MA431 Spectral Graph Theory: Lecture 4

Ahmad Abdi*   Neil Olver

Last time, we saw the following application of interlacing to graphs:

Theorem 6.2. Let G be an n-vertex graph, let A be an $n \times n$ real symmetric matrix where $A_{ij} \neq 0$ only if i and j are adjacent vertices of G, and let $\theta_1 \geq \cdots \geq \theta_n$ be the spectrum of A. Assume that A has constant row sum $k > 0$. Then

$$\alpha(G) \leq n \cdot \frac{-\theta_n}{k - \theta_n}.$$

Let us discuss an application of Theorem 6.2 to Information Theory.

6.1 The Shannon capacity of C_5

Let G be a graph over vertex set $[n]$, where the vertices are thought of as letters, and an edge as a pair of letters that can be confused with one another over a noisy channel of communication. Take an integer $k \geq 1$, to be thought of as the word length, to be sent over the channel.

Denote by G^ℓ the graph with vertex set $[n]^k$ where distinct vertices $(u_1, \dots, u_k), (v_1, \dots, v_k)$ are adjacent if for each $i \in [k]$, the two vertices u_i, v_i of G are either equal or adjacent. Observe that the vertices of G^k can be thought of as the set of all possible words of length ℓ, and the adjacencies can be thought of as words that can be confused for one another when sent over the channel. Consequently, $\alpha(G^\ell)$ is the maximum number of confusion-free length-ℓ words that can be sent over the channel. The ℓ^{th} root of this number is the effective number of signals transmitted per unit time step.

The *Shannon capacity of* G is defined as

$$\Theta(G) := \sup_\ell \sqrt[\ell]{\alpha(G^\ell)}.$$

Observe that $\Theta(G)$ is a characteristic parameter of the channel, calculating the effective number of signals transmitted per unit time step for arbitrarily long words. In 1956, Claude Shannon defined and studied this parameter [7]. Let us provide an upper bound on this parameter.

The *Kronecker product* of two matrices M and N, denoted $M \otimes N$, is the matrix whose row labels and column labels are $\text{row.labels(M)} \times \text{row.labels(N)}$ and $\text{col.labels(M)} \times \text{col.labels(N)}$, respectively, and whose entry in row (i, i') and column (j, j') is $M_{ij} N_{ii'}$. For example, the adjacency matrix of G^2 is $(A(G) + I) \otimes (A(G) + I) - I$. (Why is it not $A(G)^{\otimes 2}$?) Similarly, $A(G^\ell) = (A(G) + I)^{\otimes \ell} - I$.

Proposition 6.3. Let M, N be two matrices. Then the following statements hold:

1. $Mx \otimes Ny = (M \otimes N)(x \otimes y)$ for any two vectors x, y of appropriate dimensions,
2. if M, N are real symmetric matrices with spectra Λ_M, Λ_N, then $M \otimes N$ is a real symmetric matrix with spectrum $(\theta\theta' : \theta \in \Lambda_M, \theta' \in \Lambda_N)$,
3. if M is a real symmetric matrix whose least eigenvalue, say τ, is nonnegative, then the least eigenvalue of $M^{\otimes \ell}$ is τ^ℓ.

Proof. Exercise. □

We are now ready for the main result of this subsection:

Theorem 6.4. Let G be a k-regular graph on n vertices with least eigenvalue τ. Then

$$\Theta(G) \le n \cdot \frac{-\tau}{k-\tau}.$$

Proof. Take an integer $\ell \ge 1$. It suffices to show that

$$\alpha(G^\ell) \le \left(n \cdot \frac{-\tau}{k-\tau}\right)^\ell.$$

To this end, let $A := A(G)$ and $A_\ell := (A - \tau I)^{\otimes \ell} - (-\tau)^\ell I$. We shall use Proposition 6.3 several times without reference. As the matrix $A - \tau I$ has least eigenvalue 0, so does the matrix $(A-\tau I)^{\otimes \ell}$, so A_ℓ has least eigenvalue $-(-\tau)^\ell$. Moreover,

$$A_\ell \mathbf{1} = (A - \tau I)^{\otimes \ell}\mathbf{1} - (-\tau)^\ell I \mathbf{1} = (k-\tau)^\ell \mathbf{1} - (-\tau)^\ell \mathbf{1},$$

so A_ℓ has constant row sum $(k-\tau)^\ell - (-\tau)^\ell$. Furthermore, for any two non-adjacent vertices $u = (u_1, \dots, u_\ell)$ and $v = (v_1, \dots, v_\ell)$ of G^ℓ, we have

$$(A_\ell)_{uv} = \Pi_{i=1}^{\ell}(A - \tau I)_{u_i v_i} - (-\tau)^\ell \Pi_{i=1}^{\ell} I_{u_i v_i} = \left\{ \begin{array}{ll} (-\tau)^\ell - (-\tau)^\ell & \text{if } u = v, \\ 0 - 0 & \text{otherwise.} \end{array} \right\} = 0$$

Thus, by Theorem 6.2,

$$\alpha(G^\ell) \le n^\ell \cdot \frac{(-\tau)^\ell}{(k-\tau)^\ell},$$

as required. □

In the same 1956 paper where $\Theta(G)$ was defined, Shannon asked for the capacity of C_5, the odd cycle of length five. This question was finally answered by Lovász in 1979 [5]. Let us compute this using Theorem 6.4.

Corollary 6.5. The Shannon capacity of C_5 is $\sqrt{5}$.

Proof. It can be readily seen that C_5 has spectrum $\frac{-1-\sqrt{5}}{2}^{(2)}, \frac{-1+\sqrt{5}}{2}^{(2)}, 2$. In particular, its smallest eigenvalue is $\tau := \frac{-1-\sqrt{5}}{2}$. Thus, by Theorem 6.4,

$$\Theta(C_5) \le n \cdot \frac{-\tau}{k-\tau} = 5 \cdot \frac{1+\sqrt{5}}{5+\sqrt{5}} = \sqrt{5}.$$

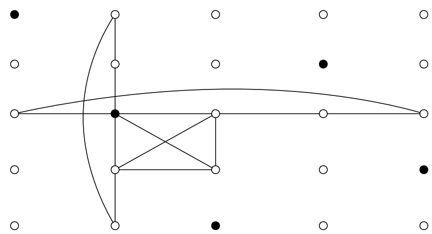

Figure 1: A partial representation of C_5^2. The filled-in vertices form a stable set of cardinality 5.

Conversely, note that C_5^2, displayed in Figure 1, has a stable set of cardinality 5, so

$$\Theta(G) \geq \sqrt{\alpha(C_5^2)} \geq \sqrt{5},$$

thereby proving the result. □

Exercises

1. Use Cauchy's Interlacing Theorem to prove that the Petersen graph has no Hamilton cycle.

2. Determine the graphs with smallest eigenvalue at least -1.

3. Prove Theorem 4.3.

4. Prove Lemma 4.5.

5. Prove Cauchy's Interlacing Theorem by using the Courant-Hilbert-Haemers Theorem.

6. Let $G = (V, E)$ be a k-regular graph with spectrum $k \geq \theta_2 \geq \cdots \geq \theta_n$. Prove that

$$\alpha(G) \leq n \cdot \frac{-\theta_n}{k - \theta_n}.$$

   Moreover, prove that if S is a stable set meeting this bound, then every vertex outside of S has exactly $-\theta_n$ neighbours inside S.

7. Let G be a graph on n vertices with minimum degree δ, and let $\theta_1 \geq \cdots \geq \theta_n$ denote its spectrum. Prove that

$$\alpha(G) \leq n \cdot \frac{-\theta_1\theta_n}{\delta^2 - \theta_1\theta_n}.$$

8. Let G be a graph on n vertices with at least one edge whose spectrum is $\theta_1 \geq \cdots \geq \theta_n$, and let x be an arbitrary eigenvector of $A := A(G)$. Let $X_1, \ldots, X_k$ be a partition of the vertex set into k nonempty stable sets, and let B be the $k \times k$ matrix where

$$B_{ij} = \frac{1}{\sum_{u \in X_i} x_u^2} \cdot \sum \left(x_u x_v : u \in X_i, v \in X_j,\ u, v \text{ are adjacent}\right).$$

(a) Prove that the spectrum of B interlaces the spectrum of A.

(b) Prove that $k \geq 1 - \frac{\theta_1}{\theta_n}$.

(c) Conclude that G has chromatic number at least $1 - \frac{\theta_1}{\theta_n}$.

9. Prove Proposition 6.3.

10. Let $G = (V, E)$ be a regular graph. Suppose S is a stable set such that every vertex in $V \setminus S$ has a unique neighbour in S. Prove that -1 is an eigenvalue of $A(G)$.

11. Take an odd integer $n \geq 7$.

(a) What is the spectrum of the cycle C_n of length n?

(b) Compute the upper bound given in Theorem 6.4 on the Shannon capacity of C_n.

7 The Laplacian matrix and spectrum

Let $G = (V, E)$ be a graph (recall that loops are not allowed by parallel edges are). Denote by $\Delta(G)$ the diagonal matrix corresponding to the vertex degrees of G. That is, the rows and columns of $\Delta(G)$ are indexed by V, and for each vertex $u \in V$, the uu-entry of $\Delta(G)$ is equal to $\deg(u)$. Recall that $A(G)$ is the adjacency matrix of G.

Definition 7.1. The *Laplacian matrix of* G is the real symmetric matrix $\Delta(G) - A(G)$.

An *orientation of* G is a directed graph D that is obtained from G by orienting every edge in an arbitrary direction. The *incidence matrix of* D is the $0, \pm 1$ matrix whose rows and columns are indexed by the vertices and arcs, respectively, where column (v, u) is equal to $e_u - e_v$.

Proposition 7.2. Let L be the Laplacian matrix of G. Then

1. $L = MM^\top$, where M is the incidence matrix of any orientation of G,

2. $L = \sum_{\{u,v\} \in E} (e_u - e_v)(e_u - e_v)^\top$,

3. for every $x \in \mathbb{R}^V$,
$$x^\top L x = \sum_{\{u,v\} \in E} (x_u - x_v)^2.$$

In particular, L is a positive semidefinite matrix.

Proof. Exercise. □

Definition 7.3. The *Laplacian spectrum of* G is the spectrum of its Laplacian matrix. If G has n vertices, then its spectrum is denoted $\lambda_1(G) \leq \cdots \leq \lambda_n(G)$.[1]

[1] Note that for the Laplacian spectrum, λ_1 denotes the least eigenvalue, while for the usual spectrum, θ_1 denotes the largest eigenvalue.

For general graphs, the Laplacian spectrum and the spectrum are not related; for example, it is possible for two cospectral graphs to have different Laplacian spectra (see Exercise 2). For regular graphs, however, the situation is different:

Theorem 7.4. Let G be an n-vertex graph that is k-regular. If G has spectrum $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$, then its Laplacian spectrum is $k - \theta_1 \leq k - \theta_2 \leq \ldots \leq k - \theta_n$.

Proof. Let $A := A(G)$, and let $v_1, \ldots, v_n$ be eigenvectors of A with eigenvalues $\theta_1, \ldots, \theta_n$, respectively. Let $L := \Delta(G) - A$ be the Laplacian matrix of G. As G is k-regular, $\Delta(G) = kI$, so $L = kI - A$. Subsequently,

$$Lv_i = (kI - A)v_i = (k - \theta_i)v_i,$$

implying in turn that $v_1, \ldots, v_n$ are also eigenvectors of L with eigenvalues $k - \theta_1, \ldots, k - \theta_n$, as claimed. □

Given that the Laplacian matrix is positive semidefinite, its eigenvalues are nonnegative. In fact, the least eigenvalue of the Laplacian spectrum is guaranteed to be 0:

Proposition 7.5. Let G be a graph with c connected components, let L be its Laplacian matrix, and let $\lambda_1 \leq \cdots \leq \lambda_n$ be the Laplacian spectrum. Then the following statements hold:

1. $L\mathbf{1} = \mathbf{0}$, that is, $\mathbf{1}$ is an eigenvector with eigenvalue 0. In particular, $\lambda_1 = 0$.

2. If $Lx = \mathbf{0}$, then x takes the same value on the vertices of each connected component of G.

3. $\mathrm{rank}(L) = n - c$. Equivalently, the eigenvalue 0 of L has multiplicity c.

Proof. **(1)** follows immediately from the definition of the Laplacian matrix.

(2) Let x be a vector such that $Lx = \mathbf{0}$. By Proposition 7.2,

$$0 = x^\top Lx = \sum_{\{u,v\} \in E} (x_u - x_v)^2,$$

implying that $x_u = x_v$ whenever u, v are adjacent. Thus x takes the same value on the vertices of each connected component, as required.

(3) Let $V_1, \ldots, V_c$ be the vertex sets of the connected components of G. For each $i \in [c]$, let $v_i \in \{0,1\}^V$ be the incidence vector of V_i. We claim that $v_1, \ldots, v_c$ is a basis for the null space of L, i.e. $\{x : Lx = \mathbf{0}\}$. Clearly, $Lv_i = \mathbf{0}$, and the v_i are linearly independent. Now choose a vector x such that $Lx = \mathbf{0}$. Then, by (2), x is a linear combination of $v_1, \ldots, v_c$. Thus, $v_1, \ldots, v_c$ is a basis for the null space of L, implying in turn that $\mathrm{rank}(L) = n - c$. □

Given that the least Laplacian eigenvalue is zero, one may ask questions about the second least Laplacian eigenvalue of a graph. Fiedler [2] calls $\lambda_2(G)$ the *algebraic connectivity of* G.

One can also get an upper-bound of n on the largest eigenvalue $\lambda_n(G)$ of the Laplacian of a simple graph G – see Exercise 3.

8 The Matrix-Tree Theorem

Let $G = (V, E)$ be an n-vertex graph, and let L be the Laplacian matrix. Denote by $T(G)$ the number of spanning trees of a graph – so if G is not connected, this number is zero. In this section, we prove Kirchhoff's *Matrix-Tree Theorem*, which states that $T(G)$ is equal to the determinant of any $(n-1) \times (n-1)$ principal submatrix of L.

The Matrix-Tree Theorem is by and large a consequence of the Laplace (cofactor) expansion for the determinant, combined with a powerful *deletion-contraction* recursive formula for $T(G)$. To elaborate on the latter, let G be a graph, and let e be an edge. The *deletion* $G \setminus e$ is the graph obtained from G after removing the edge e. The *contraction* G/e is the graph obtained after identifying the ends of G, and deleting all the loops created.[2] Observe that contracting may create (additional) parallel edges.

Lemma 8.1. Let G be a graph. Then for every edge e,

$$T(G) = T(G/e) + T(G \setminus e).$$

Proof. The spanning trees of G can be separated into two groups, those that contain the edge e, and those that do not. The ones in the second group are precisely the spanning trees of $G \setminus e$. The ones are in the first group, however, are in correspondence with the spanning trees of G/e. More precisely, if T' is a spanning tree of G/e then $T' \cup \{e\}$ is a spanning tree of G containing e, and if T is a spanning tree of G containing e then $T - \{e\}$ is a spanning tree of $G \setminus e$. The formula above is an immediate consequence of this grouping of the spanning trees of G. $\square$

We are now ready for the main result of this section:

Theorem 8.2 (Matrix-Tree Theorem)**.** Let G be an n-vertex graph, and let L be its Laplacian matrix. Then $T(G)$ is equal to the determinant of any $(n-1) \times (n-1)$ principal submatrix of L.

Proof. We shall proceed by induction on the number of edges of G. If G has no edge, then $T(G) = 0$, and since L is the zero matrix, the result follows. If $n = 2$, given that G has m parallel edges between the two vertices, we have $T(G) = m$, so the result follows since $L = \left(\begin{smallmatrix} m & -m \\ -m & m \end{smallmatrix}\right)$. For the induction step, assume that G has at least one edge, and $n \geq 3$. For $i \in [n]$, denote by $L[i]$ the principal submatrix of L obtained after removing row i and column i. It suffices to prove that $\det(L[n]) = T(G)$.

If G is not connected, then Proposition 7.5 implies that L has rank at most $n-2$, so $L[n]$ is a singular matrix, implying that $\det(L[n]) = 0 = T(G)$.

Otherwise, G is connected. Pick an edge e incident with n, say $e = \{n-1, n\}$. Consider the deletion $G \setminus e$ and the contraction G/e. For the latter, denote by $n-1$ the vertex obtained from identifying $n-1, n$. Denote by L^d, L^c the Laplacian matrices of $G \setminus e, G/e$, respectively. By the induction hypothesis, $T(G \setminus e) = \det(L^d[n])$ and $T(G/e) = \det(L^c[n-1])$. Let us recalculate the two determinants in terms of subdeterminants of L.

[2] In general, loops are not deleted after edge contractions, but in our context we must.

First, observe that $L^c[n-1] = L[n][n-1]$, so

$$\det(L^c[n-1]) = \det(L[n][n-1]).$$

Secondly, observe that L^d, L differ in only four entries, namely, $L^d_{n-1,n-1} = L_{n-1,n-1} - 1$, $L^d_{n,n} = L_{n,n} - 1$, $L^d_{n-1,n} = L_{n-1,n} + 1$ and $L^d_{n,n-1} = L_{n,n-1} + 1$. Subsequently, by a Laplace expansion along row $n-1$ of $L^d[n]$, we see that

$$\det(L^d[n]) = \det(L[n]) - \det(L[n][n-1]).$$

Consequently,

$$T(G \setminus e) + T(G/e) = \det(L^d[n]) + \det(L^c[n-1]) = \det(L[n]).$$

By Lemma 8.1, however, the LHS is equal to $T(G)$, so $T(G) = \det(L[n])$, thereby completing the induction step. □

As a consequence, we get a proof of *Cayley's formula*:

Corollary 8.3. For every integer $n \geq 2$, the number of spanning trees K_n is n^{n-2}.

Proof. Let L be the Laplacian matrix of K_n. Then $L = (n-1)I - (J - I) = nI - J$, where J is the all-ones matrix. As a result, any $(n-1) \times (n-1)$ principal submatrix of L is equal to $nI_{n-1} - J_{n-1}$. The spectrum of this submatrix is $1, n^{(n-2)}$, implying in turn that it has determinant n^{n-2}. The result now follows from the Matrix-Tree Theorem. □

We also have the following consequence of the Matrix-Tree Theorem:

Theorem 8.4. Let G be a graph, and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be its Laplacian spectrum. Then

$$T(G) = \frac{1}{n} \prod_{i=2}^{n} \lambda_i.$$

Proof. Exercise. □

For another application of the Matrix-Tree Theorem, see Exercise 6.

9 Extensions to weighted graphs

The Matrix-Tree Theorem has a useful extension to the weighted setting. Let $G = (V, E)$ be a graph. The *Kirchhoff polynomial of* G is the following polynomial with variables $(x_e : e \in E)$:

$$\mathrm{Kir}(G; x) := \sum_{T \text{ a spanning tree}} \prod_{e \in T} x_e.$$

Observe that if $w \in \mathbb{R}^E$ is a set of edge weights, then $\mathrm{Kir}(G; w)$ computes the sum of the "multiplicative weights" of the spanning trees. In particular, $\mathrm{Kir}(G; \mathbf{1})$ is nothing but the number of spanning trees of the graph G, i.e. $\mathrm{Kir}(G; \mathbf{1}) = T(G)$. Much like $T(G)$, the Kirchhoff polynomial has a powerful recursive formula.

Lemma 9.1. For every edge,

$$\mathrm{Kir}(G; x) = x_e \cdot \mathrm{Kir}(G/e; x^e) + \mathrm{Kir}(G \setminus e; x^e),$$

where x^e denotes the vector obtained from x after dropping the coordinate corresponding to e.

Proof. Exercise. □

Define $L(G, x)$ to be the matrix whose rows and columns are indexed by the vertices, and whose entries are defined as follows:

1. for each vertex u, the uu-entry is $\sum(x_e : e$ is an edge incident with $u)$,
2. for adjacent vertices u, v, the uv-entry is $-\sum(x_e : e$ has ends $u, v)$,
3. for non-adjacent vertices u, v, the uv-entry is 0.

Theorem 9.2. Let G be an n-vertex graph, let $w \in \mathbb{R}^E$, and let $L_w := L(G, w)$. Then $\mathrm{Kir}(G; w)$ is equal to the determinant of any $(n-1) \times (n-1)$ principal submatrix of L_w.

Proof. Exercise. □

We are now ready to define the Laplacian matrix in a particular weighted setting:

Definition 9.3. Let $G = (V, E)$ be a graph, and let $w \in \mathbb{R}^E_+$. The *Laplacian matrix* of the weighted graph (G, w) is the matrix $L(G, w)$.

Observe that the Laplacian matrix of the weighted graph $(G, \mathbf{1})$ is just the Laplacian matrix of the graph G. The nonnegativity of the edge weights is needed in order to guarantee the positive semidefinite-ness of the Laplacian matrix. More generally, we have the following:

Proposition 9.4. Let $G = (V, E)$ be a graph, let $w \in \mathbb{R}^E_+$, and let L_w be the Laplacian of the weighted graph (G, w). Then the following statements hold:

1. $L_w = \sum_{e=\{u,v\}\in E} w_e \cdot (e_u - e_v)(e_u - e_v)^\top$,
2. for each $x \in \mathbb{R}^V$, $x^\top L_w x = \sum_{e=\{u,v\}\in E} w_e(x_u - x_v)^2$,
3. L_w is a positive semidefinite matrix,
4. $\mathbf{1}$ is an eigenvector of L_w with eigenvalue 0,
5. if every edge has nonzero weight, then L_w has rank $n - c$, and 0 as an eigenvalue has multiplicity c, where c is the number of connected components of G

Proof. Exercise. □

In the weighted setting, for all intents and purposes, we may assume that the graph G is simple (i.e. it has no loops or parallel edges), and every edge has a strictly positive weight. These two assumptions can be made after deleting all edges of weight zero, and after collapsing all parallel edges to a single edge whose weight is the sum of the previous weights.

Exercises

1. Prove Proposition 7.2.

2. Recall the cospectral pair of graphs from Lecture 1, displayed in Figure 2. Find the Laplacian spectrum of

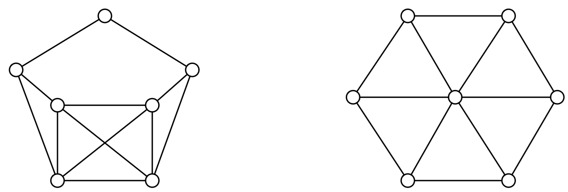

Figure 2: Two cospectral graphs with different Laplacian spectra.

each graph. Then conclude that cospectral graphs may not necessarily have the same Laplacian spectra.

3. Let G be an n-vertex simple graph, and let $\overline{G}$ be its complement. Prove the following statements:

   (a) $\lambda_i(\overline{G}) = n - \lambda_{n-i+2}(G)$ for $2 \leq i \leq n$,

   (b) $\lambda_n(G) \leq n$,

   (c) if $\overline{G}$ has $\overline{c}$ connected components, and $\overline{c} \geq 2$, then $\lambda_n(G) = n$ and its multiplicity is $\overline{c} - 1$.

4. Let $G = (V, E)$ be a connected graph, let $\lambda_2 := \lambda_2(G)$, and let $f \in \mathbb{R}^V$ be a corresponding eigenvector.

   (a) Call a path $(u_1, \ldots, u_r)$ *strictly decreasing* if $f_{u_1} > \cdots > f_{u_r}$. Prove that if $f_u > 0$, then it is joined by a strictly decreasing path to some vertex v such that $f_v \leq 0$.

   (b) Prove that for any $c \leq 0$, the graph induced on the vertex set $\{v \in V : f_v \geq c\}$ is connected.

5. Prove Theorem 8.4.

6. Let M be an $n \times n$, and let C be its cofactor matrix. Recall that C is an $n \times n$ matrix whose ij-entry is $(-1)^{i+j}$ times the determinant of the submatrix of M obtained after removing row i and column j. By a Laplace expansion along any row of M, we get the matrix equation $C^\top M = \det(M)I$. The matrix $C^\top$ is called the *adjugate of* M, and denoted $\operatorname{adj}(M)$.

   Let G be a graph, and let L be its Laplacian matrix. Prove that every entry of $\operatorname{adj}(L)$ is equal to $T(G)$.

7. Prove Lemma 9.1.

8. Prove Theorem 9.2.

9. Prove Proposition 9.4.

10. Let G be a connected graph on n vertices. Prove that

$$\lambda_2(G) = \min_x \frac{n \sum_{ij \in E}(x_i - x_j)^2}{\sum_{i<j}(x_i - x_j)^2}$$

where the minimum is taken over all non-constant vectors x.

11. Let T be a tree. Prove that $\lambda_2(T) \leq 1$, and equality holds if and only if T is a star.

12. The *Cartesian product* of two graphs G, H, denoted $G \square H$, is the graph over vertex set $V(G) \times V(H)$, where (u_1, u_2) and (v_1, v_2) are adjacent if $u_1 = v_1$ and u_2, v_2 are adjacent in H, or $u_2 = v_2$ and u_1, v_1 are adjacent in G.

    Prove that $\lambda_2(G \square H) = \min\{\lambda_2(G), \lambda_2(H)\}$.

13. Recall from Lecture 2 that Q_n is the skeleton graph of the n-dimensional unit hypercube. Prove that $\lambda_2(Q_n) = 2$.

14. Let G be a connected graph on n vertices and with diameter d. Prove that $\lambda_2(G) \geq \frac{1}{nd}$.

15. Let L_1, L_2 be positive semidefinite matrices of the same dimensions such that $L_1 \succcurlyeq L_2$. Prove that the k^{th} largest eigenvalue of L_1 in its spectrum is greater than or equal to the k^{th} largest eigenvalue of L_2 in its spectrum.

16. Let $(G, w), (H, w')$ be weighted graphs on the same number of vertices and with positive edge weights. Let L_1, L_2 be the Laplacian matrices of $(G, w), (H, w')$, respectively. We write $(G, w) \succcurlyeq (H, w')$ if $L_1 \succcurlyeq L_2$.

    Prove that

$$(P_n, (n-1)\mathbf{1}) \succcurlyeq (A_n, \mathbf{1})$$

    where the weighted graph on the left is the path on vertices $\{1, 2, \ldots, n\}$ with edges $\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}$, whose weights are equal to $n-1$, and the weighted graph on the right is the graph on the same vertex set with just one edge, $\{1, n\}$, whose weight is 1.

17. (a) Prove that $\lambda_2(K_n) = n$.

    (b) Prove that $\lambda_2(P_n) \geq \frac{6}{(n+1)(n-1)}$.

Acknowledgements

The presentation of §7 and §8 is inspired by [3], Chapter 13.

The Matrix-Tree Theorem dates back to the 1800s. Gustav Kirchhoff proved the "dual" of it in 1847 [4], but it was James Maxwell who stated the result explicitly in *A Treatise on Electricity and Magnetism, I* [6] (see

Part II, Chapter 6, pp. 329-337). The theorem, as is, was stated and proved by Trent [8]. See also [1] for other references.

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