# MA431 Spectral Graph Theory: Lecture 3

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February 3, 2021

Last time, we saw the following theorem and saw some applications to graph theory.

**Theorem 3.3** (Cauchy's Interlacing Theorem)**.** Given an $n \times n$ real symmetric matrix $A$ with spectrum $\theta_1 \geq \cdots \geq \theta_n$ and an $m \times m$ principal submatrix $B$ with spectrum $\mu_1 \geq \cdots \geq \mu_m$, we have that

$$\theta_i \geq \mu_i \geq \theta_{n-m+i} \qquad i = 1, \ldots, m.$$

## 4.1 On the Sensitivity Conjecture

Let us see a final application of Cauchy's Interlacing Theorem, this time to Boolean functions. Recall that the spectral radius of a real symmetric matrix $A$, denoted by $\rho(A)$, is the maximum absolute value of its eigenvalues.

**Lemma 4.5.** Let $G$ be an $n$-vertex graph, and let $A$ be an $n \times n$ real symmetric matrix such that $|A| \leq A(G)$. Then $\Delta(G) \geq \rho(A)$.

*Proof.* Exercise. □

Denote by $Q_n$ the skeleton graph of the unit hypercube $[0,1]^n$. That is, $Q_n$ has vertex set $\{0,1\}^n$ where two vertices are adjacent if they differ in exactly one coordinate, that is, if their Hamming distance is one. The adjacency graph of $Q_n$ is defined recursively as follows: $A(Q_1) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and for each integer $n \geq 2$, $A(Q_n) = \left(\begin{smallmatrix} A(Q_{n-1}) & I \\ I & A(Q_{n-1}) \end{smallmatrix}\right)$.

We shall use Cauchy's Interlacing Theorem in a clever way to prove that every induced subgraph of $Q_n$ on at least $2^{n-1}+1$ has a vertex of degree at least $\sqrt{n}$. We will need to work with an appropriate signing of the adjacency matrix $A(Q_n)$. Let $A_1 := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and for each integer $n \geq 2$, let $A_n := \left(\begin{smallmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{smallmatrix}\right)$.

**Lemma 4.6.** The following statements hold:

1. $A_n$ is a $2^n \times 2^n$ real symmetric matrix,
2. $|A_n| = A(Q_n)$,
3. $A_n^2 = nI$,
4. $A_n$ has spectrum $-\sqrt{n}$ and $\sqrt{n}$, each with multiplicity $2^{n-1}$.

*Proof.* **(1)** and **(2)** are immediate. **(3)** follows from induction combined with $A_n^2 = \begin{pmatrix} A_{n-1}^2+I & \mathbf{0} \\ \mathbf{0} & A_{n-1}^2+I \end{pmatrix}$. **(4)** It follows from (3) that every eigenvalue of $A_n$ is either $\sqrt{n}$ or $-\sqrt{n}$. Since $\operatorname{tr}(A_n) = 0$, the result follows. □

As a consequence,

**Theorem 4.7.** Let $G$ be an induced subgraph of $Q_n$ with $2^{n-1}+1$ vertices. Then $\Delta(G) \geq \sqrt{n}$.

*Proof.* Note that $A(G)$ is a principal submatrix of $A(Q_n)$; let $A$ be the corresponding principal submatrix of $A_n$. Then by Lemma 4.6, $|A| = A(G)$, so $\Delta(G) \geq \rho(A)$ by Lemma 4.5.

By Cauchy's Interlacing Theorem, the spectrum of $A$, say $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{|V(G)|}$, interlaces the spectrum of $A_n$, say $\theta_1 \geq \cdots \geq \theta_{2^n}$. By Lemma 4.6, the first $2^{n-1}$ $\theta_i$'s are equal to $\sqrt{n}$, while the last $2^{n-1}$ are equal to $-\sqrt{n}$. Thus, since $|V(G)| = 2^{n-1}+1$, interlacing implies that $\mu_1 \geq \sqrt{n}$.

Since $\rho(A) \geq \mu_1$, the two inequalities obtained imply that $\Delta(G) \geq \sqrt{n}$, as required. □

In the 90s, Gotsman and Linial proved that the statement above has a deep implication on the "sensitivity" of Boolean functions [1]. To elaborate, every Boolean function $f : \{0,1\}^n \to \{0,1\}$ can be uniquely expressed as a multilinear polynomial of degree at most $n$ over the reals. The *degree* of $f$ is then simply the degree of this polynomial. The *sensitivity* of $f : \{0,1\}^n \to \{0,1\}$ is the maximum over all inputs $x \in \{0,1\}^n$ of the number of coordinates which, when flipped in $x$, change $f$. One form of the so-called *Sensitivity Conjecture* claims that the degree of a Boolean function is polynomially upper-bounded by its sensitivity [4]. What Gotsman and Linial proved is that this conjecture is implied from Theorem 4.7. The theorem above was proved quite recently by Huang [3], and the proof we gave is almost identical to his proof.

# 5 The Courant-Hilbert-Haemers Theorem

We shall need the following Courant-Fischer Theorem from Lecture 0. (Parts (5) and (6) are omitted.)

**Theorem** (CFT). Let $A$ be an $n \times n$ real symmetric matrix with eigenvalues $\theta_1 \geq \cdots \geq \theta_n$. Then the following statements hold:

1. $\min\{x^\top Ax : x^\top x = 1\} = \theta_n$. Moreover, equality is achieved only by vectors in the $\theta_n$-eigenspace.

2. $\max\{x^\top Ax : x^\top x = 1\} = \theta_1$. Moreover, equality is achieved only by vectors in the $\theta_1$-eigenspace.

Let $u_1, \ldots, u_n$ be an orthogonal basis of eigenvectors corresponding to the eigenvalues $\theta_1, \ldots, \theta_n$, respectively. For each $j \in \{1, \ldots, n-1\}$, let $U_j$ be the subspace spanned by $u_1, \ldots, u_j$. Then we have the following *Rayleigh inequalities*:

3. $\min\{x^\top Ax : x^\top x = 1, x \in U_j\} = \theta_j$. Moreover, equality is achieved only by vectors in the $\theta_j$-eigenspace.

4. $\max\{x^\top Ax : x^\top x = 1, x \in U_j^\perp\} = \theta_{j+1}$. Moreover, equality is achieved only by vectors in the $\theta_{j+1}$-eigenspace.

In this section, we state a powerful extension of Cauchy's Interlacing Theorem, and prove it using the Courant-Fischer Theorem instead of interlacing polynomials. To this end, consider a sequence $\theta_1 \geq \cdots \geq \theta_n$ and a shorter one $\mu_1 \geq \cdots \geq \mu_m$ that interlaces the longer sequence. The interlacing is *tight* if, for some $j$,

$$\mu_i = \begin{cases} \theta_i & \text{for } i \leq j \\ \theta_{n-m+i} & \text{for } i \geq j+1. \end{cases}$$

In particular, the first $j$ values of the shorter sequence are as large as possible, while the remaining $m-j$ values are as small as possible.

**Theorem 5.1** (Courant-Hilbert-Haemers Theorem)**.** Take an integer $n \geq 2$, and let $A$ be an $n \times n$ real symmetric matrix with eigenvalues $\theta_1 \geq \cdots \geq \theta_n$. For some integer $1 \leq m < n$, let $S$ be an $n \times m$ real matrix such that $S^\top S = I_m$, and let $B := S^\top AS$. Let $v_1, \ldots, v_m$ be orthogonal eigenvectors for $B$ corresponding to eigenvalues $\mu_1 \geq \cdots \geq \mu_m$, respectively. Then the following statements hold:

1. the eigenvalues of $B$ interlace those of $A$,
2. if $\mu_i = \theta_i$ (resp. $\mu_i = \theta_{n-m+i}$), then $B$ has a $\mu_i$-eigenvector $v$ such that $Sv$ is a $\mu_i$-eigenvector for $A$,
3. if $\mu_i = \theta_i$ for $i = 1, \ldots, j$ (resp. $\mu_i = \theta_{n-m+i}$ for $i = j, \ldots, m$), then $Sv_i$ is a $\mu_i$-eigenvector for $A$ for $i = 1, \ldots, j$ (resp. $i = j, \ldots, m$),
4. if the interlacing is tight, then $SB = AS$.

*Proof.* **(1)** Let $u_1, \ldots, u_n$ be an orthogonal basis of eigenvectors for $A$ with eigenvalues $\theta_1, \ldots, \theta_n$, respectively. For each $i \in [m]$, take a nonzero vector $w_i \in \mathbb{R}^m$ in

$$\langle v_1, \ldots, v_i \rangle \cap \langle S^\top u_1, \ldots, S^\top u_{i-1} \rangle^\perp.$$

(For $i = 1$, the RHS is $\langle v_1 \rangle$.) As $w_i \in \langle v_1, \ldots, v_i \rangle$, CFT (1) and (3) applied to $B$ implies that

$$\frac{w_i^\top B w_i}{w_i^\top w_i} \geq \mu_i.$$

Our choice of $w_i$ implies that $Sw_i \in \langle u_1, \ldots, u_{i-1} \rangle^\perp$, so CFT (2) and (4) applied to $A$ implies that

$$\frac{w_i^\top S^\top A S w_i}{w_i^\top S^\top S w_i} \leq \theta_i.$$

Since $B = S^\top AS$ and $S^\top S = I_m$, we have

$$\theta_i \geq \frac{w_i^\top S^\top A S w_i}{w_i^\top S^\top S w_i} = \frac{w_i^\top B w_i}{w_i^\top w_i} \geq \mu_i.$$

A similar argument applied to $-A$ and $-B$ implies that $\mu_i \geq \theta_{n-m+i}$, thereby proving (1).

**(2)** If $\theta_i = \mu_i$, then equality holds throughout, so it follows from CFT parts (3) and (4) that $w_i$ is a $\mu_i$-eigenvector of $B$, while $Sw_i$ is a $\mu_i$-eigenvector of $A$, thereby proving (2).

**(3)** We proceed by induction on $j$. By the induction hypothesis, we may pick $u_i = Sv_i$ for $i = 1, \dots, j-1$.[1] We may therefore pick $w_j = v_j$. Since $\theta_j = \mu_j$, it follows from (2) that $Sv_j$ is a $\mu_j$-eigenvector, thereby completing the induction step.

**(4)** For some $j$,

$$\mu_i = \begin{cases} \theta_i & \text{for } i \leq j \\ \theta_{n-m+i} & \text{for } i \geq j+1. \end{cases}$$

Thus, by applying (3) twice, we get that $Sv_i$ is a $\mu_i$-eigenvector for $A$ for all $i \in [m]$. Consequently,

$$(SB - AS)v_i = SS^\top ASv_i - ASv_i = \mu_i SS^\top Sv_i - \mu Sv_i = 0 \quad \forall i \in [m].$$

Since $v_1, \dots, v_m$ is a basis for $\mathbb{R}^m$, it follows that $SB = AS$, as required. □

# 6 Applications of the Courant-Hilbert-Haemers Theorem

We leave it as an exercise for the reader to prove Cauchy's Interlacing Theorem as an application of the Courant-Hilbert-Haemers Theorem. For now, let us see another application. Let $A$ be an $n \times n$ real symmetric matrix, whose rows and columns

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix}$$

are partitioned according to a partition $X_1, \dots, X_m$ of $[n]$ into nonempty parts. Observe that $A_{ij} = A_{ji}^\top$. Denote by $S'$ the $n \times m$ matrix whose entries are defined as follows: $S'_{ij} = 1$ if $i \in X_j$, and $S'_{ij} = 0$ if $i \notin X_j$. The *quotient matrix* of the partition is the $m \times m$ matrix $B'$ whose $ij$-entry is equal to the average row sum of the block $A_{ij}$, that is,

$$B'_{ij} = \frac{1}{|X_i|}\mathbf{1}^\top A_{ij}\mathbf{1} = \frac{1}{|X_i|}(S'^\top AS')_{ij}.$$

The partition is *equitable* if each block $A_{ij}$ has constant row sum, that is, if $AS' = S'B'$. Since $A_{ij} = A_{ji}^\top$, each block of an equitable partition has constant column sum.

**Theorem 6.1.** Let $A$ be a real symmetric matrix that is partitioned symmetrically, and let $B'$ be the quotient matrix of the partition. Then the following statements hold:

1. the eigenvalues of $B'$ interlace those of $A$,
2. if the interlacing is tight, then the partition is equitable.

*Proof.* Let $D := \operatorname{Diag}(|X_1|, |X_2|, \dots, |X_m|)$. Then $DB' = S'^\top AS'$, implying in turn that $D^{\frac{1}{2}}B'D^{-\frac{1}{2}} = D^{-\frac{1}{2}}S'^\top AS'D^{-\frac{1}{2}}$. Thus, for

$$B := D^{\frac{1}{2}}B'D^{-\frac{1}{2}}$$
$$S := S'D^{-\frac{1}{2}},$$

[1]By Lecture 0, Exercise 4 (c), for any integer $1 \leq m < n$, any set of $m$ orthogonal eigenvectors can be extended to $n$ orthogonal eigenvectors.

we have $B = S^\top AS$. As $S^\top S = D^{-\frac{1}{2}} S'^\top S' D^{-\frac{1}{2}} = D^{-\frac{1}{2}} D D^{-\frac{1}{2}} = I_m$, it follows from the Courant-Hilbert-Haemers Theorem that the eigenvalues of $B$ interlace those of $A$, and if the interlacing is tight, then $SB = AS$, and so $S'B' = AS'$, meaning the partition is equitable. As $B, B'$ are similar matrices, they have the same eigenvalues, so the theorem follows. $\square$

Let us now present a powerful upper-bound on the stability number of a graph:

**Theorem 6.2.** Let $G$ be an $n$-vertex graph, let $A$ be an $n \times n$ real symmetric matrix where $A_{ij} \neq 0$ only if $i$ and $j$ are adjacent vertices of $G$, and let $\theta_1 \geq \cdots \geq \theta_n$ be the spectrum of $A$. Assume that $A$ has constant row sum $k > 0$. Then

$$\alpha(G) \leq n \cdot \frac{-\theta_n}{k - \theta_n}.$$

*Proof.* Let $S \subseteq V$ be a stable set of cardinality $\alpha := \alpha(G)$. Consider the symmetric partitioning of $A$ according to the partition $S, V \setminus S$ of the vertex set. The quotient matrix of this partition is

$$B' = \begin{pmatrix} 0 & k \\ \frac{k\alpha}{n-\alpha} & k - \frac{k\alpha}{n-\alpha} \end{pmatrix},$$

which has spectrum $\mu_1 = k, \mu_2 = -\frac{k\alpha}{n-\alpha}$ (note that $k$ is the constant row sum of $B'$, while the other eigenvalue is $\mathrm{tr}(B') - k$). By Theorem 6.1, $\mu_2 \geq \theta_{n-2+2} = \theta_n$, which in turn implies the desired inequality. $\square$

In the next lecture, we see how to use this bound to find the *Shannon capacity* of a 5-hole.


## Acknowledgements

The presentation of §5 and §6 follows Haemers [2] closely.


## References


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