# London Taught Course Centre

**2017 examination**

# Graph Theory

**Instructions to candidates**

This open-book exam has 3 questions, all of which are in some way about proper vertex-colourings. Some parts are harder than others, and you should not expect to be able to answer all questions completely. Substantial credit will be given for partial answers and ideas which you cannot justify, provided that you clearly distinguish between statements which you believe but do not see how to prove, statements which you believe you have proved, and statements you think are obvious enough not to need a proof.

You may wish to use Internet searches in addition to the lecture notes. This is allowed.

**1** Given a graph $G$, the *square* of $G$, written $G^2$, is the graph on $V(G)$ with edges $xy \in E(G^2)$ if and only if there is a path with at most two edges in $G$ from $x$ to $y$. Recall that $\Delta(G)$ is the maximum degree of $G$, and $\chi(G)$ is the chromatic number of $G$.

(a) Show that if $T$ is a tree, then $\chi\big(T^2\big) = \Delta(T) + 1$.

(b) Show that if $G$ is a planar graph, and $\Delta(G) \geq 5$, then $\chi(G^2) \leq 9\Delta(G) - 19$.

(c) Show that if $G$ is a planar graph, then $\chi\big(G^2\big) \leq \frac{3}{2}\Delta(G) + 5$.

**2** (a) Show that it is NP-complete to decide whether a graph $G$ can be properly vertex-coloured with three colours.

(b) Show that for each $\gamma > 0$ there is a polynomial-time algorithm which takes as input a graph $G$ on $n$ vertices and which has the following behaviour. If $G$ has a proper 3-colouring, then the algorithm must return 'Yes'. If, for every $S \subset E(G)$ with $|S| \leq \gamma n^2$, the graph $G - S$ does not have a proper 3-colouring, then the algorithm must return 'No'. If neither condition is satisfied, the algorithm may answer either 'Yes' or 'No'. *You may use the fact that there is an algorithm which takes as input a graph $G$ and a parameter $\varepsilon > 0$, which returns an $\varepsilon$-regular partition of $V(G)$ with between $\varepsilon^{-1}$ and $K$ parts, and whose running time, for any fixed $\varepsilon > 0$, is polynomial in $n$. Here $K = K(\varepsilon)$ does not depend on $n$.*

**3** (a) Show that if $H$ is an $m$-vertex graph which does not contain $K_3$ and all of whose vertices have degree strictly greater than $\frac{2m}{5}$, then $H$ can be properly vertex-coloured with two colours.

(b) Show that if $G$ is an $n$-vertex graph which does not contain $K_4$ and all of whose vertices have degree strictly greater than $\frac{5n}{8}$, then $G$ can be properly vertex-coloured with three colours.

(c) Show that for each integer $C$ and each sufficiently large $n$ (depending on $C$) there exists an $n$-vertex graph $G$ which does not contain $K_3$, all of whose vertices have degree at least $\frac{n}{\log n}$, and which cannot be properly vertex-coloured with $C$ colours.