MA431 Spectral Graph Theory: Lecture 10

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Abstract

In this lecture, we prove a remarkable theorem of Tutte on how to draw a 3-connected planar graph through a connection to the Laplacian. Then we prove that for such a graph, the second smallest eigenvalue of the Laplacian has multiplicity at most 3.

23 Tutte drawings of planar graphs

All the graphs in this section and the next, unless stated otherwise, are assumed simple. Let $G = (V, E)$ be a 3-connected planar graph. We exhibit an elegant way to embed G on the plane (so that no two edges cross), by solving a matrix equation involving the Laplacian matrix of G. This drawing is originally due to Tutte.

23.1 Plane embeddings and peripheral circuits

The reason we work with planar graphs that are 3-connected is because such graphs essentially have a unique embedding, and for such graphs, we can characterise the facial circuits independently of the embedding. To elaborate, let $G = (V, E)$ be a connected graph. A circuit $C \subseteq E$ is *peripheral* if G/C has no cut-vertex. We need the following classic result on planarity:

Theorem 23.1. Let $G = (V, E)$ be a 3-connected planar graph. Then the following statements hold:

1. Every edge belongs to exactly two peripheral circuits, and the two circuits are internally vertex-disjoint.
2. Any two plane embeddings of G are homeomorphically equivalent.
3. Given a plane embedding of G, the facial circuits are precisely the peripheral circuits of G.

Given disjoint edge sets $I, J \subseteq E$, the *minor* $G \setminus I/J$ is the (not necessarily simple) graph obtained from deleting the edges in I and contracting the edges in J. Observe that every minor of a planar graph is also planar, so because $K_{3,3}$, the complete bipartite graph with 3 vertices on each side, is not planar, we get the following useful, widely known fact:

Remark 23.2. A planar graph has no $K_{3,3}$ minor.

We also need the following lemma:

Lemma 23.3 (Key Lemma)**.** Let $G = (V, E)$ be a 3-connected planar graph, let $e = uv \in E$, and let C_1, C_2 be the two peripheral circuits of G containing e. Then every path connecting $V(C_1) - \{u, v\}$ to $V(C_2) - \{u, v\}$ and every uv-path different from $\{e\}$ have a vertex in common.

Proof. Let us embed G on the plane. By Theorem 23.1, C_1, C_2 are the boundaries of the two faces containing edge e; call the two faces $F_1, F_2 \subseteq \mathbb{R}^2$, respectively. Let $F := \mathbb{R}^2 \setminus (F_1 \cup F_2)$. Observe that F is homeomorphic to a disc, and its boundary is the circuit $C := C_1 \triangle C_2$.

Let $P \subseteq E$ be a path connecting $V(C_1) - \{u, v\}$ to $V(C_2) - \{u, v\}$, and let Q be a uv-path different from $\{e\}$. We need to prove that P, Q have a vertex in common. If $V(P) \cap \{u, v\} \neq \emptyset$, we are done. Otherwise, $V(P) \cap \{u, v\} = \emptyset$. By moving to a subpath of P, if necessary, we may assume that P is internally vertex-disjoint from $V(C_1) \cup V(C_2) = V(C)$. Suppose P is an ab-path, where $a \in V(C_1) - \{u, v\}$ and $b \in V(C_2) - \{u, v\}$.

Since F_1, F_2 are faces of the embedding, the two paths P, Q are disjoint from them and therefore contained in F. We know that P joins the two boundary points a, b of F, and Q joins the two boundary points u, v. Since the four points appear on the boundary of F, which is C, as a, u, b, v clockwise or anticlockwise, the two paths P, Q must have a point, which must be a vertex, in common. □

23.2 Locally convex embeddings

Let $G = (V, E)$ be a 3-connected graph, and let $C \subseteq E$ be a peripheral circuit. A *locally convex embedding* of the pair (G, C) is a straight-line plane drawing of G as specified by $\phi : V \to \mathbb{R}^2$, where

1. ϕ maps C to a strictly convex polygon,
2. for every vertex $v \in V - V(C)$, $\phi(v) \in \text{relint}(\text{conv}(N(v)))$, that is, v is mapped to the relative interior of the convex hull of the images of the neighbours of v (here, we assume that the relative interior of a single point is the point itself).

We see in the next subsection why such an embedding exists. For now, let us prove the following:

Theorem 23.4. Let $G = (V, E)$ be a 3-connected planar graph, and let C be a peripheral circuit. Then every locally convex embedding of (G, C) is a plane embedding of G.

Proof. Let $\phi : V \to \mathbb{R}^2$ be a locally convex embedding of (G, C).

Claim 1. $\phi(v) \in \text{int}(\text{conv}(\phi(V(C))))$ for every vertex $v \in V - V(C)$.

Proof of Claim. Exercise. ◊

Claim 2. Let $a \in \mathbb{R}^2$ and $\beta \in \mathbb{R}$, and let $X := \{v \in V : a^\top \phi(v) > \beta\}$. If $X \neq \emptyset$, then the induced subgraph $G[X]$ is connected.

Proof of Claim. Suppose $X \neq \emptyset$. By Claim 1, $X \cap V(C) \neq \emptyset$, and clearly, every pair of vertices in $X \cap V(C)$ are connected via a subpath of C that appears in $G[X]$. Let $K \subseteq X$ be the connected component of $G[X]$

containing the vertices of C. Let $\beta^\star = \max\{a^\top \phi(v) : v \notin K\}$, and let L collect the vertices that attain this maximum:

$$L = \{v \notin K : a^\top \phi(v) = \beta^\star\}.$$

Suppose for a contradiction $K \neq X$. Then $\beta^\star > \beta$ and so $L \subseteq X - K$. Since G is connected, $\delta(L) \neq \emptyset$, so there exists an edge uv where $u \in L$ and $v \notin L$. Since there is no edge between L and K, we have $v \notin K$, so our choice of $\beta^\star$ implies that $a^\top \phi(v) < \beta^\star$. By local convexity, there must exist another $w \in N(u)$ for which $a^\top \phi(w) > \beta^\star$. Our choice of $\beta^\star$ would then imply that $w \in K$, which is a contradiction as there is no edge between L and K. Thus, $K = X$, thereby proving the claim. $\Diamond$

Claim 3 (Nondegeneracy)**.** For every vertex $u \in V$, the points $\{\phi(v) : v \in N(u)\}$ are not collinear.

Proof of Claim. Suppose otherwise. That is, there exist $a \in \mathbb{R}^2$ and $\beta \in \mathbb{R}$ such that $a^\top \phi(v) = \beta$ for every $v \in N(u)$. Clearly, $u \in V - V(C)$. Thus $\phi(u) \in \operatorname{relint}(\operatorname{conv}(N(u)))$, so we also have $a^\top \phi(u) = \beta$. Let

$$A_1 := \{w \in V : a^\top \phi(w) > \beta\}$$
$$A_2 := \{w \in V : a^\top \phi(w) < \beta\}.$$

By Claim 1, these sets are nonempty, and by Claim 2, $G[A_1], G[A_2]$ are connected subgraphs. Let $A_3' \subseteq V - (A_1 \cup A_2)$ be the set of all vertices w such that $a^\top \phi(w') = \beta$ for all $w' \in N(w)$. That is, A_3' collects all the vertices w such that neither w nor its neighbours belong to $A_1 \cup A_2$. By definition, $u \in A_3'$. Let

$$A_3 := \text{the connected component of } G[A_3'] \text{ containing } u \subseteq A_3'.$$

Next, let $B := V - (A_1 \cup A_2 \cup A_3')$. That is, B collects vertices w such that $a^\top \phi(w) = \beta$ but $a^\top \phi(w') \neq \beta$ for some $w' \in N(w)$. In fact, local convexity implies that each vertex in B has a neighbour in A_1 as well as a neighbour in A_2. Moreover, B *cuts* the vertex set A_3 from the rest of the graph, that is, $G \setminus B$ has A_3 as a connected component. Subsequently, since G is 3-connected, B must contain at least three vertices b_1, b_2, b_3 each of which has a neighbour in A_3.

To summarise, we have found three vertex-disjoint connected subgraphs, $G[A_1], G[A_2], G[A_3]$, and three vertices outside, b_1, b_2, b_3, such that each b_i has a neighbour in each A_j. Consequently, G has a $K_{3,3}$ minor, a contradiction to Remark 23.2 as G is planar. $\Diamond$

Recall from Theorem 23.1 that every edge of G belongs to exactly two peripheral circuits, that G has a unique embedding in the plane, and that the peripheral circuits correspond precisely to the face boundaries of the unique embedding.

Claim 4. Let $e = uv \in E - C$, and let C_1, C_2 be the two peripheral circuits containing e. Then the line through $\phi(u), \phi(v)$ strictly separates $\{\phi(w) : w \in V(C_1) - \{u, v\}\}$ from $\{\phi(w') : w' \in V(C_2) - \{u, v\}\}$.

Proof of Claim. Suppose otherwise. Then there exist $a \in \mathbb{R}^2$ and $\beta \in \mathbb{R}$ such that $a^\top \phi(u) = a^\top \phi(v) = \beta$, and both $V(C_1) - \{u, v\}, V(C_2) - \{u, v\}$ intersect $\{w \in V : a^\top \phi(w) \geq \beta\}$. We shall construct paths P, Q on different sides of the line and are therefore vertex-disjoint, where P connects $V(C_1)-\{u,v\}$ to $V(C_2)-\{u,v\}$, and Q is a uv-path different from $\{e\}$, thereby contradicting the Key Lemma, Lemma 23.3.

To this end, let $X := \{w \in V : a^\top \phi(w) > \beta\}$ and $Y := \{w \in V : a^\top \phi(w) < \beta\}$. Since $e = uv \in E - C$, it follows (from Claim 1) that X, Y are nonempty. By Claim 2, $G[X], G[Y]$ are connected subgraphs. Nondegeneracy, together with local convexity, implies that each of u, v has a neighbour in Y. As a result, since $G[Y]$ is connected, there exists a uv-path Q different from $\{e\}$ whose internal vertices are in Y. The path P is constructed in a similar fashion on the X side. More precisely, let w_1 be a vertex of $V(C_1) - \{u,v\}$ such that $a^\top \phi(w_1) \geq \beta$, and let w_2 be a vertex of $V(C_2) - \{u,v\}$ such that $a^\top \phi(w_2) \geq \beta$. A similar argument as above tells us that there exists an w_1w_2-path P whose internal vertices are in X. Since $\{w_1, w_2\} \cap \{u,v\} = \emptyset$ and $X \cap Y = \emptyset$, it follows that P, Q are the desired vertex-disjoint paths, thereby yielding a contradiction. $\Diamond$

Claim 5. Every peripheral circuit is mapped, under ϕ, to a strictly convex polygon.

Proof of Claim. This follows from Claim 4. $\Diamond$

Let S be the set of points in $\mathrm{conv}(\phi(V(C)))$ that do not lie on any line segment connecting the ϕ-images of adjacent vertices. For each point $p \in S$, let $n(p)$ be the number of peripheral circuits $C' \neq C$ such that $p \in \mathrm{relint}(\mathrm{conv}(\phi(V(C'))))$.

Claim 6. $n(p) = 1$ for each $p \in S$.

Proof of Claim. Consider the ϕ-image of C, a strictly convex polygon. Pick a non-corner point q on the boundary of this polygon. Observe that $n(p) = 1$ for every point $p \in S$ that is sufficiently close to q: $n(p) \geq 1$ follows from the fact that the edge of G containing q belongs to a peripheral circuit different from C, while $n(p) \leq 1$ follows from Claim 1. To prove the claim for every point in S, we shall leverage Claim 4.

Pick an arbitrary point $p \in S$. Pick a line ℓ through p that does not pass through any $\phi(v), v \in V$, and let q_1, q_2 be the intersection points of ℓ with the ϕ-image of C. Observe that $p \in \ell[q_1, q_2]$. What we just observed tells us $n(p') = 1$ for every $p' \in S \cap \ell[q_1, q_2]$ that is sufficiently close to q_1 or q_2. As we move in $S \cap \ell[q_1, q_2]$ from a neighbourhood of q_1 to a neighbourhood of q_2, the function $n(\cdot)$ does not change unless we "jump" over $(\phi(u), \phi(v))$ for some $uv \in E - C$. However, by Claim 4, $n(\cdot)$ remains the same whenever we jump over any $(\phi(u), \phi(v)), uv \in E - C$. Consequently, $n(\cdot)$ remains the same, 1, as we move in $S \cap \ell[q_1, q_2]$ from a neighbourhood of q_1 to a neighbourhood of q_2, implying in turn that $n(p) = 1$, as required. $\Diamond$

Claim 7. The ϕ-images of any two distinct edges of G are internally disjoint.

Proof of Claim. Exercise. $\Diamond$

Claim 7 finishes the proof. $\square$

23.3 Barycentric extensions

Let $G = (V, E)$ be a connected graph, let $X \subseteq V$ be nonempty, and let $\phi' : X \to \mathbb{R}^2$ be an embedding of the vertices of X. A function $\phi : V \to \mathbb{R}^2$ is a *barycentric extension of* ϕ' if for every $v \in V$,

$$\phi(v) = \begin{cases} \phi'(v) & \text{if } v \in X, \\ \frac{1}{\deg(v)} \sum_{u \in N(v)} \phi(u) & \text{otherwise.} \end{cases}$$

Lemma 23.5. Let $G = (V, E)$ be a connected graph, let $X \subseteq V$ be nonempty, and let $\phi' : X \to \mathbb{R}^2$ be an embedding of the vertices of X. Then a barycentric extension of ϕ' exists uniquely.

Proof. Let L be the Laplacian of G. For subsets $S_1, S_2 \subseteq V$, denote by $L[S_1, S_2]$ the submatrix of L whose rows correspond to S_1 and whose columns correspond to S_2. Let $\phi : V \to \mathbb{R}^2$ be an arbitrary extension of ϕ', and let A be the $V \times \{x, y\}$ matrix whose rows are $\phi(v), v \in V$. Observe that ϕ is a barycentric extension of ϕ' if, and only if, $\deg(v) \cdot \phi(v) - \sum_{u \in N(v)} \phi(u) = \mathbf{0}$ for all $v \in V - X$. Writing the latter in matrix form, we see that ϕ is a barycentric extension if, and only if, $L[V - X, V] \cdot A = \mathbf{0}$, which can be rewritten as

$$L[V - X, V - X] \cdot A[V - X, \{x, y\}] = -L[V - X, X] \cdot A[X, \{x, y\}].$$

In the matrix equation above, the values of $A[X, \{x, y\}]$ are given by the coordinates of ϕ', whereas the values of $A[V - X, \{x, y\}]$ are determined by the unknown coordinates of ϕ on $V - X$. As $X \neq \emptyset$, $L[V - X, V - X]$ is a proper principal submatrix of L. Moreover, since G is a connected graph, $L[V - X, V - X]$ is nonsingular matrix (see Exercise 4). Thus, the values of ϕ on $V - X$ are determined uniquely as follows:

$$A[V - X, \{x, y\}] = -L[V - X, V - X]^{-1} \cdot L[V - X, X] \cdot A[X, \{x, y\}].$$

This finishes the proof. □

We may therefore speak of *the* barycentric extension of a partial mapping.

Theorem 23.6. Let $G = (V, E)$ be a 3-connected planar graph, let $C \subseteq E$ be a peripheral circuit, and assume that $\phi' : V(C) \to \mathbb{R}^2$ maps C to a strictly convex polygon. Then the barycentric extension of ϕ' yields a locally convex embedding of (G, C), and thus a straight-line plane embedding of G.

Proof. That the barycentric extension of ϕ' yields a locally convex embedding of (G, C) is immediate. Once there, Theorem 23.4 tells us that this embedding must be a plane embedding of G. □

24 The multiplicity of λ_2 for planar graphs

Let $G = (V, E)$ be a 3-connected planar graph, let L be the Laplacian matrix, and let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be its Laplacian spectrum. In this section, we prove that the second eigenvalue, λ_2, has multiplicity at most 3.

Lemma 24.1. Let $G = (V, E)$ be a connected graph, let L be its Laplacian matrix, and let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be its Laplacian spectrum. Let $f \in \mathbb{R}_+^V$ be a λ_2-eigenvector whose support is minimal amongst all λ_2-eigenvectors of L. Let $U_+ := \{u \in V : f_u > 0\}$, $U_- := \{u \in V : f_u < 0\}$, and $U := U_+ \cup U_-$. Then the following statements hold:

1. every vertex of $V - U$ with a neighbour in one of U_+, U_- has a neighbour in other set,
2. $G[U_+], G[U_-]$ are connected subgraphs.

Proof. As $\mathbf{1}^\top f = 0$, the sets U_+, U_- are nonempty; this will be a useful fact in our proof. For every vertex $u \in V$, we have $\deg(u) \cdot f_u = \sum_{v \in N(u)} f_v$. These equalities imply **(1)** immediately. **(2)** We will show that $G[U_+]$ is connected; that $G[U_-]$ is connected follows from applying a similar argument to $-f$. Suppose for a contradiction $G[U_+]$ is not connected. Then there exists a partition of U_+ into nonempty parts I, J such that there is no edge between the two parts. Define the nonzero vector $g \in \mathbb{R}^V$ as follows:

$$g_u := \begin{cases} f_u & \text{if } u \in I \\ -\alpha \cdot f_u & \text{if } u \in J \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha \in \mathbb{R}$ is chosen so that $\mathbf{1}^\top g = 0$. We claim that g is a λ_2-eigenvector of L, thereby contradicting the minimality of the support of f.

Claim. $\frac{g^\top L g}{g^\top g} \leq \lambda_2$.

Proof of Claim. For subsets $S_1, S_2 \subseteq V$, denote by $L[S_1, S_2]$ the submatrix of L whose rows correspond to S_1 and whose columns correspond to S_2, and by v_{S_1} the subvector of $v \in \mathbb{R}^V$ restricted to the coordinates in S_1. Then

$$\begin{aligned} g^\top L g &= g_I^\top L[I, I] g_I + g_J^\top L[J, J] g_J && \text{because } L[I, J] = \mathbf{0} \\ &= f_I^\top L[I, I] f_I + \alpha^2 f_J^\top L[J, J] f_J \\ &= f_I^\top (\lambda_2 f_I - L[I, U_-] f_{U_-}) + \alpha^2 f_J^\top (\lambda_2 f_J - L[J, U_-] f_{U_-}) && \text{because } Lf = \lambda_2 f \\ &= \lambda_2 g^\top g - f_I^\top L[I, U_-] f_{U_-} - \alpha^2 f_J^\top L[J, U_-] f_{U_-} \\ &\leq \lambda_2 g^\top g \end{aligned}$$

where the last inequality follows from the inequalities $f_I, f_J > \mathbf{0}$, $f_{U_-} < \mathbf{0}$, and the fact that $L[I, U_-], L[J, U_-]$ have nonpositive entries. ♢

However, as $g \in \langle \mathbf{1} \rangle^\perp$, CFT (3) implies that $\frac{g^\top L g}{g^\top g} \geq \lambda_2$, and equality is achieved only for vectors g in the λ_2-eigenspace. The claim above implies that indeed equality is achieved, and so g must be a λ_2-eigenvector, thereby contradicting the support minimality of f. □

We need the following classic result from Graph Theory:

Theorem 24.2 (Menger's Theorem)**.** Let $G = (V, E)$ be a graph, and let s, t be distinct vertices. Then the following statements are equivalent:

1. there exist k internally vertex-disjoint st-paths,
2. for all $X \subseteq V - \{s, t\}$ such that $|X| < k$, the vertices s, t belong to the same connected component of $G \setminus X$.

We are now ready for the main result of this section:

Theorem 24.3. Let $G = (V, E)$ be a 3-connected planar graph, let L be the Laplacian matrix, and let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be its Laplacian spectrum. Then λ_2 has multiplicity at most 3.

Proof. Suppose for a contradiction λ_2 has multiplicity at least 4. Embed G on the plane; let C be a facial (i.e. peripheral) circuit, and let v_1, v_2, v_3 be distinct vertices of $V(C)$. Our contrary assumption implies that there exists a λ_2-eigenvector f such that $f_{v_1} = f_{v_2} = f_{v_3} = 0$. We may assume that f is support minimal amongst all λ_2-eigenvectors. Let $U_+ := \{u \in V : f_u > 0\}, U_- := \{u \in V : f_u < 0\}$, and $U := U_+ \cup U_-$.

As G is 3-connected, we may apply Menger's Theorem and conclude that there exist vertex-disjoint paths P_1, P_2, P_3 such that for each $i \in [3]$,

- P_i is a $u_i v_i$-path in $G[V - U]$, and
- u_i has a neighbour in U.

To see this, let G' be the graph obtained from G after introducing a new vertex, t, with neighbours v_1, v_2, v_3. Observe that G' remains 3-connected. Now, pick an arbitrary vertex $s \in U$, and find three internally vertex-disjoint st-paths in G', whose existence is guaranteed by Menger's Theorem. The three paths P_1, P_2, P_3 are appropriate subpaths of these st-paths.

Moving forward, note that by placing t in the face bounded by C, we get a plane embedding of G' as well. Our contradiction will come from the fact that G' has a $K_{3,3}$ minor, which is at odds with the planarity of G' by Remark 23.2.

By Lemma 24.1, in G, each u_i has a neighbour in U_+ and a neighbour in U_-, and $G[U_+], G[U_-]$ are disjoint connected subgraphs. Thus, by contracting $G[U_+], G[U_-]$ to single vertices u_+, u_-, respectively, and by contracting P_1, P_2, P_3, we obtain a (not necessarily simple) minor of G' where each of u_+, u_-, t is a neighbour of each of v_1, v_2, v_3, implying in turn that G' has a $K_{3,3}$ minor, which is a contradiction. $\square$

See Exercises 7, 8, 9 for an extension of the results of this section to a larger class of matrices, called *generalised Laplacians*.

Acknowledgements

Proofs of the preliminaries in §23.1 can be looked up, for example, in [2], Chapter 4 (peripheral circuits are referred to as *non-separating induced cycles*). The drawing method of §23 originally comes from Tutte [4]. Our

presentation closely followed Geelen [3]. The main result of §24 is due to Colin de Verdière [1], but the short proof is due to van der Holst [5].

References

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Exercises

1. Prove Theorem 23.4, Claim 1.

2. Prove Theorem 23.4, Claim 7.

3. Let $v_1, \ldots, v_k$ be k points in $\mathbb{R}^n$. Prove that $x^\star = \frac{1}{k}\sum_{i=1}^k v_i$ is the unique minimiser of the function $f(x) = \sum_{i=1}^k \|x - v_i\|^2$.

4. Let G be a connected graph, and let L be its Laplacian matrix. Prove that every proper principal submatrix of L is nonsingular.

5. Based on the results of this lecture, describe an algorithm that given a 3-connected graph $G = (V, E)$ runs in time polynomial in $|V|$ and outputs a straight-line embedding of G or certifies that G is not planar.

6. Let $G = (V, E)$ be a 2-connected graph, and let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be its Laplacian spectrum.

   (a) Prove that if G is a path, then λ_2 has multiplicity at most 1.

   (b) G is *outerplanar* if it has a plane embedding where every vertex belongs to the boundary of the same face. Prove that if G is outerplanar, then λ_2 has multiplicity at most 2.

7. Let $G = (V, E)$ be a connected graph. A *generalised Laplacian* is a symmetric $V \times V$ matrix Q such that for all $u, v \in V$,

$$Q_{uv} \begin{cases} < & 0 & \text{if } u, v \text{ are adjacent} \\ = & 0 & \text{if } u, v \text{ are nonadjacent and distinct,} \end{cases}$$

   Let λ be the smallest eigenvalue of Q. Prove that λ is a simple eigenvalue, and each associated eigenvector has nonzero entries of the same sign.

8. Let G be an n-vertex connected graph, let Q be a generalised Laplacian, and let $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be the spectrum of Q. Let $f \in \mathbb{R}_+^V$ be a λ_2-eigenvector whose support is minimal amongst all λ_2-eigenvectors of Q. Let $U_+ := \{u \in V : f_u > 0\}$, $U_- := \{u \in V : f_u < 0\}$, and $U := U_+ \cup U_-$. Prove that $G[U_+], G[U_-]$ are connected subgraphs.

9. Let G be an n-vertex connected graph, let Q be a generalised Laplacian, and let $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be the spectrum of Q. Prove that if G is 3-connected and planar, then λ_2 has multiplicity at most 3.