MA431 Assignment 2

Due Friday, February 12 at 12:01pm GMT

Collect as many points as you can. The threshold for 100% is 35 points (20 points reserved for mandatory exercises, 15 points for the optional ones).

Please do not look up the solutions. You can collaborate with other students, and if you do, you should acknowledge it in writing. Please make individual submissions, i.e. no group submissions.

Email your solutions by the due date to `a.abdi1@lse.ac.uk`.

Q1. (Mandatory, 5 points)

Let G be an n-vertex graph, and let A be an $n \times n$ real symmetric matrix such that $|A| \leq A(G)$. Prove that $\Delta(G) \geq \rho(A)$, where $\Delta(G)$ denotes the maximum degree of G.

Q2. (Mandatory, 10 points)

Let G be a graph on n vertices with at least one edge whose spectrum is $\theta_1 \geq \cdots \geq \theta_n$, and let x be an arbitrary eigenvector of $A := A(G)$. Let $X_1, \ldots, X_k$ be a partition of the vertex set into k nonempty stable sets, and let B be the $k \times k$ matrix where

$$B_{ij} = \frac{1}{\sum_{u \in X_i} x_u^2} \cdot \sum \left(x_u x_v : u \in X_i, v \in X_j,\ u, v \text{ are adjacent} \right).$$

(a) Prove that the spectrum of B interlaces the spectrum of A.

(b) Prove that $k \geq 1 - \frac{\theta_1}{\theta_n}$.

(c) Conclude that G has chromatic number at least $1 - \frac{\theta_1}{\theta_n}$.

Q3. (Mandatory, 5 points)

Use Cauchy's Interlacing Theorem to prove that the Petersen graph has no Hamilton cycle. (You may use a solver to compute a graph spectrum.)

Q4. (Optional, 5 points)

Let A be an $n \times n$ matrix. Recall that $p_A(x) = \det(xI - A)$. Choose $\sigma_0(A), \sigma_1(A), \ldots, \sigma_n(A)$ such that

$$p_A(x) = \sum_{k=0}^{n} (-1)^k \sigma_k(A) x^{n-k}.$$

Prove the following statements for each k:

(a) $\sigma_k(A)$ is the sum of the product of any k eigenvalues, counted according to their algebraic multiplicity. That is, if $\lambda_1, \ldots, \lambda_n$ are the n eigenvalues of A, repeated according to the algebraic multiplicity of the eigenvalues, then

$$\sigma_k(A) = \sum_{S \subseteq [n], |S| = k} \prod_{i \in S} \lambda_i.$$

(b) $\sigma_k(A)$ is the sum of the determinants of all principal $k \times k$ submatrices. That is,

$$\sigma_k(A) = \sum \left(\det(B) : B \text{ is a } k \times k \text{ principal submatrix of } A \right).$$

Q5. (Optional, 5 points)

Let $G = (V, E)$ be a k-regular graph with spectrum $k \geq \theta_2 \geq \cdots \geq \theta_n$. Prove that $\alpha(G) \leq n \cdot \frac{-\theta_n}{k - \theta_n}$. Moreover, prove that if S is a stable set meeting this bound, then every vertex outside of S has exactly $-\theta_n$ neighbours inside S.

Q6. (Optional, 5 points)

Take an odd integer $n \geq 7$.

(a) What is the spectrum of the cycle C_n of length n?

(b) Compute the upper bound given in Theorem 6.4 on the Shannon capacity of C_n.

Q7. (Optional, 5 points)

Let M, N be two matrices. Prove the following statements:

(a) $Mx \otimes Ny = (M \otimes N)(x \otimes y)$ for any two vectors x, y of appropriate dimensions,

(b) if M, N are real symmetric matrices with spectra Λ_M, Λ_N, then $M \otimes N$ is a real symmetric matrix with spectrum $(\theta\theta' : \theta \in \Lambda_M, \theta' \in \Lambda_N)$,

(c) if M is a real symmetric matrix whose least eigenvalue, say τ, is nonnegative, then the least eigenvalue of $M^{\otimes \ell}$ is τ^ℓ.

Q8. (Optional, 5 points)

The *Laplacian matrix* of a graph $G = (V, E)$ is the matrix $\Delta(G) - A(G)$, where $\Delta(G)$ is the diagonal matrix whose diagonal entries correspond to vertex degrees, and $A(G)$ is the adjacency matrix. The *Laplacian spectrum* of G is the spectrum of its Laplacian matrix.

Find the Laplacian spectra of the graphs below. Show your work. (You may use a solver.)