that $f \in C$ and $x^\star(C) = 1$. Thus, we may strictly decrease the f^{th} coordinate of $x^\star$ until we get another fractional extreme point $\bar{x}$ of P. Clearly, $I_{\bar{x}} = I_{x^\star} - \{f\}$, so by the induction hypothesis, Q has a fractional extreme point. This completes the induction step. $\square$

For a clutter $\mathcal{C}$, denote by $P(\mathcal{C})$ the set covering polytope $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$. Notice that the covers of $\mathcal{C}$ are precisely the integer extreme points of $P(\mathcal{C})$. (Every integer point of $P(\mathcal{C})$ is also an extreme point.) Moreover, the minors of $\mathcal{C}$ have a natural geometric interpretation in terms of $P(\mathcal{C})$:

Remark 9.8. *Let $\mathcal{C}$ be a clutter over ground set E, and take an element $e \in E$. Then the following statements hold:*

- *$P(\mathcal{C} \setminus e)$ is the restriction $P(\mathcal{C}) \cap \{x : x_e = 1\}$ after dropping coordinate x_e.*
- *$P(\mathcal{C}/e)$ is the restriction $P(\mathcal{C}) \cap \{x : x_e = 0\}$ after dropping coordinate x_e.*

We can now dive into Lehman's characterizations. First up is a brilliant lemma that will be very useful. Take an integer $n \geq 2$, and let A be an $n \times n$ matrix with $0-1$ entries and without a row or a column of all ones. We say that A is *cross regular* if whenever $A_{ij} = 0$, the number of ones in column j is equal to the number of ones in row i.

Lemma 9.9 (Lehman 1990)**.** *The following statements hold:*

(1) Take an integer $n \geq 2$, and let A be a $0-1$ $n \times n$ matrix without a row or a column of all ones, and whenever $A_{ij} = 0$, the number of ones in column j is greater than or equal to the number of ones in row i. Then A is cross regular.

(2) Cross regular matrices cannot differ in just one row.

Proof. **(1)** Suppose A is an $n \times n$ matrix. For each row $i \in [n]$ and column $j \in [n]$, let r_i denote the number of ones in row i and let c_j denote the number of ones in column j. Then

$$\sum_{j\in[n]} c_j = \sum_{j\in[n]} \sum_{i\in[n]:A_{ij}=0} \frac{c_j}{n-c_j} \geq \sum_{j\in[n]} \sum_{i\in[n]:A_{ij}=0} \frac{r_i}{n-r_i} = \sum_{i\in[n]} \sum_{j\in[n]:A_{ij}=0} \frac{r_i}{n-r_i} = \sum_{i\in[n]} r_i.$$

As the left- and right-hand side terms are equal, equality must hold throughout, implying in turn that whenever $A_{ij} = 0$, then $r_i = c_j$. Thus, A is cross regular. **(2)** Suppose for a contradiction that $\begin{pmatrix} B \\ a \end{pmatrix}, \begin{pmatrix} B \\ a' \end{pmatrix}$ are cross regular matrices and $a \neq a'$. We may assume that $a_1 = 1$ and $a'_1 = 0$. Since $\begin{pmatrix} B \\ a \end{pmatrix}$ is cross regular, the first column of B has a zero entry, say it is the first entry. Let $k \geq 0$ be the number of ones in the first column of B. Then as $\begin{pmatrix} B \\ a \end{pmatrix}$ is cross regular, the first row of B has $k+1$ ones. However, as $\begin{pmatrix} B \\ a' \end{pmatrix}$ is also cross regular, the first row of B must have k ones, a contradiction. $\square$

Given a full-dimensional polytope $P \subseteq \mathbb{R}^n$ and a vertex $x^\star$, we say that $x^\star$ is *simple* if it belongs to exactly n facets. Recall that if $x^\star$ is simple, then there are exactly n edges emanating from $x^\star$, each of which is defined uniquely by $n-1$ many of the tight facets. As a result, if $x^\star$ is simple, then it has exactly n adjacent vertices. Lehman proved the following geometric characterization of the mni clutters different from the deltas:

Theorem 9.10 (Lehman 1990)**.** *Let $\mathcal{C}$ be a minimally non-ideal clutter over ground set E that is not a delta, and let $n := |E|$. Let $x^\star$ be a fractional extreme point of $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$. Then the following statements hold:*

(1) $\mathbf{0} < x^\star < \mathbf{1}$*,*

(2) $x^\star$ *lies on exactly* n *facets, that correspond to members* $C_1, \dots, C_n \in \mathcal{C}$ *– so* $x^\star$ *is a simple vertex,*

(3) the n *neighbors of* $x^\star$ *are integral vertices, that correspond to covers* $B_1, \dots, B_n$ *labeled so that for distinct* $i, j \in [n]$, $|C_i \cap B_i| > 1$ *and* $|C_i \cap B_j| = 1$*,*

(4) $B_1, \dots, B_n$ *are minimal covers,*

(5) $C_1, \dots, C_n$ *are precisely the minimum cardinality members of* $\mathcal{C}$*,*

(6) $x^\star$ *is the unique fractional extreme point of* $\{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$*,*

(7) there is an integer $d \geq 1$ *such that for each* $i \in [n]$, $|C_i \cap B_i| = 1 + d$*.*

In particular, $x^\star$ *is the unique fractional extreme point of* $\{x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$*.*

Proof. Let $P := P(\mathcal{C}) = \{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$. Then for each element $e \in E$, the clutters $\mathcal{C}/e, \mathcal{C} \setminus e$ are ideal, so the polytopes $P \cap \{x : x_e = 0\}$ and $P \cap \{x : x_e = 1\}$ are integral by Remark 9.8, implying in turn for each fractional extreme point $x^\star$ that $0 < x^\star_e < 1$, so **(1)** holds. (The fact that $\mathcal{C}$ is different from a delta will be first used in Claim 4.)

Claim 1. *Let $x^\star$ be a fractional extreme point of P, and let A be an $n \times n$ non-singular submatrix of $M(\mathcal{C})$ such that $Ax^\star = \mathbf{1}$. Then A is cross regular.*

Proof of Claim. Clearly, A has no all ones row, and since $x^\star$ is the unique solution to $Ax^\star = \mathbf{1}$, A has no all ones column either. To prove that A is cross regular, assume that $A_{11} = 0$. Let C be the member corresponding to the first row of A. By Lemma 9.9 (1), it suffices to show that the number of ones in the first column is greater than or equal to $|C|$. To this end, let $\widehat{x} := (1, x^\star_2, \dots, x^\star_n) \in P \cap \{x : x_1 = 1\}$. Let F be the smallest face of the polytope $P \cap \{x : x_1 = 1\}$ containing $\widehat{x}$. Notice that $a^\top \widehat{x} = 1$ for every row a of A whose first entry is 0. As these rows are linearly independent, and as $\widehat{x}_1 = 1$, it follows that

$$\dim(F) \leq n - \text{ number of 0s in the first column } - 1 = \text{ number of 1s in the first column } - 1.$$

On the other hand, as $P \cap \{x : x_1 = 1\}$ is an integral polytope, F is also an integral polytope, so

$$\widehat{x} = \sum_{i=1}^{k} \lambda_i \chi_{B_i}$$

for some extreme points $\chi_{B_1}, \dots, \chi_{B_k}$ of F and some $\lambda > \mathbf{0}$ such that $\sum_{i=1}^k \lambda_i = 1$. Notice for each $i \in [k]$ that B_i is a cover, and as $\widehat{x}(C) = 1$, we get that $|B_i \cap C| = 1$. Since $\widehat{x} > \mathbf{0}$, each element of C appears in at least one B_i, so the matrix whose rows are the χ_{B_i}'s has rank at least $|C|$, implying in turn that the affine dimension of the χ_{B_i}'s is at least $|C|-1$. As a result,

$$\dim(F) \geq |C| - 1.$$

Putting the last two inequalities gives the desired inequality, as desired. $\Diamond$

Claim 2. *Every fractional extreme point of P is simple, that is, it lies on exactly n facets. Thus,* **(2)** *holds.*

Proof of Claim. Suppose for a contradiction that P has a non-simple fractional extreme point $x^\star$. Let A be an $n \times n$ non-singular submatrix of $M(\mathcal{C})$ such that $Ax^\star = \mathbf{1}$. As $x^\star$ is non-simple, there is another row a' of $M(\mathcal{C})$ such that ${a'}^\top x^\star = 1$. Pick a row a of A such that the matrix A' obtained by replacing a and a' is non-singular. (To find a, write a' as a linear combination of the rows of A, and pick the row a whose coefficient is non-zero.) Then by Claim 1, both A and A' are cross regular, a contradiction to Lemma 9.9 (2) as A and A' differ in exactly one row. $\Diamond$

Claim 3. *P does not have neighboring fractional extreme points. Thus,* **(3)** *holds.*

Proof of Claim. Suppose for a contradiction that P has neighboring fractional extreme points $x^\star, y^\star$. Then there are $n \times n$ non-singular submatrices A, B of $M(\mathcal{C})$ that differ in exactly one row such that $Ax^\star = \mathbf{1} = By^\star$. By Claim 1, both A and B are cross regular, a contradiction to Lemma 9.9 (2). $\Diamond$

Now pick a fractional extreme point $x^\star$ of P. By Claims 2 and 3, $x^\star$ lies on n facets and has precisely n neighbors, all of which are integral. Let $C_1, \ldots, C_n \in \mathcal{C}$ be the members corresponding to the facets $x^\star$ sits on, and let $B_1, \ldots, B_n$ be the covers corresponding to the neighbors of $x^\star$, where our labeling satisfies for $i, j \in [n]$ the following:

$$|C_i \cap B_j| \begin{cases} > 1 & \text{if } i = j \\ = 1 & \text{if } i \neq j. \end{cases}$$

Let A (resp. B) be the $0-1$ matrix whose columns are labeled by E and whose rows are the incidence vectors of $C_1, \ldots, C_n$ (resp. $B_1, \ldots, B_n$). Then the equations above imply that

$$AB^\top = J + \operatorname{diag}\left(|C_1 \cap B_1| - 1, \ldots, |C_n \cap B_n| - 1\right).$$

In particular, $AB^\top$ is non-singular, implying in turn that B is non-singular. Moreover, by Claim 1, A is cross regular. Let G be the bipartite representation of A, where column e and row C are adjacent if $e \notin C$. Since A is cross regular, it follows that adjacent vertices of G have the same degree. In particular, every component of G is regular and so it has the same number of vertices in each bipartition.

Claim 4. *G is connected.*

Proof of Claim. Suppose for a contradiction that G is not connected. Then there exist a partition of the rows of A into non-empty parts X_1, X_2 and a partition of the columns of A into non-empty parts $Y_1, Y_2 \subseteq E$ such that $|X_1| = |Y_1|$, $|X_2| = |Y_2|$, and the (X_2, Y_1) and (X_1, Y_2) blocks of A are submatrices of all ones. If $|Y_1| = 1$

or $|Y_2| = 1$, then A has a row with $n-1$ ones, so $\mathcal{C}$ has a delta minor by Theorem 9.3, implying in turn by minimality that $\mathcal{C}$ is a delta, a contradiction as $\mathcal{C}$ is not a delta. Otherwise, $|X_1| = |Y_1| \geq 2$ and $|X_2| = |Y_2| \geq 2$. As a result, for each $i \in [n]$, $|B_i \cap Y_1| = |B_i \cap Y_2| = 1$, implying in turn that the columns of B corresponding to Y_1 have the same sum as the columns of B corresponding to Y_2, a contradiction as B is non-singular. $\Diamond$

In particular, G is a regular graph, implying in turn that for some integer $r \geq 2$, every row and every column of A has exactly r ones – this has two consequences. Firstly, each B_i is a minimal cover. For if not, then $B_i - \{e\}$ is a cover for some $e \in B_i$, implying in turn that column e of A has at least $n-1$ zero entries, implying in turn that $r \leq 1$, which is not the case. Thus **(4)** holds. Secondly, since A is non-singular, it follows that $x^\star = \left(\frac{1}{r}\ \frac{1}{r}\ \cdots \frac{1}{r}\right)$. As a result, as $x^\star \in P$, every row of $M(\mathcal{C})$ has at least r ones, and as $x^\star$ is simple, every row of $M(\mathcal{C})$ not in A has at least $r+1$ ones, so **(5)** holds. In particular, we cannot run this argument for another fractional extreme point, so $x^\star$ is the unique fractional extreme point of P, so **(6)** holds. Finally, for each $i \in [n]$, let $d_i := |C_i \cap B_i| - 1 \in \{1, \ldots, r-1\}$, and let $D := \mathrm{diag}(d_1, \ldots, d_n)$. Then

$$(n+d_1, n+d_2, \ldots, n+d_n) = \mathbf{1}^\top (J+D) = \mathbf{1}^\top (AB^\top) = (\mathbf{1}^\top A)B^\top = r \cdot (B\mathbf{1})^\top.$$

Since there is at most one multiple of r in $\{n+1, \ldots, n+r-1\}$, it follows that $d := d_1 = d_2 = \cdots = d_n$, implying in turn that **(7)** holds, thereby finishing the proof. $\square$

For an integer $k \geq 1$, a square $0-1$ matrix is *k-regular* if every row and every column has exactly k ones. We will need the following tool:

Theorem 9.11 (Bridges and Ryser 1969)**.** *Take an integer $n \geq 3$, and let A, B be $n \times n$ matrices with $0-1$ entries such that*

$$AB = J + dI$$

for some integer $d \geq 1$. Then A, B are non-singular matrices that commute

$$BA = J + dI,$$

and for some integers $r, s \geq 2$ such that $rs = n + d$, A is r-regular and B is s-regular.

Proof. As $J + dI$ is non-singular, it follows that both A, B are non-singular matrices. In particular, neither A nor B has a zero row or a zero column. We have

$$I = (J + dI)\left(\frac{1}{d}I - \frac{1}{d(n+d)}J\right) = (AB)\left(\frac{1}{d}I - \frac{1}{d(n+d)}J\right) = A\left(\frac{1}{d}B - \frac{1}{d(n+d)}BJ\right),$$

so A and $\frac{1}{d}B - \frac{1}{d(n+d)}BJ$ are inverses of one another. Thus,

$$I = \left(\frac{1}{d}B - \frac{1}{d(n+d)}BJ\right)A = \frac{1}{d}BA - \frac{1}{d(n+d)}(B\mathbf{1})(A^\top \mathbf{1})^\top,$$

so

$$BA = \frac{1}{n+d}(B\mathbf{1})(A^\top \mathbf{1})^\top + dI.$$

For each $i \in [n]$, denote by $s_i \in \{1, 2, \ldots, n\}$ the number of ones in row i of B, and by $r_i \in \{1, 2, \ldots, n\}$ the number of ones in column i of A. Then the previous equation implies that

(1) for all $i, j \in [n]$, $n + d \mid s_i r_j$.

As $\mathrm{trace}(AB) = \mathrm{trace}(BA)$, it follows that

$$n + nd = \frac{1}{n+d}\sum_{i=1}^{n} s_i r_i + nd,$$

so

$$n(n+d) = \sum_{i=1}^{n} s_i r_i \geq n(n+d),$$

implying in turn that

(2) for each $i \in [n]$, $n + d = s_i r_i$.

(1) and (2) imply that $r := r_1 = r_2 = \cdots = r_n$ and $s := s_1 = s_2 = \cdots = s_n$. As a consequence,

$$BA = \frac{1}{n+d}(B\mathbf{1})(A^\top \mathbf{1})^\top + dI = J + dI = AB.$$

Analyzing the equation $AB = J + dI$, we proved that every row of B has the same s number of ones, and every column of A has the same r number of ones. The same argument on the equation $BA = J + dI$ implies that every row of A has the same number of ones, and the number inevitably has to be r, while every column of B has the same number of ones, and the number inevitably has to be s. In particular, A is r-regular and B is s-regular. As $rs = n + d$ and $r, s < n + d$, it follows that $r, s \geq 2$, thereby finishing the proof. $\square$

We are now ready for Lehman's combinatorial characterization of the mni clutters different from the deltas:

Theorem 9.12 (Lehman 1990). *Suppose $\mathcal{C}$ is a minimally non-ideal clutter over ground set E that is not a delta, and let $\mathcal{B} := b(\mathcal{C})$. Denote by $\overline{\mathcal{C}}, \overline{\mathcal{B}}$ the clutters over ground set E of the minimum cardinality members of $\mathcal{C}, \mathcal{B}$, respectively. Then*

(1) $M(\overline{\mathcal{C}})$ and $M(\overline{\mathcal{B}})$ are square and non-singular matrices,

(2) for some integers $r \geq 2$ and $s \geq 2$, $M(\overline{\mathcal{C}})$ is r-regular and $M(\overline{\mathcal{B}})$ is s-regular,

(3) for $n := |E|$, $rs \geq n + 1$,

(4) after possibly permuting the rows of $M(\overline{\mathcal{B}})$, we have

$$M(\overline{\mathcal{C}})M(\overline{\mathcal{B}})^\top = J + (rs - n)I = M(\overline{\mathcal{B}})^\top M(\overline{\mathcal{C}}),$$

that is, there is a labeling $C_1, \ldots, C_n$ of the members of $\overline{\mathcal{C}}$ and a labeling $B_1, \ldots, B_n$ of the members of $\overline{\mathcal{B}}$ such that for all $i, j \in [n]$,

$$|C_i \cap B_j| = \begin{cases} rs - n + 1 & \text{if } i = j \\ 1 & \text{if } i \neq j, \end{cases}$$

and for all elements $g, h \in E(\mathcal{C})$,

$$\big|\{i \in [n] : g \in C_i, h \in B_i\}\big| = \begin{cases} rs - n + 1 & \text{if } g = h \\ 1 & \text{if } g \neq h. \end{cases}$$

Proof. Let $x^\star \in [0,1]^E$ be a fractional extreme point of $P(\mathcal{C})$. After applying Theorem 9.10 to the mni clutter $\mathcal{C}$, we get the following implications. The point $x^\star \in [0,1]^E$ is the unique fractional extreme point of $P(\mathcal{C})$, $\mathbf{1} > x^\star > \mathbf{0}$ and $x^\star$ is simple. Let A be the submatrix of $M(\mathcal{C})$ such that $Ax^\star = \mathbf{1}$. We have that $A = M(\overline{\mathcal{C}})$. Let $B_1, \ldots, B_n$ be the minimal covers that correspond to the neighbors of $x^\star$, and let B be the matrix whose rows are the incidence vectors of $B_1, \ldots, B_n$. Then after possibly permuting the rows of B, $AB^\top = J + dI$ for some integer $d \geq 1$.

It now follows from Theorem 9.11 that A, B are non-singular matrices such that $AB^\top = J + dI = B^\top A$, and for some integers $r, s \geq 2$ such that $rs = n + d$, A is r-regular and B is s-regular. To finish the proof, it remains to show that $B = M(\overline{\mathcal{B}})$. To this end, notice that $x^\star$ is equal to $(\frac{1}{r} \ \cdots \ \frac{1}{r})$, and the neighbors of $x^\star$ lie on the hyperplane $\sum_{i=1}^n x_i = s$. Therefore, the inequality $\sum_{i=1}^n x_i \geq s$ is valid for all the integer extreme points of P, implying in turn that every member of $\mathcal{B}$ has cardinality at least s. As a result, $(\frac{1}{s} \ \cdots \ \frac{1}{s})$ is a fractional extreme point of $P(\mathcal{B})$. Applying Theorem 9.10 to the mni clutter $\mathcal{B}$, we see that $(\frac{1}{s} \ \cdots \ \frac{1}{s})$ must be the unique fractional extreme point of $P(\mathcal{B})$ and $B = M(\overline{\mathcal{B}})$, as required. □

9.3 Immediate applications

The first application of Theorem 9.12 is that the deltas (with the exception of Δ_3) are the only mni clutters requiring unequal weights to violate the width-length inequality. The following application is the true analogue of the max-max inequality, Theorem 5.6:

Theorem 9.13. *A clutter without a delta minor is ideal if, and only if, for each minor $\mathcal{C}$,*

$$\min\{|C| : C \in \mathcal{C}\} \cdot \min\{|B| : B \in b(\mathcal{C})\} \leq |E(\mathcal{C})|.$$

Proof. If the clutter is ideal, then the inequality follows from the width-length inequality of Theorem 7.7. Conversely, it suffices to prove that for an mni clutter $\mathcal{C}$ that is not a delta,

$$\min\{|C| : C \in \mathcal{C}\} \cdot \min\{|B| : B \in b(\mathcal{C})\} > |E(\mathcal{C})|.$$

Let n, r, s be the parameters as in Theorem 9.12. Then the inequality $rs \geq n + 1$ implies the inequality above, as required. □

(Notice that the theorem can be extended to clutters without a minor in $\{\Delta_n : n \geq 4\}$.) A second application of Theorem 9.12 is the following truly remarkable result that, to test integrality of an n-dimensional set covering polyhedron, it is sufficient to test just 3^n directions:

Theorem 9.14. *If $\mathcal{C}$ is a minimally non-ideal clutter, then*

$$\min\{\mathbf{1}^\top x : M(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}\}$$

has no integral optimal solution. As a consequence, if $\mathcal{C}$ is a non-ideal clutter over ground set E, then there exists $w \in \{0, 1, +\infty\}^E$ such that

$$\min\{w^\top x : M(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}\}$$

has no integral optimal solution.

Proof. If $\mathcal{C}$ is a delta, then the result follows from Theorem 9.2 (2). Otherwise, $\mathcal{C}$ is not a delta, and let n, r, s be as in Theorem 9.12. As every member has cardinality at least r, it follows that $x^\star := \left(\frac{1}{r}\ \frac{1}{r}\ \cdots \frac{1}{r}\right)$ is a feasible solution, and its objective value is $\frac{n}{r} \leq \frac{rs-1}{r} < s$. However, the minimum cardinality of a cover is s, so $\min\{\mathbf{1}^\top x : M(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}\}$ has no integral optimal solution. The second part follows from the first part after applying Remark 7.10. $\square$

A clutter $\mathcal{C}$ *fractionally packs* if it has a fractional packing of value $\tau(\mathcal{C})$. It follows from the preceding theorem that an mni clutter does not fractionally pack. Thus,

Theorem 9.15. *A clutter is ideal if, and only if, every minor fractionally packs.*

We say that a clutter has the *packing property* if every minor packs. An immediate consequence of the preceding theorem is that,

Corollary 9.16. *If a clutter has the packing property, then it is ideal.*

Conforti and Cornuéjols (1993) conjecture that if a clutter has the packing property, then it must be Mengerian.

10 Weakly bipartite graphs

Let $G = (V, E)$ be a graph. A subset $F \subseteq E$ is *bipartite* if the vertices can be bicolored so that every edge of F gets both colors. Let P be the convex hull of the incidence vectors

$$\{\chi_F : F \text{ is bipartite}\} \subseteq \{0,1\}^E.$$

Notice that the polytope P carries information about the cuts of G. For instance, for $w \in \mathbb{R}^E_+$, the optimization problem $\max\{w^\top x : x \in P\}$ seeks the maximum weight of a cut of G. As the latter is a fundamental NP-complete optimization problem, studying the polytope P is certainly worthwhile. We will be after a polyhedral description of P. Observe that an edge subset is bipartite if, and only if, it contains no odd-length circuit. As a result,

$$P \subseteq \left\{x \in [0,1]^E : \sum (x_e : e \in C) \le |C| - 1 \;\; \forall \text{ odd-length circuits } C\right\}.$$

Observe that equality holds above if, and only if, the polytope on the right is integral. Following Grötschel and Pulleyblank (1981), a graph $G = (V, E)$ is *weakly bipartite* if the polytope

$$\left\{x \in [0,1]^E : \sum (x_e : e \in C) \le |C| - 1 \;\; \forall \text{ odd-length circuits } C\right\}$$

is integral. After a change of variables $x \mapsto \mathbf{1} - x$, we see that G is weakly bipartite if, and only if, the set covering polytope

$$\left\{x \in [0,1]^E : \sum (x_e : e \in C) \ge 1 \;\; \forall \text{ odd-length circuits } C\right\}$$

is integral. Hence, by Proposition 9.7, a graph is weakly bipartite if, and only if, its clutter of odd-length circuits is ideal. Bipartite graphs are vacuously weakly bipartite. A non-trivial example is provided below:

Theorem 10.1 (Hadlock 1975, Barahona 1980)**.** *Planar graphs are weakly bipartite.*

Proof. Let $G = (V, E)$ be a plane graph. Notice that

> ($\star$) every circuit has an inside and an outside; the circuit can be written as the symmetric difference of the facial circuits that are inside (or outside); the circuit is odd-length if and only if the number of odd-length facial circuits used in the sum is odd.

Consider now the plane dual $G^\star = (V^\star, E)$, and let $T \subseteq V^\star$ denote the odd-degree vertices. Observe that T is simply the odd-length facial circuits of G. Notice that the cycles of G are the cuts of $G^\star$, and so the circuits of G are the minimal cuts of $G^\star$. Moreover, it follows from ($\star$) that the odd-length circuits of G are the minimal T-cuts of $G^\star$. As the clutter of minimal T-cuts of $G^\star$ is ideal by Theorem 8.11 (3), it follows that the clutter of odd-length circuits of G is ideal, so G is weakly bipartite. □

Thus the class of weakly bipartite graphs is quite rich. Let us analyze the two non-planar graphs K_5 and $K_{3,3}$. As $K_{3,3}$ is bipartite, it is also weakly bipartite. K_5 however is not weakly bipartite. To see this, let us look at the set covering polytope associated with the odd-length circuits of K_5:

$$\left\{x \in [0,1]^{E(K_5)} : \sum (x_e : e \in C) \ge 1 \;\; \forall \text{ odd-length circuits } C \text{ of } K_5\right\}.$$

Observe that K_5 has precisely 10 triangles, whose incidence vectors are linearly independent in $\mathbb{R}^{E(K_5)} \cong \mathbb{R}^{10}$, and that the other odd-length circuits all have length 5. As a result, the fractional point $(\frac{1}{3}\ \frac{1}{3}\ \cdots\ \frac{1}{3}) \in \mathbb{R}^{E(K_5)}$ belongs to the polytope and is an extreme point. Consequently, the clutter of odd-length circuits of K_5 is non-ideal, so K_5 is not weakly bipartite.

What are the weakly bipartite graphs? Whatever the class is, it must contain both bipartite and planar graphs. These rich classes suggest that a complete classification of the weakly bipartite graphs is a difficult problem, and indeed, this is still an open problem. We may however take another approach. The question we asked may be posed as, when is the clutter of odd-length circuits of a graph ideal? The advantage with this question is that idealness is a minor-closed property, so we may look for an excluded minor characterization. Let $G = (V, E)$ be a graph, and let $\mathcal{C}$ be its clutter of odd-length circuits. Take an edge $e \in E$. What do the minors $\mathcal{C} \setminus e, \mathcal{C}/e$ correspond to in terms of G? Recall that $\mathcal{C} \setminus e$ is the clutter of odd-length circuits of G avoiding e, so it is the clutter of odd-length circuits of $G \setminus e$. However,

$$\mathcal{C}/e = \text{ the minimal sets of } \{C - \{e\} : C \text{ is an odd-length circuit of } G\}$$

is *not* the clutter of odd-length circuits of G/e. (For instance, we could have that G is bipartite while G/e is non-bipartite.) It is not clear what $\mathcal{C}/e$ corresponds to in terms of the graph G. To make sense of this, we will need to change our framework.

10.1 Signed graphs

Let $G = (V, E)$ be a graph, and take a subset $\Sigma \subseteq E$. The pair (G, Σ) is called a *signed graph*. In (G, Σ), an *odd cycle* is a cycle $C \subseteq E$ such that $|C \cap \Sigma|$ is odd, and an *even cycle* is a cycle $C \subseteq E$ such that $|C \cap \Sigma|$ is even. Observe that for sets $C_1, C_2 \subseteq E$ we have

$$(C_1 \triangle C_2) \cap \Sigma = (C_1 \cap \Sigma) \triangle (C_2 \cap \Sigma).$$

In particular, if C_1, C_2 are cycles of parities $p_1, p_2 \in \{0, 1\}$, then $C_1 \triangle C_2$ is a cycle of parity $p_1 + p_2 \pmod 2$. In (G, Σ), an *odd circuit* is a circuit $C \subseteq E$ such that $|C \cap \Sigma|$ is odd, and an *even circuit* is a circuit $C \subseteq E$ such that $|C \cap \Sigma|$ is even. We leave the following as an exercise:

Remark 10.2. *Let (G, Σ) be a signed graph, and take a subset $C \subseteq E(G)$. The following statements are equivalent:*

- *C is a even cycle,*
- *C is a disjoint union of circuits, an even number of which are odd circuits,*

and the following statements are equivalent:

- *C is an odd cycle,*
- *C is a disjoint union of circuits, an odd number of which are odd circuits.*

We will use this useful observation without reference. To *resign* (G, Σ) is to replace it by the signed graph $(G, \Sigma \triangle \delta(U))$ for some $U \subseteq V$.

Remark 10.3. *Resigning preserves the parity of a cycle.*

Proof. Let (G, Σ) be a signed graph, and let $(G, \Sigma \triangle \delta(U))$ be a resigning. Let C be a cycle. As $|C \cap \delta(U)|$ is even, it follows that

$$|C \cap (\Sigma \triangle \delta(U))| = |(C \cap \Sigma) \triangle (C \cap \delta(U))| \equiv |C \cap \Sigma| + |C \cap \delta(U)| \equiv |C \cap \Sigma| \pmod 2.$$

Thus, C has the same parity in both (G, Σ) and $(G, \Sigma \triangle \delta(U))$, thereby finishing the proof. □

A *signature* of (G, Σ) is any set of the form $\Sigma \triangle \delta(U), U \subseteq V$.

Proposition 10.4 (Zaslavsky 1982)**.** *If (G, Σ) has no odd cycle, then $\emptyset$ is a signature.*

Proof. Let A be the $0-1$ matrix whose columns are labeled by the edges, and whose first $|V|$ many rows are the incidence vectors of $\delta(v), v \in V$ and whose last row is the incidence vector of Σ. Let b be the column vector whose first $|V|$ many coordinates are 0 and whose last entry is 1. As (G, Σ) has no odd cycle, the system

$$Ax \equiv b \pmod 2$$

has no $0-1$ solution. By Farkas' lemma for binary spaces, there is a certificate $c \in \{0,1\}^V \times \{0,1\}$ such that

$$c^\top A \equiv \mathbf{0} \quad \text{and} \quad c^\top b \equiv 1 \pmod 2.$$

The second equation implies that the last entry of c is 1. Pick $U \subseteq V$ such that $c = (\chi_U \; 1)$. Then the first equation implies that $\Sigma = \delta(U)$, so $\emptyset = \Sigma \triangle \delta(U)$ is a signature. □

As a consequence,

Theorem 10.5. *For a signed graph, the clutter of odd circuits and the clutter of minimal signatures are blockers.*

Proof. Let $\mathcal{C}$ be the clutter of odd circuits of (G, Σ). By Remark 10.3, every minimal signature intersects every odd circuit in an odd number of edges, so every minimal signature is a cover of $\mathcal{C}$. Conversely, let B be a minimal cover of $\mathcal{C}$. Then the signed graph $(G \setminus B, \Sigma - B)$ has no odd circuit by definition, implying in turn that it has no odd cycle. It therefore follows from Proposition 10.4 that $\Sigma - B = \delta_{G \setminus B}(U)$ for some $U \subseteq V$. Then $\Sigma \triangle \delta(U) \subseteq B$, so B contains a signature of (G, Σ). It follows Remark 6.6 that $b(\mathcal{C})$ is the clutter of minimal signatures, as required. □

Take disjoint edge subsets I, J of (G, Σ). By Theorem 10.5, J does not contain an odd cycle if, and only if, there is a signature disjoint from J. Let

$$(G, \Sigma) \setminus I/J := \begin{cases} (G \setminus I/J, \emptyset) & \text{if } J \text{ contains an odd cycle,} \\ (G \setminus I/J, B - I) & B \text{ is a signature disjoint from } J. \end{cases}$$

We refer to $(G,\Sigma)\setminus I/J$ as a *minor* of (G,Σ) obtained after deleting I and contracting J.[13] Observe that $(G,\Sigma)\setminus I/J$ is defined up to resigning. In contrast to the unsigned graph case, we have the following:

Proposition 10.6. *Let (G,Σ) be a signed graph and $\mathcal{C}$ the clutter of its odd circuits. Take disjoint edge subsets I,J such that J does not contain an odd cycle. Then $\mathcal{C}\setminus I/J$ is the clutter of odd circuits of $(G,\Sigma)\setminus I/J$.*

Proof. Let B be a signature of (G,Σ) disjoint from J. Then $(G\setminus I/J, B-I)=(G,\Sigma)\setminus I/J$. By Remark 6.6, it suffices to show that every odd circuit of $(G\setminus I/J, B-I)$ contains a member of $\mathcal{C}\setminus I/J$, and every member of $\mathcal{C}\setminus I/J$ contains an odd circuit of $(G\setminus I/J, B-I)$.

Let C' be an odd circuit of $(G\setminus I/J, B-I)$. Then there is a circuit C of (G,Σ) such that $C'\subseteq C\subseteq C'\cup J$. As B is a signature of (G,Σ) disjoint from J, it follows that $B\cap C=B\cap C'=(B-I)\cap C'$, so $|B\cap C|$ is odd, implying in turn that C is an odd circuit of (G,Σ). As $C-J$ contains a member of $\mathcal{C}\setminus I/J$, it follows that C' contains a member of $\mathcal{C}\setminus I/J$.

Conversely, let C be an odd circuit of (G,Σ) such that $C\cap I=\emptyset$. Then $C-J$ is a cycle of $(G\setminus I/J,B-I)$. Since $|C\cap B|$ is odd, we get that $|(C-J)\cap(B-I)|$ is odd, so $C-J$ is an odd cycle of $(G\setminus I/J,B-I)$, so it contains an odd circuit of $(G\setminus I/J,B-I)$, as required. $\square$

We say that a signed graph is *weakly bipartite* if its clutter of odd circuits is ideal. Observe that a graph $G=(V,E)$ is weakly bipartite if, and only if, the signed graph (G,E) is weakly bipartite. Hence, as the graph K_5 is not weakly bipartite, it follows that the signed graph $(K_5,E(K_5))$ is not weakly bipartite. We will refer to the signed graph $(K_5,E(K_5))$ as an *odd-K_5*. It follows from Remark 7.11 and Proposition 10.6 that,

Remark 10.7. *If a signed graph is weakly bipartite, then it has no odd-K_5 minor.*

Seymour (1977) conjectured that the converse of this remark also holds. Over 20 years later, in his PhD thesis, Guenin (2001) proved this conjecture! His proof made a spectacular use of Lehman's powerful result, Theorem 9.12. To prove the conjecture, we will need a lemma due to Schrijver (2002).

10.2 The whirlpool lemma and pseudo-odd-K_5's

Let G be a graph, and take a cut B. As $E(G)-B=E(G)\triangle B$ is a signature for $(G,E(G))$, it follows that

$$(G,E(G))/B=(G/B,E(G/B)).$$

This observation will be useful throughout the rest of this section. The signed graph $(K_4,E(K_4))$ is called an *odd-K_4*. Schrijver (2002) found a very nice way to find an odd-K_4 minor in a signed graph. To explain his method, let W be the graph on vertices $0,1,1',2,2',3,3'$ and edges $\{0,1\},\{0,2\},\{0,3\},\{1',2'\},\{2',3'\},\{3',1'\},\{1,2'\},\{2,3'\},\{3,1'\}$. We will refer to the signed graph $(W,E(W))$ as a *whirlpool* with *central edges* $\{0,1\},\{0,2\}$, $\{0,3\}$ – see Figure 1. Observe that a whirlpool has an odd-K_4 minor using its central edges, obtained after contracting the cut $\delta(\{0,1,2,3\})$.

[13] In this setting, to contract a loop is to delete it.

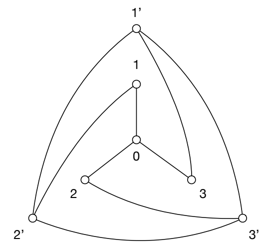

Figure 1: The whirlpool with central edges $\{0,1\},\{0,2\},\{0,3\}$. Every edge is odd.

Lemma 10.8 (Schrijver 2002)**.** *Take a graph $G=(V,E)$. Suppose that there are disjoint stable sets S_1,S_2,S_3 and distinct vertices $0,1,2,3$ such that*

- $0 \in V-(S_1\cup S_2\cup S_3)$ *and* $i\in S_i$ *for each* $i\in[3]$,
- $\{0,i\}\in E$ *for each* $i\in[3]$,
- *for distinct $i,j\in[3]$, there is an ij-path contained in $G[S_i\cup S_j]$.*

Then $(G,E(G))$ has an odd-K_4 minor using the three edges $\{0,1\},\{0,2\},\{0,3\}$.

Proof. We prove this by induction on $|V|+|E|\geq 10$. The base case $|V|+|E|=10$ is true as $(G,E(G))$ itself is an odd-K_4. For the induction step, assume that $|V|+|E|\geq 11$. For distinct $i,j\in[3]$, let $P_{ij}\subseteq E$ be an ij-path contained in $G[S_i\cup S_j]$. We may assume that $V=\{0\}\cup V(P_{12})\cup V(P_{23})\cup V(P_{31})$ and $E=\{\{0,1\},\{0,2\},\{0,3\}\}\cup P_{12}\cup P_{23}\cup P_{31}$. If G has a vertex v of degree two, then the graph $G/\delta(v)$ still satisfies the conditions of the lemma for the same vertices $0,1,2,3$ and appropriate stable sets, so by the induction hypothesis, $(G/\delta(v),E(G/\delta(v)))=(G,E(G))/\delta(v)$ has an odd-K_4 lemma using edges $\{0,1\},\{0,2\},\{0,3\}$, implying in turn that $(G,E(G))$ has an odd-K_4 lemma using edges $\{0,1\},\{0,2\},\{0,3\}$. We may therefore assume that G does not have a vertex of degree two. This implies in turn that

> $(\star)$ for every permutation i,j,k of $1,2,3$ we have $S_i=V(P_{ij})\cap V(P_{ik})$, and that $|S_1|=|S_2|=|S_3|\geq 2$,

as $|V|+|E|\geq 11$. Let $2'\in S_2$ be the second vertex of the 12-path P_{12}, $3'\in S_3$ the second vertex of the 23-path P_{23}, and $1'\in S_1$ the second vertex of the 31-path P_{31}. Notice that $1'\neq 1, 2'\neq 2, 3'\neq 3$. Let $H:=G/\delta(0)$, and let $0'$ be the vertex corresponding to $0,1,2,3$. Notice that $\{0',1'\},\{0',2'\},\{0',3'\}\in E(H)$. For each $i\in[3]$, let $S_i':=S_i-\{i\}$. Then for each $i\in[3]$, S_i' is stable in H and $i'\in S_i'$. Moreover, for distinct $i,j\in[3]$, the vertices i',j' lie on the path P_{ij} by $(\star)$, so $H[S_i'\cup S_j']$ contains an $i'j'$-path. It therefore follows from the induction hypothesis that $(G,E(G))/\delta(0)=(H,E(H))$ has an odd-K_4 minor using $\{0',1'\},\{0',2'\},\{0',3'\}$. After decontracting $\delta(0)$, we get that $(G,E(G))$ has a whirlpool minor with central edges $\{0,1\},\{0,2\},\{0,3\}$, which has an odd-K_4 minor using the central edges. Consequently, $(G,E(G))$ has an odd-K_4 minor using the edges $\{0,1\},\{0,2\},\{0,3\}$, thereby completing the induction step. $\square$

This lemma is also helpful for finding an odd-K_5 minor. A *pseudo-odd-K_5* is a signed graph $(G, E(G))$ for which the following statements hold: there exist a partition of $V(G)$ into parts S_0, S_1, S_2, S_3 and distinct vertices $x, y \in S_0$ such that

- there is an edge $e \in E$ whose ends are x, y, and for each $i \in \{0, 1, 2, 3\}$, S_i is stable in $G \setminus e$,
- $G \setminus e$ has internally vertex-disjoint xy-paths P_1, P_2, P_3, where for each $i \in [3]$, $V(P_i) \subseteq S_0 \cup S_i$,
- for distinct $i, j \in [3]$, $G[S_i \cup S_j]$ has a path with one end in $V(P_i)$ and the other in $V(P_j)$.

As a consequence of the Whirlpool Lemma 10.8, we get that,

Theorem 10.9. *A pseudo-odd-K_5 has an odd-K_5 minor.*

Proof. If $f \in E - (\{e\} \cup P_1 \cup P_2 \cup P_3)$ is an edge with an end in S_0, then $(G \setminus f, E(G \setminus f)) = (G, E(G)) \setminus f$ is still a pseudo-odd-K_5. We may therefore assume that each edge of $E - (\{e\} \cup P_1 \cup P_2 \cup P_3)$ has both ends in $S_1 \cup S_2 \cup S_3$. If $u \in S_0$ is an internal vertex of one of P_1, P_2, P_3, then as S_0 is stable, $(G/\delta(u), E(G/\delta(u))) = (G, E(G))/\delta(u)$ is still a pseudo-odd-K_5. We may therefore assume that P_1, P_2, P_3 do not have any internal vertices in S_0. Subsequently, as S_1, S_2, S_3 are stable, it follows that for each $i \in [3]$, $V(P_i) = \{x, y, v_i\}$ for some vertex $v_i \in S_i$. Let $(H, E(H)) := (G, E(G)) \setminus \delta(y)$. Then by the Whirlpool Lemma 10.8, $(H, E(H))$ has an odd-K_4 minor using edges $\{x, v_1\}, \{x, v_2\}, \{x, v_3\}$. Adding vertex y and its incident edges back in, we see that $(G, E(G))$ has an odd-K_5 minor, as required. □

10.3 A signed graph without an odd-K_5 minor is weakly bipartite.

Let $(G = (V, E), \Sigma)$ be a signed graph. Let $U, U' \subseteq V$ be different components of G, if any, and let H be the graph obtained from G by identifying a vertex of U with a vertex of U'. Notice that G, H have the same edge sets, and that the odd circuits of (G, Σ) are precisely the odd circuits of (H, Σ). Thus, (G, Σ) is weakly bipartite if, and only if, (H, Σ) is weakly bipartite. Moreover, because K_5 does not have a cut-vertex, if (H, Σ) has an odd-K_5 minor, then so does (G, Σ). We will use these observations in the proof below, due to Schrijver (2002).

Theorem 10.10 (Guenin 2001)**.** *A signed graph without an odd-K_5 minor is weakly bipartite.*

Proof. Let $(G = (V, E), \Sigma)$ be a signed graph that is not weakly bipartite. We will show that (G, Σ) has an odd-K_5 minor. To this end, we may assume that G is connected, and that every proper minor of (G, Σ) is weakly bipartite. Let $\mathcal{C}$ be the clutter of odd circuits of (G, Σ). It then follows from Proposition 10.6 that $\mathcal{C}$ is a minimally non-ideal clutter. Take an edge $e \in E$. Using Lehman's Theorem 9.12, we prove the following:

Claim 1. *There are minimum odd circuits C_1, C_2, C_3 and minimum signatures B_1, B_2, B_3 such that for distinct $i, j \in [3]$,*

(C1) $|C_i \cap B_i| \geq 3$ *and* $C_i \cap B_j = \{e\}$*,*

(C2) $C_i \cap C_j = \{e\} = B_i \cap B_j$*,*

(C3) the only odd cycles contained in $C_i \cup C_j$ are C_i, C_j,

(C4) the only signatures contained in $B_i \cup B_j$ are B_i, B_j.

Proof of Claim. Let $n := |E|$ and let $\mathcal{B}$ be the clutter of minimal signatures. By Theorem 10.5, we have $\mathcal{B} = b(\mathcal{C})$. Let M (resp. N) be the row submatrix of $M(\mathcal{C})$ (resp. $M(\mathcal{B})$) corresponding to the minimum odd circuits (resp. minimum signatures). By Theorem 9.12, M (resp. N) is a square and non-singular matrix that is r-regular (resp. s-regular) for some integers $r, s \geq 2$ such that $rs \geq n+1$. Moreover, for some labeling $C_1, \ldots, C_n$ of the minimum odd circuits and labeling $B_1, \ldots, B_n$ of the minimum signatures, we have that for all $i, j \in [n]$,

$$|C_i \cap B_j| = \begin{cases} rs - n + 1 & \text{if } i = j \\ 1 & \text{if } i \neq j, \end{cases}$$

and for all elements $g, h \in E$,

$$\big|\{i \in [n] : g \in C_i, h \in B_i\}\big| = \begin{cases} rs - n + 1 & \text{if } g = h \\ 1 & \text{if } g \neq h. \end{cases}$$

As signatures and odd circuits intersect in an odd number of edges, and $rs-n+1 \geq 2$, it follows that $rs-n+1 \geq 3$. By the previous equation, after possibly re-indexing the C_i and B_i's, we have that

$$e \in C_i \cap B_i \qquad i = 1, \ldots, rs - n + 1.$$

Consider C_1, C_2, C_3 and B_1, B_2, B_3. We will show that these are the desired sets. **(C1)** clearly holds. **(C2)** If $f \in (C_1 \cap C_2) - \{e\}$, then $\{i \in [n] : f \in C_i, e \in B_i\} \supseteq \{1, 2\}$, which is not the case. This shows that $C_1 \cap C_2 = \{e\}$ and similarly, $C_2 \cap C_3 = C_3 \cap C_1 = \{e\}$. Moreover, if $g \in (B_1 \cap B_2) - \{e\}$, then $\{i \in [n] : e \in C_i, g \in B_i\} \supseteq \{1, 2\}$, which is not the case. Thus, $B_1 \cap B_2 = \{e\}$ and similarly, $B_2 \cap B_3 = B_3 \cap B_1 = \{e\}$. **(C3)** Let C be an odd cycle contained in $C_i \cup C_j$. Then $C' := C_i \triangle C_j \triangle C$ is an odd cycle. As $C \cup C' \subseteq C_i \cup C_j$ and $C \cap C' \subseteq C_i \cap C_j$, it follows that

$$2r = |C_i| + |C_j| = |C_i \cup C_j| + |C_i \cap C_j| \geq |C \cup C'| + |C \cap C'| = |C| + |C'| \geq 2r,$$

so equality holds throughout. That is, C, C' are minimum odd circuits and $\chi_{C_i} + \chi_{C_j} = \chi_C + \chi_{C'}$. As M is non-singular, it follows that $\{C, C'\} = \{C_i, C_j\}$, as required. **(C4)** Let B be a signature contained in $B_i \cup B_j$. Pick $W_i, W_j, W \subseteq V$ such that $B_i = \Sigma \triangle \delta(W_i), B_j = \Sigma \triangle \delta(W_j)$ and $B = \Sigma \triangle \delta(W)$. Then for $W' := W_i \triangle W_j \triangle W$ we have

$$B' := B_i \triangle B_j \triangle B = \Sigma \triangle \delta(W_i) \triangle \Sigma \triangle \delta(W_j) \triangle \Sigma \triangle \delta(W) = \Sigma \triangle \delta(W'),$$

so B' is also a signature. As $B \cup B' \subseteq B_i \cup B_j$ and $B \cap B' \subseteq B_i \cap B_j$, it follows that

$$2s = |B_i| + |B_j| = |B_i \cup B_j| + |B_i \cap B_j| \geq |B \cup B'| + |B \cap B'| = |B| + |B'| \geq 2s,$$

so equality holds throughout. That is, B, B' are minimum signatures and $\chi_{B_i} + \chi_{B_j} = \chi_B + \chi_{B'}$. As N is non-singular, it follows that $\{B, B'\} = \{B_i, B_j\}$, as required. $\Diamond$

We will not be using Lehman's Theorem 9.12 anymore. Let x, y be the ends of e. For each $i \in [3]$, let $P_i := C_i - \{e\}$. Notice that P_1, P_2, P_3 are xy-paths that are (edge-)disjoint by (C2).

Claim 2. *For distinct $i, j \in [3]$, P_i and P_j are internally vertex-disjoint xy-paths.*

Proof of Claim. Suppose for a contradiction that P_1, P_2 have a vertex v other than x, y in common. Let $C := P_1[x, v] \cup P_2[v, y] \cup \{e\}$. Observe that C is a cycle, and because for the signature B_3 we have $B_3 \cap C = \{e\}$ by (C1), it follows that C is an odd cycle. However, C is an odd cycle contained in $C_1 \cup C_2$ that is different from C_1, C_2, a contradiction to (C3). Thus, P_1, P_2 are internally vertex-disjoint, and similarly, for distinct $i, j \in [3]$, P_i and P_j are internally vertex-disjoint. $\Diamond$

For distinct $i, j \in [3]$, pick $U_{ij} \subseteq V - \{x\}$ such that $B_i \triangle B_j = \delta(U_{ij})$ – as $e \notin B_i \triangle B_j$, it follows that $U_{ij} \subseteq V - \{x, y\}$.

Claim 3. *There are disjoint vertex subsets $U_1, U_2, U_3 \subseteq V$ such that for every permutation i, j, k of $1, 2, 3$,*

(C5) $U_{ij} = U_i \cup U_j$, and

(C6) each edge with an end in U_i and the other in U_j belongs to B_k, each edge with an end in U_k and the other in $V - (U_1 \cup U_2 \cup U_3)$ also belongs to B_k, and $B_k - \{e\}$ has no other edges.

Proof of Claim. Observe that

$$\emptyset = (B_1 \triangle B_2) \triangle (B_2 \triangle B_3) \triangle (B_3 \triangle B_1) = \delta(U_{12}) \triangle \delta(U_{23}) \triangle \delta(U_{31}) = \delta(U_{12} \triangle U_{23} \triangle U_{31}).$$

As G is connected, and $x, y \notin U_{12} \triangle U_{23} \triangle U_{31}$, it follows that $U_{12} \triangle U_{23} \triangle U_{31} = \emptyset$. This implies that there are disjoint vertex subsets $U_1, U_2, U_3 \subseteq V$ such that $U_{ij} = U_i \cup U_j$ for distinct $i, j \in [3]$. This proves **(C5)**. **(C6)** follows from the definition of U_1, U_2, U_3 and the fact (C2) that $B_1 \cap B_2 = B_2 \cap B_3 = B_3 \cap B_1 = \{e\}$. $\Diamond$

Claim 4. *For every permutation i, j, k of $1, 2, 3$, we have*

(C7) $V(P_i) \cap (U_j \cup U_k) = \emptyset$ and $V(P_i) \cap U_i \neq \emptyset$, and

(C8) $G[U_i \cup U_j]$ is connected.

Proof of Claim. **(C7)** As $P_i \cap B_j = P_i \cap B_k = \emptyset$, and P_i is an xy-path, it follows from (C6) that $V(P_i) \cap (U_j \cup U_k) = \emptyset$. Moreover, by (C1), $P_i \cap B_i \neq \emptyset$, so $V(P_i) \cap U_i \neq \emptyset$. **(C8)** Suppose otherwise. Then there is a non-empty and proper subset U of $U_i \cup U_j$ such that $\delta(U) \subseteq \delta(U_i \cup U_j) = \delta(U_{ij}) = B_i \triangle B_j$. Moreover, as G is connected, it follows that $\delta(U)$ is a non-empty and proper subset of $B_i \triangle B_j$. Then $B_i \triangle \delta(U)$ is a signature contained in $B_i \cup B_j$, so by (C4), $B_i \triangle \delta(U)$ is either B_i or B_j, implying in turn that $\delta(U)$ is either $\emptyset$ or $B_i \triangle B_j$, a contradiction. $\Diamond$

Let $B := B_1 \triangle B_2 \triangle B_3 = B_1 \cup B_2 \cup B_3$. Notice that B is also a signature as $B = B_1 \triangle \delta(U_2 \cup U_3)$, so (G, B) is a resigning of (G, Σ). Let H be the graph obtained from G after contracting all the edges in each $G[U_i]$ and each $C_i - B_i$, and deleting all the remaining edges outside $B_1 \cup B_2 \cup B_3$. Observe that $E(H) = B$, and so $(H, E(H))$ is a minor of (G, Σ). For each $i \in [3]$, let P'_i be an xy-path in $P_i \cap B_i$ and let U'_i be the vertices of H corresponding to the vertices U_i of G. Let $U'_0 := V(H) - (U'_1 \cup U'_2 \cup U'_3)$. Notice that P'_1, P'_2, P'_3 are internally vertex-disjoint xy-paths of H, that U'_0, U'_1, U'_2, U'_3 form a partition of $V(H)$ into stable sets of $H \setminus e$ by (C6), that for each $i \in [3]$ we have $V(P'_i) \subseteq U'_0 \cup U'_i$ and $V(P'_i) \cap U'_i \neq \emptyset$ by (C7), and for distinct $i, j \in [3]$, $H[U'_i \cup U'_j]$ is connected by (C8). In particular, for distinct $i, j \in [3]$, $H[U'_i \cup U'_j]$ contains a path with one end in $V(P'_i)$ and the other in $V(P'_j)$. As a result, $(H, E(H))$ is a pseudo-odd-K_5, so by Theorem 10.9, it has an odd-K_5 minor, implying in turn that (G, Σ) has an odd-K_5 minor, as required. $\square$

As a consequence, we get the following characterization of weakly bipartite graphs:

Corollary 10.11. *Let $G = (V, E)$ be a graph. Then the following statements are equivalent:*

(i) G is not weakly bipartite,

(ii) there exist disjoint $I, J \subseteq E$ such that J forms a cut of $G \setminus I$, and $G \setminus I/J$ is a K_5.

Proof. **(ii) ⇒ (i)**: Since J forms a cut of $G \setminus I$, it follows that

$$(K_5, E(K_5)) = (G \setminus I/J, E(G \setminus I/J)) = (G \setminus I, E(G \setminus I))/J = (G, E(G)) \setminus I/J,$$

so $(G, E(G))$ has an odd-K_5 minor, implying by Remark 10.7 that $(G, E(G))$, and so G, is not weakly bipartite.
(i) ⇒ (ii): It follows that $(G, E(G))$ is not weakly bipartite, so by Theorem 10.10, there are disjoint $I, J \subseteq E$ such that $(K_5, E(K_5)) = (G, E(G)) \setminus I/J$. Let $H := G \setminus I$. Then $(K_5, E(K_5)) = (H, E(H))/J$, so $E(K_5) = E(H) - J$ is a signature of $(H, E(H))$ disjoint from J. As a result, $J = (E(H) - J) \triangle E(H)$ is a cut of H, as required. $\square$

11 Cube-ideal sets

Take an integer $n \geq 1$. We will be working over the hypercube $\{0,1\}^n$. Inequalities of the form

$$1 \geq x_i \geq 0 \qquad i \in [n]$$

are called *hypercube inequalities*. Inequalities of the form

$$\sum_{i \in I} x_i + \sum_{j \in J} (1 - x_j) \geq 1 \qquad \text{for disjoint } I, J \subseteq [n]$$

are called *generalized set covering inequalities*. Notice that generalized set covering inequalities are precisely those inequalities that cut off a sub-hypercube of $\{0,1\}^n$. Take a subset $S \subseteq \{0,1\}^n$. We say that S is *cube-ideal* if its convex hull $\mathrm{conv}(S)$ can be described by hypercube and generalized set covering inequalities. When is a set cube-ideal? This is the theme of this section.

Example. $\{111, 100, 010, 001\} \subseteq \{0,1\}^3$ *is cube-ideal as its convex hull is equal to*

$$\left\{ x \in [0,1]^3 : \begin{array}{ll} x_1 + x_2 + x_3 & \geq 1 \\ x_1 + (1 - x_2) + (1 - x_3) & \geq 1 \\ (1 - x_1) + x_2 + (1 - x_3) & \geq 1 \\ (1 - x_1) + (1 - x_2) + x_3 & \geq 1 \end{array} \right\}.$$

Given two vectors $a, b \in \{0,1\}^n$, let $a \triangle b := a + b \pmod 2$. Given a coordinate $i \in [n]$, to *twist coordinate* i *of* S is to replace S by

$$S \triangle e_i := \{x \triangle e_i : x \in S\}.$$

So to twist coordinate i is to make the change of variables $x_i \mapsto 1 - x_i$. Since hypercube and generalized set covering inequalities are closed under this change of variables, it follows that,

Remark 11.1. *Take an integer $n \geq 1$ and a subset $S \subseteq \{0,1\}^n$. If S is cube-ideal, then so is any set obtained after twisting some coordinates.*

The *cuboid of* S, denoted $\mathrm{cuboid}(S)$, is the clutter over ground set $[2n]$ whose members have incidence vectors

$$(x_1, 1 - x_1, x_2, 1 - x_2, \ldots, x_n, 1 - x_n) \qquad x \in S.$$

Notice that $\{2i-1, 2i\}, i \in [n]$ are covers of $\mathrm{cuboid}(S)$, and that every member of $\mathrm{cuboid}(S)$ has cardinality n.

Example. *The cuboid of* $\{111, 100, 010, 001\} \subseteq \{0,1\}^3$ *has incidence matrix*

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix},$$

which is just the incidence matrix of Q_6. Thus, Q_6 is a cuboid.

We saw that $\{111, 100, 010, 001\}$ is cube-ideal, and that its cuboid is Q_6, which we know is an ideal clutter. In fact, we will show next that in general, a set is cube-ideal if and only if its cuboid is ideal.

11.1 Ideal cuboids

Let $\mathcal{C}$ be a clutter over ground set E. Denote by $Q(\mathcal{C})$ the set covering polyhedron

$$\left\{x \in \mathbb{R}_+^E : x(C) \geq 1 \;\; C \in \mathcal{C}\right\}.$$

Here, $x(C) = \sum (x_e : e \in C)$. Two elements of a clutter are *coexclusive* if they are never used together in a minimal cover. We will need the following basic result on coexclusive elements:

Theorem 11.2 (Abdi, Cornuéjols, Pashkovich 2018)**.** *Let $\mathcal{C}$ be a clutter and take distinct elements e, f. The following statements are equivalent:*

(i) e, f are coexclusive,

(ii) for all members C_e, C_f such that $C_e \cap \{e, f\} = \{e\}$ and $C_f \cap \{e, f\} = \{f\}$, $(C_e \cup C_f) - \{e, f\}$ contains another member,

(iii) for every extreme point $x^\star$ of $Q(\mathcal{C})$, $x_e^\star + x_f^\star \leq 1$.

Proof. **(i) ⇒ (ii)**: Suppose e, f are coexclusive elements of clutter $\mathcal{C}$. Take members C_e, C_f where $C_e \cap \{e, f\} = \{e\}$ and $C_f \cap \{e, f\} = \{f\}$. We will show that $C_e \cup C_f - \{e, f\}$ contains a member, thereby proving (ii). Suppose otherwise. Then the complement of $C_e \cup C_f - \{e, f\}$ is a cover, so it contains a minimal cover B. Since $B \cap C_e \neq \emptyset$ and $B \cap C_f \neq \emptyset$, we get that $\{e, f\} \subseteq B$, contradicting the fact that e, f are coexclusive. **(ii) ⇒ (iii)**: Take an extreme point $x^\star$ of $Q(\mathcal{C})$. We will show that $x_e^\star + x_f^\star \leq 1$, proving (iii). If $x_e^\star = 0$ or $x_f^\star = 0$, then clearly $x_e^\star + x_f^\star \leq 1$. Otherwise, there is a member C_e with $e \in C_e$ and a member C_f with $f \in C_f$ such that $x^\star(C_e) = x^\star(C_f) = 1$. If $\{e, f\} \subseteq C_e$, then $x_e^\star + x_f^\star \leq x^\star(C_e) = 1$. We may therefore assume that $C_e \cap \{e, f\} = \{e\}$ and, similarly, $C_f \cap \{e, f\} = \{f\}$. It now follows from (ii) that there is a member $C \subseteq C_e \cup C_f - \{e, f\}$. Then

$$x_e^\star + x_f^\star + 1 \leq x_e^\star + x_f^\star + x^\star(C) \leq x^\star(C_e) + x^\star(C_f) = 2,$$

proving (iii). **(iii) ⇒ (i)**: Since the incidence vector of every minimal cover B is an extreme point $x^\star$ of $Q(\mathcal{C})$, we get from $x_e^\star + x_f^\star \leq 1$ that B contains at most one e, f. So e, f are coexclusive, proving (i). □

Recall that if $\mathcal{C}$ is ideal, then the extreme points of $Q(\mathcal{C})$ are precisely the incidence vectors of the minimal covers, so $Q(\mathcal{C}) = \mathrm{conv}\left(\{\chi_B : B \in b(\mathcal{C})\}\right) + \mathbb{R}_+^E$. We will need this below:

Lemma 11.3 (Guenin 1998, Nobili and Sassano 1998)**.** *Take a clutter $\mathcal{C}$ over ground set $E = \{e_1, f_1, \ldots, e_n, f_n\}$, where for each $i \in [n]$, $\{e_i, f_i\}$ intersects every member exactly once. Then the following statements are equivalent:*

(i) $b(\mathcal{C})$ is ideal,

(ii) $\mathrm{conv}\{\chi_C : C \in \mathcal{C}\} = Q\big(b(\mathcal{C})\big) \cap \{x : x_{e_i} + x_{f_i} = 1 \; \forall i \in [n]\}$.

Proof. **(i)** $\Rightarrow$ **(ii)**: Since $\chi_C \in \{x : x_{e_i} + x_{f_i} = 1 \; \forall i \in [n]\}$ for every member C, the inclusion $\subseteq$ holds. Let us prove the reverse inclusion $\supseteq$. Since $b(\mathcal{C})$ is ideal, we get that

$$Q\big(b(\mathcal{C})\big) = \operatorname{conv}\{\chi_C : C \in \mathcal{C}\} + \mathbb{R}^E_+.$$

It is easy to see that this equation implies the reverse inclusion. **(ii)** $\Rightarrow$ **(i)**: Let $x^\star$ be an extreme point of $Q\big(b(\mathcal{C})\big)$. It suffices to show that $x^\star$ is integral. Since $\{e_i, f_i\}$ is a cover of $\mathcal{C}$, it contains a member of $b(\mathcal{C})$, so $x^\star_{e_i} + x^\star_{f_i} \geq 1$. Moreover, since e_i, f_i are exclusive in $\mathcal{C}$, they are coexclusive in $b(\mathcal{C})$, so by Theorem 11.2 (iii), $x^\star_{e_i} + x^\star_{f_i} \leq 1$. So for each $i \in [n]$, $x^\star_{e_i} + x^\star_{f_i} = 1$, implying in turn by (ii) that $x^\star \in \operatorname{conv}\{\chi_C : C \in \mathcal{C}\}$. Since $x^\star$ is an extreme point, it must be one of the incidence vectors and hence integral, as required. $\square$

We are now ready to prove the following:

Theorem 11.4 (Abdi, Cornuéjols, Guričanová, Lee 2018+). *Take an integer $n \geq 1$ and a subset $S \subseteq \{0,1\}^n$. Then S is cube-ideal if, and only if, $\operatorname{cuboid}(S)$ is an ideal clutter.*

Proof. Let $\mathcal{C} := \operatorname{cuboid}(S)$. Notice that $\mathcal{C}$ is over ground set $E = \{1, 2, \ldots, 2n-1, 2n\}$, where for each $i \in [n]$, $\{2i-1, 2i\}$ intersects every member exactly once. We may therefore apply Lemma 11.3. ($\Leftarrow$) Assume that $\mathcal{C}$ is ideal. It follows from Theorem 7.8 that $b(\mathcal{C})$ is an ideal clutter also. Thus by Lemma 11.3, we have that

$$\operatorname{conv}\{\chi_C : C \in \mathcal{C}\} = Q\big(b(\mathcal{C})\big) \cap \{x : x_{2i-1} + x_{2i} = 1 \; \forall i \in [n]\}.$$

Eliminating the even coordinates using the Fourier-Motzkin elimination method, we get that

$$\operatorname{conv}(S) = \left\{ y \in [0,1]^n : \sum \big(y_i : 2i-1 \in B\big) + \sum \big(1 - y_j : 2j \in B\big) \geq 1 \quad \forall B \in b(\mathcal{C}) \right\}.$$

As a result, S is cube-ideal. ($\Rightarrow$) Assume conversely that S is cube-ideal, so

$$\operatorname{conv}(S) = \left\{ y \in [0,1]^n : \sum \big(y_i : i \in I\big) + \sum \big(1 - y_j : j \in J\big) \geq 1 \quad \forall\, (I,J) \in \mathcal{V} \right\},$$

for some appropriate set $\mathcal{V}$. We may assume that for each $(I,J) \in \mathcal{V}$, $I \cap J = \emptyset$. After the change of variables $y_i \mapsto x_{2i-1}$ and $1 - y_i \mapsto x_{2i}$ to the equation above, we get that

$$\operatorname{conv}\{\chi_C : C \in \mathcal{C}\} = \left\{ x \in \mathbb{R}^{2n}_+ : \begin{array}{l} \sum \big(x_{2i-1} : i \in I\big) + \sum \big(x_{2j} : j \in J\big) \geq 1 \quad \forall\, (I,J) \in \mathcal{V} \\ x_{2i-1} + x_{2i} = 1 \quad \forall\, i \in [n] \end{array} \right\}.$$

Together with Lemma 11.3, this equation implies that $b(\mathcal{C})$ is an ideal clutter, so by Theorem 7.8, $\mathcal{C}$ is an ideal clutter, as required. $\square$

11.2 The sums of circuits property

Take an integer $n \geq 1$ and a set $S \subseteq \{0,1\}^n$. We say that S is a *binary space* (or a *vector space over $GF(2)$*) if

- $\mathbf{0} \in S$, and
- if $a, b \in S$ then $a \triangle b \in S$.

When is a binary space cube-ideal? To answer this question, we need to introduce some terminology. The *orthogonal complement of* S is

$$S^\perp := \left\{d \in \{0,1\}^n : d^\top c \equiv 0 \pmod 2 \quad \forall c \in S\right\}.$$

It is clear that $S^\perp$ is another binary space, and it is widely known that $(S^\perp)^\perp = S$. To describe $S^\perp$ explicitly, we first write

$$S = \left\{x \in \{0,1\}^n : Ax \equiv \mathbf{0} \pmod 2\right\}$$

for some $m \times n$ matrix A with $0-1$ entries. Then $S^\perp$ is equal to the row space of A modulo 2:

$$S^\perp = \left\{A^\top x : x \in \{0,1\}^m\right\}.$$

Denote by E the column labels of A. We say that a subset $C \subseteq E$ is a *cycle* if $\chi_C \in S$, and that a subset $D \subseteq E$ is a *cocycle* if $\chi_D \in S^\perp$. Notice that a cycle and a cocycle will always have an even number elements in common.

Example. *Let $G = (V, E)$ be a graph where loops are viewed as vertex-less edges. Then*

$$S := \{\chi_C : C \subseteq E \text{ is a graph cycle}\} \subseteq \{0,1\}^E$$

is a binary space, because for graph cycles C_1, C_2, their symmetric difference $C_1 \triangle C_2$ is also a graph cycle. We can represent S as

$$S = \left\{x \in \{0,1\}^E : Ax \equiv \mathbf{0} \pmod 2\right\}$$

where A is the vertex-edge incidence matrix of G. As a result, the cocycles of S correspond to the points in the row space of A modulo 2, implying in turn that the cocycles of S are precisely the cuts of G.

The following gives a partial characterization of the cube-ideal binary spaces:

Theorem 11.5 (Abdi, Cornuéjols, Guričanová, Lee 2017)**.** *Take an integer $n \geq 1$ and a binary space $S \subseteq \{0,1\}^n$. Then S is cube-ideal if, and only if,*

$$\mathrm{conv}(S) = \left\{x \in [0,1]^n : x(F) - x(D-F) \leq |F| - 1 \quad \forall \text{ cocycles } D \text{ and odd subsets } F \subseteq D\right\}.$$

Proof. ($\Leftarrow$) Notice that each inequality $x(F) - x(D-F) \leq |F| - 1$ can be rewritten as

$$\sum_{i \in D-F} x_i + \sum_{j \in F} (1 - x_j) \geq 1,$$

which is a generalized set covering inequality. Thus S is cube-ideal. ($\Rightarrow$) Suppose coversely that S is cube-ideal. We first prove that

$$\mathrm{conv}(S) \subseteq \left\{x \in [0,1]^n : x(F) - x(D-F) \leq |F| - 1 \quad \forall \text{ cocycles } D \text{ and odd subsets } F \subseteq D\right\}.$$

Denote by P the polytope on the right. To prove this inclusion, it suffices to show that for every cycle C, χ_C belongs to P. Well, for every cocycle D and odd subset $F \subseteq D$, we have $C \cap D \neq F$ because $|C \cap D|$ is even, implying in turn that

$$\chi_C(F) - \chi_C(D - F) \leq 1.$$

Thus, $\chi_C \in P$. To prove the reverse inclusion, it suffices to prove that every inequality defining $\mathrm{conv}(S)$ is valid for P. Since S is cube-ideal, $\mathrm{conv}(S)$ is described by hypercube inequalities – which are valid for P – and by generalized set covering inequalities. Take disjoint subsets $I, J \subseteq [n]$ such that $\sum_{i \in I} x_i + \sum_{j \in J}(1 - x_j) \geq 1$ is a defining inequality of $\mathrm{conv}(S)$.

Claim. *There is a cocycle D such that $D \subseteq I \cup J$ and $|D \cap J|$ is odd.*

Proof of Claim. To see this, write

$$S = \left\{x \in \{0,1\}^{[n]} : Ax \equiv \mathbf{0} \pmod 2\right\}$$

for some $0-1$ matrix A. Let d be the sum of the columns in J of A, and let B be the submatrix of A obtained after dropping columns $I \cup J$. Since $\sum_{i \in I} x_i + \sum_{j \in J}(1 - x_j) \geq 1$ is valid for every point of S, the binary system

$$By \equiv d \pmod 2$$

has no $0-1$ solution. (For if y is a solution, then by setting the coordinates in I to 0 and the coordinates in J to 1, we can extend y to a cycle x for which $\sum_{i \in I} x_i + \sum_{j \in J}(1 - x_j) = 0$, which is not the case.) By Farkas' lemma for binary spaces, there is a $0-1$ vector c such that $c^\top B \equiv \mathbf{0} \pmod 2$ and $c^\top d \equiv 1 \pmod 2$. Consider the cocycle $\chi_D := c^\top A$. Then the first equation implies that $D \subseteq I \cup J$, while the second equation implies that $|D \cap J|$ is odd, as required. $\Diamond$

Let $F := D \cap J$. Then F is an odd subset of the cocycle D. Observe that the inequality

$$\sum_{i \in I} x_i + \sum_{j \in J}(1 - x_j) \geq 1$$

is dominated by the inequality

$$\sum_{i \in D-F} x_i + \sum_{j \in F}(1 - x_j) \geq 1 \qquad \text{which is equivalent to} \qquad x(F) - x(D - F) \leq |F| - 1,$$

because $D - F \subseteq I$ and $F \subseteq J$. As a result, every inequality defining $\mathrm{conv}(S)$ is valid for P, so $\mathrm{conv}(S) \supseteq P$. Hence, $\mathrm{conv}(S) = P$, thereby finishing the proof. $\square$

Consider the polyhedral cone generated by S:

$$\mathrm{cone}(S) = \left\{\sum_{x \in S} \alpha_x x : \alpha \in \mathbb{R}^S_+\right\} \subseteq \{0,1\}^n.$$

Barahona and Grötschel (1986) showed that due to the transitivity of S, to describe the facets of $\mathrm{conv}(S)$, it suffices to have a facet description of $\mathrm{cone}(S)$.

Theorem 11.6 (Barahona and Grötschel 1986)**.** *Take an integer $n \geq 1$ and a binary space $S \subseteq \{0,1\}^n$. Then*

$$(1) \qquad \mathrm{conv}(S) = \big\{x \in [0,1]^n : x(F) - x(D-F) \leq |F| - 1 \quad \forall \text{ cocycles } D \text{ and odd subsets } F \subseteq D\big\}$$

if, and only if,

$$(2) \qquad \mathrm{cone}(S) = \big\{x \in \mathbb{R}^n_+ : x_f - x(D - \{f\}) \leq 0 \quad \forall \text{ cocycles } D \text{ and } f \in D\big\}.$$

Proof. ($\Rightarrow$) Suppose that (1) holds. As $\mathbf{0} \in S$, the facets of $\mathrm{conv}(S)$ tight at $\mathbf{0}$ describe the conic hull of S. Since the cocycle inequality

$$x(F) - x(D-F) \leq |F| - 1 \quad \text{cocycle } D \text{ and odd subset } F \subseteq D$$

is tight at $\mathbf{0}$ if and only if $|F| = 1$, it follows that (2) holds. ($\Leftarrow$) Conversely, suppose that (2) holds. To prove that (1) holds, let

$$(\diamond) \qquad \sum_{i \in I} a_i x_i + \sum_{j \in [n]-I} a_j(1-x_j) \geq b \qquad a \in \mathbb{R}^n_+, b \in \mathbb{R}$$

be a facet-defining inequality for $\mathrm{conv}(S)$. It suffices to show that ($\diamond$) is equivalent to a cocycle inequality. To this end, take a point $u \in S$ that lies on this facet. Consider the change of variables $x_i \mapsto 1 - x_i$ for the indices in $\big\{i \in [n] : u_i = 1\big\}$; this mapping sends the above inequality to the inequality

$$(\star) \qquad \sum_{i \in I : u_i = 0} a_i x_i + \sum_{i \in I : u_i = 1} a_i(1-x_i) + \sum_{j \in [n]-I : u_j = 0} a_j(1-x_j) + \sum_{j \in [n]-I : u_j = 1} a_j x_j \geq b,$$

and the set S to the set $S \triangle u := \{x \triangle u : x \in S\}$. Then ($\star$) is a facet-defining inequality for $S \triangle u$ and the facet contains the point $\mathbf{0} = u \triangle u \in S \triangle u$. Hence, ($\star$) also defines a facet for $\mathrm{cone}(S \triangle u)$. However, since S is a binary space, $S \triangle u$ is just the original set S, so ($\star$) defines a facet of $\mathrm{cone}(S)$. By (2), there is a cocycle $D \subseteq [n]$ and an element $f \in D$ such that ($\star$) is equivalent to the inequality

$$x_f - x(D - \{f\}) \leq 0.$$

Take the cycle $C \subseteq [n]$ such that $u = \chi_C$. Reverting back the change of variables, we see that ($\diamond$) is equivalent to

$$x(F) - x(D-F) \leq |F| - 1$$

where $F = (C \cap D) \triangle \{f\}$. Since $|C \cap D|$ is even, it follows that $|F|$ is odd, so ($\diamond$) is equivalent to a cocycle inequality, as required. $\square$

Take a finite set E and a binary space $S \subseteq \{0,1\}^E$. A *circuit* is a non-empty cycle that does not properly contain another non-empty cycle. Notice that $\{\chi_C : C \text{ is a circuit}\}$ are precisely the non-zero points in S of minimal support.

Remark 11.7. *Take a finite set E and a binary space $S \subseteq \{0,1\}^E$, and take a non-empty subset $C \subseteq E$. Then C is a cycle if, and only if, C is a disjoint union of circuits.*

Proof. ($\Leftarrow$) If C is a disjoint union of circuits, then it is also a symmetric difference of circuits, so C is a cycle. ($\Rightarrow$) We prove the converse by induction on $|C|$. As cycles of length 1 are already circuits, the base case $|C| = 1$ holds. For the induction step, assume that $|C| \geq 2$. If C does not properly contain a non-empty cycle, then it is already a circuit, and we are done. Otherwise, there is a non-empty cycle C' such that $C' \subsetneq C$. Let $C'' := C \triangle C'$. Notice that C is the disjoint union of the non-empty cycles C', C''. By the induction hypothesis, each of C', C'' is the disjoint union of circuits, implying in turn that C is a disjoint union of circuits, thereby completing the induction step. $\square$

It follows from Remark 11.7 that $\mathrm{cone}(S)$ is the polyhedral cone generated by the circuits, that is,

$$\mathrm{cone}(S) = \left\{ \sum_{C \text{ is a circuit}} y_C \cdot \chi_C : y \in \mathbb{R}_+^{\text{circuits}} \right\}.$$

The binary space S has the *sums of circuits property* if

$$\left\{ \sum_{C \text{ is a circuit}} y_C \cdot \chi_C : y \in \mathbb{R}_+^{\text{circuits}} \right\} = \left\{ x \in \mathbb{R}_+^n : x_f - x(D - \{f\}) \leq 0 \quad \forall \text{ cocycles } D \text{ and } f \in D \right\}.$$

Equivalently, a binary space S has the sums of circuits property if for each $w \in \mathbb{R}_+^E$ satisfying

$$w(D - \{f\}) \geq w_f \qquad \forall \text{ cocycles } D \text{ and } f \in D,$$

there is a vector $y \in \mathbb{R}_+^{\text{circuits}}$ such that $w = \sum (y_C \cdot \chi_C : C \text{ is a circuit})$. Combining Theorems 11.5 and 11.6, we get the following:

Corollary 11.8. *A binary space is cube-ideal if, and only if, it has the sums of circuits property.*

The deep concept of the sums of circuits property was introduced by Seymour (1979), where he showed that the binary space corresponding to the cycles of a graph has the sums of circuits property. (This result is in fact equivalent to Corollary 8.12.) That is, given a graph $G = (V, E)$, if a vector $w \in \mathbb{Z}_+^E$ satisfies

$$w(D - \{f\}) \geq w_f \qquad \forall \text{ cuts } D \text{ and } f \in D,$$

then there is an assignment $y \in \mathbb{R}_+^{\text{circuits}}$ such that

$$w = \sum (y_C \cdot \chi_C : C \text{ is a circuit}).$$

In particular, if $G = (V, E)$ is a bridgeless graph, then there is an assignment $y \in \mathbb{R}_+^{\text{circuits}}$ such that

$$\mathbf{1} = \sum (y_C \cdot \chi_C : C \text{ is a circuit}).$$

Szekeres (1973), and independently Seymour (1979), conjecture that y can be chosen to be half-integral; this is known as the notoriously difficult *cycle double cover conjecture*.