Spectral graph theory

Lecturers: Dr Ahmad Abdi, Dr Neil Olver

Permanent meeting room coordinates: click here.

Meeting times: Lectures are held every **Wednesday, 9-11am**. Seminars are held every **Tuesday, 11-12pm**.

Spectral graph theory is concerned with how combinatorial properties of graphs relate to the algebraic structure of certain matrices associated with the graph. One can look at the adjacency matrix of an undirected graph, which is a symmetric matrix, and consider the list of its eigenvalues, called the *spectrum*, along with the corresponding eigenvectors. The spectrum gives us important insight about the graph and its induced subgraphs, and perhaps surprisingly, this viewpoint can be used in the design of graph algorithms, such as network flow problems, plane drawings of planar graphs, isomorphism testing, etc.

The course will consist of (live) lectures on Zoom. Additional reading may be prescribed. There will be opportunities to discuss the course material as well as the homework assignments.

There will be one 2h lecture and one 1h seminar per week. The lecture/seminar schedule will be made in consultation with the students taking the course.

Prerequisites

This year, the course will be offered primarily to PhD students, and we are targeting the course with this in mind. Basics of linear algebra and graph theory will be assumed, but we expect that PhD students in the department working on combinatorics, optimisation, game theory, etc. will have the necessary background. Some background reading on linear algebra may be provided before the course starts, since a refresher may be helpful.

Outline

The course will begin with a rigorous but accessible treatment of core topics in spectral graph theory.

- Introduction to the adjacency matrix and its spectrum. The Perron-Frobenius theorem.
- Interlacing of eigenvalues. Application to the sensitivity conjecture.
- The Laplacian. The matrix tree theorem.
- The connection to random walks, electrical networks, and spanning trees.
- Graph partitioning: sparse cuts and Cheeger's inequality.

The latter part of the course will build on the basics and cover some more advanced topics. *These topics are subject to change.*

- The *matching polynomial* is a polynomial whose coefficients count matchings of various sizes. Despite this combinatorial definition, this polynomial amazingly has real roots! We consider the proof of this theorem and why it is important.
- *Expander graphs* are sparse graphs that are very well connected: the number of edges crossing any large set S is large. This expansion property is related to the spectrum of the Laplacian (in particular, the *spectral gap*). *Ramanujan graphs* are regular graphs with *optimal* spectral gap, and so are in a certain sense the best possible expanders.

  We discuss a proof of the existence of bipartite Ramanujan graphs of arbitrary degree, using *interlacing families* of polynomials.

Time permitting, we may also cover:

- Tutte's plane embedding theorem: how to find a plane embedding of a planar graph.
- Given two graphs, are they isomorphic? This is a notorious problem from an algorithmic perspective. Spectral methods can resolve it in some special cases.

Assessment

There will be written assignments during the term that will constitute the mark for the course. There is no final exam.

Discussing with your fellow students is allowed, but you *must* state on your assignment everyone you discussed with, as well as any references you used (aside from the references we have pointed you to). Your solutions must, however, be written completely alone.

References

Various references will be used, including provided lecture notes.

- Spectral and algebraic graph theory (draft) by Daniel Spielman. This book is quite accessible, but (being a draft) somewhat rough in places.
- Algebraic graph theory (Springer, 2001) by Chris Godsil and Gordon Royle. This book gives a rigorous treatment of fundamental notions in Algebraic Graph Theory.