London Taught Course Centre

2017 examination

Graph Theory

Answers

1 (a) Given a tree T, we obtain an order $V(T) = \{v_1, \ldots, v_n\}$ by removing a leaf (degree one vertex) v_n from T and repeating this in the resulting tree $T_{n-1} := T \setminus \{v_n\}$, and so on until T_1 is a single vertex v_1.

Observe that the vertices of T^2 adjacent to v_i which come before v_i in this order are the unique v_j such that $v_i v_j \in E(T)$ and $j < i$, and the set $N(v_j) \cap \{v_1, \ldots, v_{i-1}\}$.

In total at most $\Delta(T)$ neighbours in T^2 of v_i precede v_i in the ordering; so the greedy colouring algorithm, run on T^2 in this order, uses at most $\Delta(T) + 1$ colours. In the other direction, letting v be a vertex of T with $d(v) = \Delta(T)$, the set $\{v\} \cup N(v)$ is a set of $\Delta(T) + 1$ vertices which form a clique in T^2, hence $\chi(T^2) \geq \Delta(T) + 1$.

(b) This is obtained by modifying the tree argument from (a): namely, given a planar graph G, order $V(G)$ such that each vertex has at most five predecessors in the order (it was proved in lectures that this is possible). Colouring greedily in this order, when we come to colour v_i, each of its at most five predecessors has at most $\Delta(G) - 1$ neighbours other than v_i, which could all be before v_i in the order; in addition the at most $\Delta(G) - 5$ neighbours of v_i which come after it in the order have at most five predecessors each in the order, one of which is v_i but the rest of which could all come before v_i. No other vertex at distance one or two in G from v_i can precede v_i, so we conclude that in G^2 at most $5\Delta(G) + 4(\Delta(G) - 5)$ neighbours of v_i precede v_i in the order. The greedy algorithm thus uses at most $9\Delta(G) - 19$ colours, as desired.

(c) This is an open problem. More accurately, there are planar graphs G with $\Delta(G) \leq 3$ such that $\chi(G^2) = 7$, for $4 \leq k \leq 7$ there are planar graphs G with $\Delta(G) \leq k$ such that $\chi(G^2) = k + 5$, and for $k \geq 8$ that there are planar graphs G with $\Delta(G) \leq k$ such that $\chi(G^2) = \lfloor \frac{3\Delta(G)}{2} \rfloor + 1$, and it is conjectured (by Wegner) that for each value of k there are no planar graphs G with $\Delta(G) \leq k$ such that G^2 has larger chromatic number. A proof for the $k = 3$ case was announced in April 2016 by Hartke, Jahanbekam and Thomas (a previous 2006 announcement by Thomassen of the same seems to have suffered from a flaw in the proof); all other cases remain open at the time of writing. However it is known that the conjecture is at least close to true: Havet, van den Heuvel, McDiarmid and Reed showed in 2008 that for each $\gamma > 0$, if k is large enough then all planar graphs G with $\Delta(G) \leq k$ are such that G^2 can be properly vertex-coloured with $\left(\frac{3}{2} + \gamma\right)k$ colours.

Note that the cases $\Delta(G) = 1, 2$ are fairly easy to settle; the square of a matching ($\Delta(G) = 1$) is a matching and has chromatic number 2; the square of a path or cycle ($\Delta(G) = 2$) is always 4-colourable except for the square of C_5, which is K_5 and needs five colours. In these cases the restriction to planar graphs is superfluous.

For the specific (non-optimal but still an open problem) bound I asked for, you can also do $\Delta(G) = 3$ fairly quickly (thanks to the student who pointed this out—I didn't spot it). If G is planar and has maximum degree 3, we want to show $\chi(G^2) \leq \frac{3}{2} \cdot 3 + 5$, in other words $\chi(G^2) \leq 9$. We can assume G is connected (otherwise we can colour components of G^2 independently). Now G^2 has maximum degree at most 9, because any given $v \in G$ has at most three neighbours each of which has at most two neighbours other than v. So by Brooks' Theorem, we are done unless G^2 is K_{10}. But the only way we can get $G^2 = K_{10}$ is if each vertex of G^2 has degree 9; in other words, for each vertex $v \in G$, we have $d(v) = 3$, the neighbours of v form an independent set, and there are six second neighbours of v (the neighbours of the neighbours of v which aren't v), which means that G is a 3-regular graph on 10 vertices which doesn't contain a triangle or a C_4. There is only one such graph. To see this, fix a vertex v, its neighbours x, y, z and their respective neighbours $x', x'', y', y'', z', z''$. The only edges we don't already know in G are those between $x', x'', y', y'', z', z''$ which have to form a 2-regular graph. There are only two 2-regular graphs on six vertices, namely C_6 and two disjoint K_3. But G doesn't contain a triangle, so we have to have C_6. And because G doesn't contain a triangle or C_4, the pair x', x'' have to be opposite on the C_6, and the same for y', y'' and z', z''. This fixes the graph up to isomorphism: it is the Petersen graph, which contains K_5 as a minor (check it!) and therefore is not planar.

2 (a) This is a classic NP-completeness result, mentioned as such in the lectures. If you figured it out on your own, well done—but you should have realised that this must be easy to find online, and indeed there are several different routes that show up on the first page of a Google search.

(b) Given $\gamma > 0$, choose $d = \gamma/10$ and $\varepsilon = d/10$. Now given G, take an ε-regular partition $V_0, \ldots, V_t$ of $V(G)$ with $t+1$ parts, where $\varepsilon^{-1} \le t \le K(\varepsilon)$ parts, which we are told is possible in polynomial time. Draw a graph H on $[t]$ by putting an edge ij whenever (V_i, V_j) is a pair in G of density at least d. We can construct this graph in time $O(n^2)$, as we simply have to count edges. We examine all 3-colourings of $[t]$. We answer 'Yes' if there is a 3-colouring of $[t]$ such that at most εt^2 edges are not properly coloured, and otherwise 'No'. This last step takes a constant (independent of n) time and hence the algorithm in total runs in polynomial time in n.

If G is 3-colourable, fix a proper 3-colouring c of $V(G)$. We derive a colouring c' of $[t]$ by colouring i with a majority colour used on V_i (breaking ties arbitrarily). Observe that if ij is an edge of H such that $c'(i) = c'(j)$, then (V_i, V_j) is not ε-regular in G, because otherwise by ε-regularity there would be an edge from the vertices in V_i of colour $c'(i)$ to the vertices in V_j of the same colour. Since an ε-regular partition contains at most εt^2 pairs which are not ε-regular, there are at most εt^2 edges of H which are not properly coloured by c'; so our algorithm indeed returns 'Yes'.

If our algorithm returns 'Yes', fix a colouring c' of $[t]$ such that at most εt^2 edges are not properly coloured. We construct a set S of edges of G as follows: we put into S each edge intersecting V_0, each edge in a part V_i, and each edge between a pair (V_i, V_j) such that $c'(i) = c'(j)$. Now let c on $V(G)$ be defined by $c(v) = 1$ if $v \in V_0$, and $c(v) = c'(i)$ if $v \in V_i$. By definition of S this is a proper 3-colouring of $G - S$; on the other hand, since $|V_0| \le \varepsilon n$ there are at most εn^2 edges intersecting V_0, for each i since $|V_i| \le n/t$ there are at most n^2/t^2 edges in V_i, and hence at most $n^2/t \le \varepsilon n^2$ edges within parts. Since there are at most εt^2 pairs ij which are edges of H but not properly coloured, at most $\varepsilon t^2 (n/t)^2 = \varepsilon n^2$ edges are in pairs (V_i, V_j) such that ij is an edge of H not properly coloured, and by definition less than $d(n/t)^2 \binom{t}{2} \le dn^2$ edges are in pairs (V_i, V_j) such that ij is not an edge of H. This covers all the edges which can be in S, so $|S| \le (3\varepsilon + d)n^2 < \gamma n^2$. In particular, if our algorithm returns 'Yes' then there is a set S of at most γn^2 edges of G such that $G - S$ is 3-colourable.

3 (a) Let first I_1 and I_2 be disjoint independent sets in $V(H)$ of size greater than $2m/5$. Such sets exist since we can pick an edge xy and then $N(x)$ and $N(y)$ are examples of such sets. Either I_1 and I_2 form a bipartition of H and we are done, or there is some z in neither I_1 nor I_2. If z has a neighbour w in I_1, then z has at most $m/5$ neighbours outside I_1, otherwise w is adjacent neither to the vertices of I_1 nor to the vertices $N(z)\setminus I_1$, and this leaves less than $2m/5$ possible neighbours, a contradiction. It follows that we can add z to one of I_1 and I_2 to get a larger pair of disjoint independent sets in $V(H)$; repeating this we obtain the desired bipartition of H.

(b) Observe that by (a), the graph $G\big[N(v)\big]$ is bipartite for each $v \in V(G)$. Let I_1, I_2, I_3 be pairwise disjoint independent sets in G, each of size greater than $n/4$, with $|I_1| > 5m/16$ and with $|I_2 \cup I_3| > 5n/8$. Such sets exist since we can pick $v \in V(G)$ and find a set I_1 in $N(v)$, then I_2 and I_3 are obtained as the bipartition of $N(u)$ for some $u \in I_1$.

Let xy be any edge of G. If $\big|N(x) \cap N(y)\big| \geq 3n/8$, then since $\delta(G) > 5n/8$ there is an edge in $N(x) \cap N(y)$; in other words G contains K_4. Now if x is any vertex not in I_i (for some $i \in \{1, 2, 3\}$) which has at least one, but not more than $|I_i| - n/8$, neighbours in I_i then x and a neighbour $y \in I_i$ necessarily have $\big|N(x) \cap N(y)\big| \geq 3n/8$.

If I_1, I_2 and I_3 do not give the desired three-colouring of $V(G)$, then there is some z not in any I_i. If for some i, z has no neighbours in I_i then we can add z to I_i to obtain a larger triple of sets. But if z has at least one neighbour in each I_i, then it has at most $n/8$ non-neighbours in each I_i by the last paragraph. Choose a neighbour x of z in I_1 and a common neighbour y of x and z in I_2; this is possible since y has $|I_1|$ non-neighbours in I_1, and so less than $3n/8 - |I_1|$ non-neighbours in I_2. Since $3n/8 - |I_1| + n/8 \leq |I_2|$, the desired common neighbours exist. Now there is a common neighbour of x, y and z in I_3 by similar logic: z has at most $n/8$ non-neighbours in I_3, x has at most $3n/8 - |I_1|$ such non-neighbours, and y has at most $3n/8 - |I_2|$ non-neighbours. In total there are at most $7n/8 - |I_1| - |I_2|$ vertices in I_3 which are not common neighbours of x, y and z; since $|I_1| + |I_2| + |I_3| > 15n/16$, the desired common neighbour exists. This gives a copy of K_4 in G, a contradiction.

It follows that we can sequentially add vertices I_1, I_2 and I_3 to obtain the desired tripartition of $V(G)$.

(c) There are several ways to do this. Probably the easiest is to take a vertex-minimal graph F which does not contain K_3 and which cannot be coloured with C colours; we proved such graphs exist in the course. Now for each sufficiently large n we construct an n-vertex graph G by 'blowing up' F; that is, by replacing vertices of F with independent sets and edges with complete bipartite graphs. We choose the sizes of these independent sets to be $\lfloor n/t \rfloor$ and $\lceil n/t \rceil$ in order to obtain n vertices in total. Now G does not contain a copy of K_3, and since each vertex of F is in an edge (otherwise F would not be minimal) each vertex of G has degree at least $n/(2t)$. Provided that $\log n > 2t$, which is true for sufficiently large n, the graph G is an example as desired.

A point of interest: if you try hard enough, you can find graphs G on n vertices which are triangle-free and have minimum degree $\big(\frac{1}{3} - o(1)\big)n$; this is a construction of Hajnal (in a paper of Erdős and Simonovits). This is the best you can do, as we proved in the course. This construction is quite hard, especially if you want to understand why Kneser graphs have large chromatic number—that turns out to be a topological fact.