MA431 Assignment 4

Due Friday March 12 at 12:01pm GMT

Collect as many points as you can. The threshold for 100% is 35 points (20 points reserved for mandatory exercises, 15 points for the optional ones).

Please do not look up the solutions. You can collaborate with other students, and if you do, you should acknowledge it in writing. Please make individual submissions, i.e. no group submissions.

Email your solutions by the due date to `a.abdi1@lse.ac.uk`.

Q1. (Mandatory, 5 points)

Consider an unweighted connected graph G, and fix an edge e of G. Define $R : \mathbb{R}^E_{++} \to \mathbb{R}_+$ by defining $R(r)$ to be the effective resistance of edge e in the weighted graph (G, w), where $r_e = 1/w_e$ for each $e \in E$. Show that R is a concave functions.

Q2. (Mandatory, 10 points)

Consider Q_n, the n-dimensional binary hypercube; identify its vertex set with $\{-1, 1\}^n$.

Determine the second eigenvalue of the Laplacian of Q_n. *(Hint: for $a \in \{-1, 1\}^n$, consider vectors $x \in \mathbb{R}^{V(Q_n)}$ of the form $x_v = (-1)^{a \cdot v}$ for all $v \in \{-1, 1\}^n$.)*

Hence show that one direction of Cheeger's inequality is tight (up to constant factors) for the hypercube. Also give bounds on the mixing time, both directly from the second eigenvalue, and from Cheeger's inequality.

Q3. (Mandatory, 5 points)

Show that the commute time between any two nodes of a connected graph with n nodes is at most n^3. Find an example where the commute time between some pair of vertices is $\Omega(n^3)$.

Q4. (Optional, 5 points)

Consider C_n, the cycle of length n. Show that one direction of Cheeger's inequality is tight for C_n up to constant factors, for all n. *(Hint: you don't need to determine the second eigenvalue of the Laplacian, a bound suffices.)*

Q5. (Optional, 10 points)

The *cover time* of a connected graph G (which we will denote by $\text{Cover}(G)$ is the maximum over all choices of starting node s, of the expected time needed for a simple random walk starting from s to visit each node of G.

Let R be the maximum over $u, v \in V$ of the effective resistance between u and v in G; also let m denote the number of edges of G. Show that

$$\text{Cover}(G) = \Omega(mR) \text{ and } \text{Cover}(G) = O(\log |V| \cdot mR).$$

Hint: Markov's inequality is very useful here.

Q6. (Optional, 10 points)

Give a "direct" proof of the Transfer Current Theorem for the case where F is a spanning tree of G, from the Matrix Tree Theorem. By "direct", I mean that your proof should not prove or rely on the Transfer Current Theorem for smaller-sized F.