LTCC Course: Graph Theory 2019/20
Solutions to Exercises 1

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Exercise 1. The case left open is: G is a Δ-regular graph, for some $\Delta \geq 3$, which is 2-connected but not 3-connected.

Since G is not 3-connected, there is a pair $\{u, v\}$ of vertices of $V(G)$ whose removal disconnects G, i.e. $V(G)$ splits up into pairwise disjoint non-empty sets X, Y and $\{u, v\}$ where there are no edges between X and Y. Let us assume that u has at least as many neighbours in X as in Y. In particular, u does not have $\Delta - 1$ neighbours in Y. If both u and v have $\Delta - 1$ neighbours in X, then both u and v have at most one neighbour in Y, and since G is 2-connected, both have at least one neighbour in Y. Let v' be the neighbour of v in Y. Since v' has degree $\Delta \geq 3$, it has at least one neighbour other than u and v (it doesn't have to be adjacent to u, but it could be!). This other neighbour cannot be in X, as there are no edges between X and Y, so it is in Y and Y has size at least two. Now $\{u, v'\}$ is a pair of vertices whose removal disconnects G into $X \cup \{v\}$ and $Y \setminus \{v'\}$, which are non-empty disjoint sets with no edges between them. And v' has just one neighbour, v, in X.

This means we can assume from now on that our disconnecting pair $\{u, v\}$ has the property that at least one of u and v has less than $\Delta - 1$ neighbours in X, and at least one has less than $\Delta - 1$ neighbours in Y.

Now we define the graphs $G_1 = G[X \cup \{u, v\}] + uv$ and $G_2 = G[Y \cup \{u, v\}] + uv$. The graph G_1 has less than n vertices, since Y is non-empty. It has maximum degree at most Δ: although we added an edge uv (which might or might not have been present already) both u and v have at least one G-neighbour in Y which is no longer a neighbour in G_1. And since at least one of u and v has less than $\Delta - 1$ neighbours in X, it follows that G_1 cannot be $K_{\Delta+1}$. Finally G_1 is connected (think about why!). So we can colour G_1 with at most Δ colours; let c be a proper Δ-colouring. All the same applies to G_2 as well; let c' be a proper Δ-colouring of G_2.

Now $c(u) \neq c(v)$, and $c'(u) \neq c'(v)$. So there is a permutation σ of $\{1, \ldots, \Delta\}$ which maps $c'(u)$ to $c(u)$ and $c'(v)$ to $c(v)$. Finally we colour G using the colouring c'' defined by

$$c''(z) = c(z) \text{ if } z \in X, \quad c''(z) = \sigma\big(c'(z)\big) \text{ if } z \in Y, \quad \text{and } c''(z) = c(z) = \sigma\big(c'(z)\big) \text{ if } z \in \{u, v\}\,.$$

This is a proper Δ-colouring of G, and we are done.

Exercise 2. (a) We are asked to consider the case that $|L(v_i)| = 2$ for each vertex v_i. We want to see if there exists an assignment $\varphi(v_i)$ for all vertices v_i so that $\varphi(v_i) \in L(v_i)$, and for all edges $v_i v_{i+1}$ we have $\varphi(v_i) \neq \varphi(v_{i+1})$.

First suppose that all lists are identical: $L(v_i) = \{a, b\}$ for all i. If we choose colour a for v_1, then we are forced to colour v_2 with b, v_3 with a, etc. If k is even, then we will colour v_k with b, and hence this way we have obtained a proper colouring. But if k is odd, then we will colour v_k with a and obtain a conflict since v_1 is also coloured a. Exactly the same will happen if we start colouring v_1 with b. So we conclude that in this case we can find a proper assignment φ if and only if k is even.

So now assume that not all lists are identical. Then there must be two adjacent vertices v_i, v_{i+1} so that $L(v_i) \neq L(v_{i+1})$. Without loss of generality we can assume $L(v_1) \neq L(v_k)$. In particular, there must be a colour $a \in L(v_1)$ so that $a \notin L(v_k)$. Now colour v_1 with a and start colouring $v_2, v_3, \ldots, v_k$ in that order. Every time we need to colour a vertex v_i, there is at most one colour forbidden (the one given to v_{i-1}). But since each vertex has a list with two colours, there is always at least one colour from its list allowed. Once we have coloured everything, then by construction for every edge of the type v_iv_{i+1} we have $\varphi(v_i) \neq \varphi(v_{i+1})$. But we also have $\varphi(v_k) \neq \varphi(v_1)$, since we coloured v_1 with a colour that wasn't in $L(v_k)$. So we found a proper assignment, no matter the parity of k.

(b) It is easy to see that none of $\chi(C_{2k}), \mathrm{ch}(C_{2k}), \ldots$ is equal to 1. Hence they all must be at least 2. In (a) we've shown that, for an even cycle, we have $\mathrm{ch}(C_{2k}) \leq 2$. We can conclude that $\mathrm{ch}(C_{2k}) = 2$.

Determining $\chi(C_{2k})$ is equivalent to list colouring in which all lists are the same. From the arguments in (a) we obtain $\chi(C_{2k}) \leq 2$, hence $\chi(C_{2k}) = 2$.

Exactly the same arguments can be used to show the values of the edge chromatic number and the edge list chromatic number.

(c) From (a) we see that $\chi(C_{2k-1}), \mathrm{ch}(C_{2k-1})$ are at least 3. On the other hand, consider the case that the vertices of C_{2k-1} are given lists $L(v)$ with at least three colours. It's easy to see that we can find sublists $L'(v) \subseteq L(v)$ so that all $L'(v)$ have two colours and are not all the same. But then we can find colours from the lists $L'(v)$ to give a proper colouring of the vertices. This shows that with three (or more) colours, a colouring is always possible, proving $\chi(C_{2k-1}), \mathrm{ch}(C_{2k-1}) \leq 3$. It follows that $\chi(C_{2k-1}) = \mathrm{ch}(C_{2k-1}) = 3$.

Exactly the same arguments work for edge (list) colouring.

Exercise 3. This is actually not such a trivial question as it seems. First notice that $\chi'(K_1) = 0$. So from now on assume $n \geq 2$. Every vertex in K_n is incident with $n-1$ edges. This gives $\Delta(K_n) = n-1$, and so by Vizing's Theorem for simple graphs, $\chi'(K_n) = n-1$ or $\chi'(K_n) = n$.

Moreover, if we could colour the edges with $n-1$ colours, then every vertex is incident with one edge of every colour. So if we look at the edges that all have the same particular colour (say colour 1), then each vertex is incident with exactly one of those edges, and hence the number of vertices would be exactly twice the number of those edges. So n must be even in that case. In particular this means that we can't have $\chi'(K_n) = n-1$ if $n \geq 3$ is odd. And hence we must have $\chi'(K_n) = n$ if $n \geq 3$ is odd.

At this point we know that $\chi'(K_n) \in \{n-1, n\}$ if n is even. We will show that in fact for n even, $\chi'(K_n) = n-1$. Hence we need to find a colouring of the edges of K_n with $n-1$ colours for n even.

If $n = 2$, this is trivial. So we assume that $n \geq 4$ is even, and write $n = 2k+2$ for some $k \geq 1$. Choose one special vertex v^* and number the other vertices from $-k$ to k. Hence the vertices of G are: $v^*, v_{-k}, v_{-k+1}, \ldots, v_{-1}, v_0, v_1, \ldots, v_k$. Give colour 1 to each of the edges $v^*v_0, v_{-1}v_1, v_{-2}v_2, \ldots, v_{-k}v_k$. For colour 2 we take the edges of colour 1, but add one to each of the indices of the v_i (v^* doesn't change), where we take $k+1 = -k$. So the edge v^*v_0 becomes the edge v_*v_1, edge $v_{-1}v_1$ becomes v_0v_2, edge $v_{-2}v_2$ becomes $v_{-1}v_3$, etc. So the edges coloured 2 are $v^*v_1, v_0v_2, v_{-1}v_3, \ldots, v_{-k+2}v_k, v_{-k+1}v_{-k}$. You should check for yourself that these edges are disjoint from those coloured 1. Now for colour 3 we add one to each of the indices of the edges with colour 2 (always replacing $k+1$ by $-k$), and continue to do this for all colours up to $2k+1$. A little bit of checking should show that this is indeed an edge colouring with $2k+1 = n-1$

colours. (For this checking, it may be useful to draw a picture of the graph in which the vertices $v_{-k}, \ldots, v_{-1}v_0, v_1, \ldots, v_k$ are placed around a circle, and the vertex v^* in the centre of that circle.)

This shows $\chi'(K_1) = 0$, $\chi'(K_n) = n-1$ if n is even, and $\chi'(K_n) = n$ if $n \geq 3$ is odd.

Exercise 4. If G is not connected, then it's obvious that we can add an edge between two different components so that the resulting graph is still planar and simple.

So assume G is connected, and let $v_1, \ldots, v_k$, $k \geq 4$, be the sequence of vertices encountered when walking along the boundary of a face f of size more than three. We can add the edge v_1v_3 in that face, so that the resulting graph is still planar. But we can't be sure that the resulting graph is still simple. That fails if there already was an edge v_1v_3 in G. But then that edge must go outside the face f. And hence the edge v_2v_4 can't be present in G (since v_1, v_2, v_3, v_4 is a path along a face and both edges v_1v_3 and v_2v_4 must be on the same side of the path). So if v_1v_3 is already an edge in G, then we can always add the edge v_2v_4 so that G is still planar and simple.

Exercise 5. Let G be a graph with maximum degree $\Delta(G)$. We will prove that $\chi'(G) \leq \frac{3}{2}\,\Delta(G)$ by induction on the number of edges of G. The result it trivially true if G has no edges.

So suppose G has at least one edge, and let $\mu(G)$ be the maximum edge multiplicity of G. Then by Vizing's result we know $\chi'(G) \leq \Delta(G) + \mu(G)$. So if $\mu(G) \leq \frac{1}{2}\,\Delta(G)$, then we are done immediately.

So suppose $\mu(G) > \frac{1}{2}\,\Delta(G)$. So there is a collection $e_1, \ldots, e_{\mu(G)}$ of $\mu(G)$ edges, all with the same pair of end-vertices u, v. Remove the edge $e_{\mu(G)}$ from G, and call the resulting graph G'. Then $\Delta(G') \leq \Delta(G)$, and by induction we can edge colour G' with $\frac{3}{2}\,\Delta(G') \leq \frac{3}{2}\,\Delta(G)$ colours. We can transfer this colouring to a colouring of the edges of G with at most $\frac{3}{2}\,\Delta(G)$ colours, except that the edge $e_{\mu(G)}$ is still uncoloured.

So let's check how many colours are forbidden for $e_{\mu(G)}$. This is certainly at most the number of edges that are incident with one or both of u, v. One set of such edges consists of the $\mu(G) - 1$ edges parallel to $e_{\mu(G)}$. A second set of edges consists of the edges incident with u but not v. Since u has degree at most $\Delta(G)$, and there are $\mu(G)$ edges in G from u to v, there are at most $\Delta(G) - \mu(G)$ edges from u to some vertex other than v. Similarly, there are at most $\Delta(G) - \mu(G)$ edges incident with v and not u. So the total number of edges that are adjacent to $e_{\mu(G)}$ is at most $\mu(G) - 1 + 2\,(\Delta(G) - \mu(G)) = 2\,\Delta(G) - \mu(G) - 1 < \frac{3}{2}\,\Delta(G) - 1$ (since $\mu(G) > \frac{1}{2}\,\Delta(G)$). So we can always find a colour from the $\frac{3}{2}\,\Delta(G)$ available colours to use for $e_{\mu(G)}$, and that way colour all edges of G with $\frac{3}{2}\,\Delta(G)$ colours.

Exercise 6. For $k \geq 3$, let $\overrightarrow{C}_k$ be the directed cycle on vertices $V = \{v_1, \ldots, v_k\}$ (so the arcs are $\overrightarrow{v_1v_2}$, $\overrightarrow{v_2v_3}, \ldots, \overrightarrow{v_{k-1}v_k}$ and $\overrightarrow{v_kv_1}$). Again, we use the convention that $v_{k+1} = v_1$. For a kernel, we need an independent set $K \subseteq V$ so that for every vertex $v_i \in V \setminus K$ there is an arc from v_i to a vertex in K. But that means for each $v_i \in V \setminus K$ we must have $v_{i+1} \in K$. In particular, for each two consecutive vertices v_i, v_{i+1}, at least one must be in K. On the other hand, there cannot be two consecutive vertices in K, since K is independent.

So we conclude that a kernel K of $\overrightarrow{C}_k$ contains exactly one vertex from each pair v_i, v_{i+1}, for $i = 1, \ldots, k$. But if k is odd, then we can't make such a choice. And hence the set of directed cycles $\overrightarrow{C}_k$, $k \geq 3$ odd, is an infinite family of directed graphs without a kernel.

Exercise 7. (a) Let v be a vertex of degree $\Delta(G)$. Say the edges incident with v are $e_1, \ldots, e_{\Delta(G)}$. Then all the elements $v, e_1, \ldots, e_{\Delta(G)}$ need a different colour in a total colouring. Hence we have $\chi''(G) \geq \Delta(G) + 1$.

It's immediate (using Proposition 1 and Vizing's Theorem), that $\chi''(G) \leq \Delta(G) + \deg(G) + 2$. We shall show something stronger, namely that $\chi''(G) \leq \max\{\Delta(G) + \deg(G), 2\,\deg(G) + 1\}$. We do this by induction on the number of vertices. If G has just one vertex, then $\chi''(G) = 1$ and $\Delta(G) = \deg(G) = 0$, so the bound is true in that case.

So consider the case that G has at least two vertices. Let v be a vertex in G with degree at most $\deg(G)$. Remove v from G and call the resulting graph G'. Then we have $\Delta(G') \leq \Delta(G)$ and $\deg(G') \leq \deg(G)$, and by induction we can find a total colouring of G' using at most $\max\{\Delta(G')+\deg(G'), 2\ \deg(G')+1\} \leq \max\{\Delta(G)+\deg(G), 2\ \deg(G)+1\}$ colours. And this gives a proper total colouring of G, except that we still need to find colours for v and for the edges incident with v.

Having coloured all vertices and edges from G' with at most $\max\{\Delta(G)+\deg(G), 2\ \deg(G)+1\}$ colours, we now colour one by one the edges incident with v and then finally the vertex v. When colouring an edge $e = uv$ incident with v, we have to take into account the colours of: the vertex u (that is one colour), the other edges incident with u (there are at most $\Delta(G) - 1$ other edges), and the other edges incident with v (there are at most $\deg(G) - 1$ such edges). So there are at most $1 + (\Delta(G) - 1) + (\deg(G) - 1) = \Delta(G) + \deg(G) - 1$ forbidden colours for e. And since we have more colours available, there is at least one free colour for e.

Now finally colour the vertex v. Forbidden colours for v are the colours of its neighbours and the colours of the edges incident with v. Both sets have size at most $\deg(G)$, so there are at most $2\ \deg(G)$ forbidden colours for G. But since we have more colours available, there is at least one free colour for v.

And then we have found a total colouring of G using at most $\max\{\Delta(G)+\deg(G), 2\ \deg(G)+1\}$ colours.

(b) Let C_n be a cycle, $n \geq 3$, and assume the vertices are $v_1, v_2, \ldots, v_n$ and the edges are $e_1 = v_1v_2$, $_2 = v_2v_3$, $\ldots$, $e_{n-1} = v_{n-1}v_n$ and $e_n = v_nv_1$. In what follows, we will always use the convention that $v_{n+1} = v_1$.

By part (a), we already know that $\chi''(C_n) \geq 3$. If we want to total colour C_n with three colours (say with $1, 2, 3$), then we can assume we start with colouring v_1 with 1, e_2 with 2, and v_2 with 3. Then we must choose 1 for e_3, 2 for v_3, etc. In other words, the sequence $v_1, e_2, v_2, e_3, v_3, \ldots$, gets coloured 1, 2, 3, 1, 2, 3, ... Continuing this way, we can only make it into a total colouring with three colours for the whole cycle if the final edge e_n gets colour 3. But this only happens if $n \equiv 0 \pmod 3$.

So if $n \not\equiv 0 \pmod 3$, then we need at least four colours. If $n \equiv 0 \pmod 2$, then the sequence of colours 1, 2, 3, 4, 1, 2, 3, 4, ... for $v_1, e_2, v_2, e_3, v_3, \ldots$ gives a proper total colouring. I'll leave it to you to find a colour scheme with four colours for the remaining cases.

All in all it follows that $\chi''(C_n) = 3$ if $n \equiv 0 \pmod 3$, and $\chi''(C_n) = 4$ if $n \not\equiv 0 \pmod 3$.

Again by part (a) we have that $\chi''(K_n) \geq n$.

First consider the case that n is odd, so $n+1$ is even. In Question 2 we gave a colouring of the edges of K_{n+1} with n colours which involves a special vertex v^*. If we consider that edge colouring of K_{n+1} and remove the vertex v^*, then we get an edge colouring of K_n with n colours and for every colour i, there is exactly one vertex that is not incident with any edge of colour i. So for a total colouring we can colour that vertex with i. The result will be a total colouring of K_n with n colours for n odd.

Finally, the even case. It's in fact not possible to have a proper total colouring of K_n with n colours only. That is trivial for K_2 and not so hard to check for K_4. But for larger n it's quite tedious, so lets forget about that. If n is even, then $n+1$ is odd. From the above we know that we can find a total colouring of K_{n+1} with $n+1$ colours. Removing one vertex gives a total colouring of K_n with $n+1$ colours.

All in all it follows that $\chi''(K_n) = n$ if n is odd, and $\chi''(K_n) = n+1$ if n is even.

(c) Let G be a graph with maximum degree $\Delta(G)$. And suppose we are given the $\Delta(G)+3$ colours $C = \{1, \ldots, \Delta(G)+3\}$. Let φ be a colouring of the vertices of G with colours from C. (That is easy since we can always colour vertices with at most $\Delta(G)+1$ colours.) For every $e = uv$ in G, set $L(e) = C \setminus \{\varphi(u), \varphi(v)\}$. Then we can give e any colour from $L(e)$ and don't cause a conflict with the colours of u and v. So we only have to worry about conflicts with other edges.

But each edge e has a list $L(e)$ of $\Delta(G)+1$ colours. We also know that $\chi(G) \leq \Delta(G)+1$ (Vizing's Theorem in the notes). So if the List Colouring Conjecture is true, then we can find a proper colouring of the edges of G using colours from each edge's list. The combination of that edge colouring and the vertex colouring φ is a proper total colouring of G.