LTCC Course: Graph Theory 2019/20
§0: Introduction

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Notes 0: Introduction

This is the first installment of notes for a short course in Graph Theory at the *London Taught Course Centre* (LTCC). The course is intended for first-year research students in Mathematics, especially those specialising in Pure Mathematics. The course consists of five two-hour sessions.

It is expected that the audience will include some students intending to specialise in Graph Theory, Combinatorics, or a closely related area, along with others who wish to broaden their mathematical knowledge and learn new techniques.

The course is not a comprehensive treatment of Graph Theory : such a thing would be impossible within the time constraints. Moreover, we are expecting that most students taking the course will have encountered the basics of the subject at some earlier point in their career. So we will not be starting from scratch with the definition of a graph, or the definition of a path or a cycle in a graph, or a proof that trees on n vertices have $n-1$ edges, etc. In the next section, we shall point the reader to various sources where such material can be found. Our plan is to concentrate on themes and techniques, rather than on results. Anyone looking for a definitive source on the main results of Graph Theory should consult a (recent) textbook: that's what they are for!

More information on the course appears on the course webpage : `http://personal.lse.ac.uk/allenpd/LTCC/`. In particular, that page contains a list of the course content.

Over the years, this course has been given by several lecturers. The current lecturer is responsible for anything unclear or just wrong in the notes, and would like to thank Julia Böttcher, Graham Brightwell, Jan van den Heuvel and Jozef Skokan for writing most of the good bits in previous years.

Every year there is some finetuning and rewriting of the material, and sometimes (for example this year) reorganisation of the topics in the course. This means that the production of notes is being carried out in accordance with the "just-in-time" paradigm : There will be weekly notes, but don't expect them to be available before the start of the lectures. Conversely, if you ask early in the course for something to be covered, it may be possible to accommodate that request.

The notes will include references to books and online material, and some exercises. Solutions to those exercises will be published on the course webpage at the appropriate time.

Pre-requisites; books and online sources

There are many introductory texts on Graph Theory. Any one of these will provide an adequate account of the basics of the subject, although the reader is warned that notation and terminology is not completely standard; different books do different things, and while the notation in the lectures will be internally consistent, there is no promise it is consistent with any given book.

Here is a list of notions that we are expecting a student on this course to be familiar with already.

- the basic definition of a *graph* G as a pair (V, E), where V is the set of *vertices* and E the set of *edges* of G;
- the meaning of terms such as *adjacent* and *neighbour*. We will write $N_G(x) := \{y \in V(G) : xy \in E(G)\}$ for the neighbourhood of a vertex, and drop the subscript whenever G is clear from the context;
- *trees*: equivalent definitions and basic results;

  definitions of a *path* and a *cycle* in a graph (no repeated vertices);
- *(vertex) degree*, and notation $d(v)$; $\delta(G)$, and $\Delta(G)$ denote the *minimum* and *maximum* degrees of the graph G;
- *connectedness*; connected *components*;
- *bipartite* graphs, and the idea of a *matching* in a bipartite graph;
- *subgraphs*, and the distinction between a subgraph and an *induced* subgraph;
- *complete* and *empty* graphs; a complete subgraph of a graph is called a *clique*, and a set of vertices inducing an empty graph is an *independent set*.

Other notions will be defined as they come up.

For those missing some or all of this background: don't worry! It's easy, and the terminology is very intuitive.

Here are some books that we particularly recommend.

- (**B&M**) J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*. North Holland (1976).

  One of the classic text books on graph theory. This book is actually out of print (and has been out of print for ages). But the full text is available online for personal use via the course website. Bondy and Murty have recently published a new book entitled *Graph Theory* – see `www.springer.com/gb/book/9781846289699`. This is not a 2nd edition of the book above, but likely to become a new classic in graph theory.
- (**Diestel**) Reinhard Diestel, *Graph Theory* (1st, 2nd, 3rd or 4th edition). Springer-Verlag (1997, 2000, 2005, 2010).

  Although this book is still in print, the author has made sure that it is available in several versions online as well. See `diestel-graph-theory.com`. The free version is not printable (at least not easily). All editions are suitable for this course. References in the notes will refer to the 4th edition (which is the same as the one you can download).
- (**Bollobás**) Béla Bollobás, *Modern Graph Theory*. Springer-Verlag (1998).

  See `www.springer.com/gb/book/9780387984889`.

  This is another classic textbook aimed at students at this level, and is suitable for the course.

Other books are particularly recommended for use in individual weeks, and these will be mentioned at the appropriate points.