MA431 Assignment 1

Due Friday, Jan 29 at 12:01pm GMT

Collect as many points as you can. The threshold for 100% is 35 points (20 points reserved for mandatory exercises, 15 points for the optional ones).

Please do not look up the solutions. You can collaborate with other students, and if you do, you should acknowledge it in writing.

Email your solutions by the due date to `a.abdi1@lse.ac.uk`.

Mandatory Exercises

Q1. Lecture 0, Exercise 5 (10 points)

Prove the Courant-Fischer Theorem (Theorem 0.2) parts (1), (3) and (5). Then apply those parts to $-A$ to prove parts (2), (4) and (6).

Q2. Lecture 1, Exercise 9 (5 points)

Let G be a connected graph, and let $A := A(G)$. Prove that the following statements are equivalent:

(a) G is bipartite,

(b) the spectrum of G is symmetric about the origin, that is, if θ belongs to the spectrum, then so does $-\theta$, and both eigenvalues have the same multiplicity,

(c) $-\rho(A)$ is an eigenvalue.

Q3. Lecture 1, Exercise 10 (5 points)

Let $G = (V, E)$ be a graph, and let $A := A(G)$. Prove that

$$\frac{\sum_{v \in V} \deg(v)}{|V|} \leq \rho(A) \leq \max\{\deg(v) : v \in V\},$$

that is, $\rho(A)$ is sandwiched between the average degree and the maximum degree of G.

Optional Exercises

Q4. Lecture 1, Exercise 2 (5 points)

Find the spectrum of the Petersen graph. Show your work. (You may use a solver.)

Q6. Lecture 1, Exercise 4 (5 points)

Let G be a k-regular graph on n vertices with no loops or parallel edges, and let $k, \theta_2, \ldots, \theta_n$ be its spectrum. Prove that G and its complement $\overline{G}$ have the same eigenvectors, and that the eigenvalues of $\overline{G}$ are $n - 1 - k, -1 - \theta_2, \ldots, -1 - \theta_n$.

January 25 update: "Prove that G and $\overline{G}$ have the same eigenvectors" needs clarification, as it is ambiguous. The problem asks for a common set $v_1, v_2, \ldots, v_n$ of eigenvectors, which form a basis of $\mathbb{R}^n$.

Q7. Lecture 1, Exercise 11 (5 points)

Let G be a connected graph with maximum degree Δ, and let $A = A(G)$. Prove that $\rho(A) = \Delta$ if, and only if, G is a Δ-regular graph.

Q8. Lecture 1, Exercise 12 (10 points)

The *diameter* of a connected graph is the smallest integer d such that there is a path of length at most d between every pair of vertices.

Let G be a connected graph whose diameter is d. Prove that the spectrum of G has at least $d + 1$ distinct elements.