MA431 Assignment 5

Due Friday March 26 at 12:01pm GMT

Collect as many points as you can. The threshold for 100% is 35 points (15 points reserved for mandatory exercises, 20 points for the optional ones).

Please do not look up the solutions. You can collaborate with other students, and if you do, you should acknowledge it in writing. Please make individual submissions, i.e. no group submissions.

Email your solutions by the due date to `a.abdi1@lse.ac.uk`.

Q1. (Mandatory, 5 points)

Prove the following weaker version of the Alon-Bopanna bound.

For any d-regular graph G with n vertices with Laplacian L,

$$\lambda_2(L) \leq d - \sqrt{d} + o(1),$$

where the $o(1)$ hides terms that go to zero as n goes to infinity with d fixed.

Do this by considering the trace of some power of the adjacency matrix of G.

Q2. (Mandatory, 5 points)

Let G be a d-regular 2-sided $(1-\epsilon)$-expander for some $0 < \epsilon < 1$.

(a) Show that an independent set of G has size at most ϵn, where n is the number of vertices of G.

(b) Show that there is a constant C depending only on d and ϵ so that if r edges are removed from G, the resulting graph has a connected component of size at least $n - Cr$.

Q3. (Mandatory, 5 points)

Fix $d \geq 3$ and $\beta > 0$. Let $G = (V, E)$ be a d-regular 1-sided β-expander of size n. For any $v \in V$, $r \leq n$, define

$$B(v, r) = \{u \in V : \text{ there is a path from } v \text{ to } u \text{ of length at most } r\}.$$

(a) Show that $|B(v,r)| \geq \min\{(1+c)^r, n/2\}$ for some constant c depending only on d and β.

(b) Deduce that the diameter of G is $O(\log n)$.

Q4. (Optional, 5 points)

Fix d. Let T_k denote the complete d-ary tree of height k. Show that $\rho(T_k) \to 2\sqrt{d-1}$ as $k \to \infty$ (where $\rho(G)$ denotes the spectral radius of the adjacency matrix of G).

Q5. (Optional, 10 points)

The following gives another proof of the real-rootedness of the matching polynomial, essentially. (An additional continuity argument is needed to handle nonnegative rather than strictly positive weights.)

Given a weighted graph (G,w) (with positive weights) and a matching M in G, we define the *weight* of M to be the product of the weights of the edges in M. We can define the matching polynomial of a weighted graph (G,w) by writing

$$\mu_G(x) = \sum_{i \geq 0} (-1)^i m_i x^{n-2i},$$

where m_i is not the number of matchings in G, but the sum of the weights of the matchings of size i.

(a) Prove that $\mu_G(x) = x\mu_{G\setminus u}(x) - \sum_{v:uv\in E} w_{uv}\mu_{G\setminus\{u,x\}}(x)$.

(b) Let (G,w) be a weighted *complete* graph on n vertices, with all weights strictly positive. Using induction and interlacing (of polynomials) give a proof that $\mu_G(x)$ has real roots.

Q6. (Optional, 10 points)

Here, we will give a (random) construction of an expander. We will only prove 1-sided expansion (though the construction does yield a 2-sided expander).

Fix d to be a constant (later, we will need to choose it large enough). Let n be large and even, and let $V = \{1, 2, \ldots, n\}$. For $i \in \{1, 2, \ldots, d\}$, let M_i be an independent, uniformly chosen perfect matching on V. Then let $E = \bigcup_{i=1}^{d} M_i$, and $G = (V, E)$.

(a) Show that G is d-regular with probability that tends to 1 as $n \to \infty$, for any fixed d.

Let $N(S)$ denote the set of neighbours of S, for any $S \subseteq V$.

(b) Consider some $k \leq n/2$, a set $S \subseteq V$ of size k, and a set $T \subseteq V$ of size $\lfloor k/6 \rfloor$ that is disjoint from S. Show that

$$\Pr[N(S) \subseteq S \cup T] \leq \left(\frac{k}{n}\right)^{dk/6}.$$

(c) Hence show that for some fixed d, and some $\alpha > 0$, the probability that G has edge expansion α tends to 1 as $n \to \infty$.

Q7. (Optional, 10 points)

Let $G = (V, E)$ be a d-regular 1-sided β-expander on n vertices, for some d and $\beta > 0$. Let d_G be the shortest path metric on V induced by G: that is, $d_G(u, v)$ is the length of a shortest path from u to v in G, for every $u, v \in V$.

Here, we will see that there is no embedding of this metric into an ℓ_1 metric (of any dimension) with distortion $o(\log n)$.

Let (V, ρ) be any ℓ_1-metric on V (that is, there exists some dimension N and map $\phi : V \to \mathbb{R}^N$ so that $\rho(u, v) = \|\phi(u) - \phi(v)\|_1$ for all $u, v \in V$) for which $\rho(u, v) \leq d_G(u, v)$ for all $u, v \in V$. You may use the following fact (that we essentially showed during the seminar) without proof.

Lemma. *There exists weights $\lambda : 2^V \to \mathbb{R}_+$ such that for all $u, v \in V$,*

$$\rho(u, v) = \sum_{S \subseteq V} \lambda(S) |\mathbf{1}_{u \in S} - \mathbf{1}_{v \in S}|.$$

(a) Show that the average distance in the metric d_G between two nodes is $\Omega(\log n)$. That is,

$$\frac{1}{n^2} \sum_{u \in V} \sum_{v \in V} d_G(u, v) = \Omega(\log n).$$

(b) Hence show that there exist $u, v \in V$ for which $\rho(u, v) \leq \frac{C}{\log n} d_G(u, v)$, where C is some constant depending only on d and β.