MA431 Assignment 3

Due Friday February 26 at 12:01pm GMT

Collect as many points as you can. The threshold for 100% is 35 points (20 points reserved for mandatory exercises, 15 points for the optional ones).

Please do not look up the solutions. You can collaborate with other students, and if you do, you should acknowledge it in writing. Please make individual submissions, i.e. no group submissions.

Email your solutions by the due date to `a.abdi1@lse.ac.uk`.

Q1. (Mandatory, 5 points)

Let G be a graph, and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be its Laplacian spectrum. Prove that the number of spanning trees of G is equal to $\frac{1}{n}\prod_{i=2}^{n}\lambda_i$.

Q2. (Mandatory, 5 points)

Let G be an n-vertex simple graph, and let $\overline{G}$ be its complement. Prove the following statements:

(a) $\lambda_i(\overline{G}) = n - \lambda_{n-i+2}(G)$ for $2 \leq i \leq n$,

(b) $\lambda_n(G) \leq n$,

(c) if $\overline{G}$ has $\bar{c}$ connected components, and $\bar{c} \geq 2$, then $\lambda_n(G) = n$ and its multiplicity is $\bar{c} - 1$.

Q3. (Mandatory, 5 points)

Let $G = (V, E)$ be a connected graph and let e be any edge of G. Let i be the unit electrical flow across the endpoints of e, i.e., $i = P_\star \chi^e$. Show that for any edge $e' \in E$, $i_{e'} \leq i_e$.

Q4. (Mandatory, 5 points)

Given a connected graph G, let $d(u, v)$ denote the effective resistance between u and v, for every $u, v \in V$. Show that $d(\cdot, \cdot)$ defines a metric.

Q5. (Optional, 5 points)

Let M be an $n \times n$, and let C be its cofactor matrix. Recall that C is an $n \times n$ matrix whose ij-entry is $(-1)^{i+j}$ times the determinant of the submatrix of M obtained after removing row i and column j. By a Laplace expansion along any row of M, we get the matrix equation $C^\top M = \det(M)I$. The matrix $C^\top$ is called the *adjugate of* M, and denoted $\text{adj}(M)$.

Let G be a graph, and let L be its Laplacian matrix. Prove that every entry of $\text{adj}(L)$ is equal to $T(G)$, the number of spanning trees of G.

Q6. (Optional, 5 points)

Let G be a connected graph on n vertices. Prove that

$$\lambda_2(G) = \min_x \frac{n \sum_{ij \in E}(x_i - x_j)^2}{\sum_{i<j}(x_i - x_j)^2}$$

where the minimum is taken over all non-constant vectors x.

Q7. (Optional, 5 points)

Let G be any connected graph with no cut vertex, i.e., there is no vertex whose removal disconnects the graph. Let e be any edge of G. Show that the cycle space of G has a basis consisting of the characteristic vectors of a set of cycles that all contain e.

Q8. (Optional, 10 points) Let M_n be the $n \times n$ 2-dimensional grid.

(a) (5 points) Show that the effective resistance between opposite corners of the grid is $\Omega(\log n)$.

(Hint: exploit Rayleigh monotonicity.)

(b) (5 points) Show that it is $O(\log n)$.