Summer 2015 Examination

MA100
Mathematical Methods
Full Unit

Suitable for all candidates

Instructions to candidates

Time allowed: 3 hours.

This paper contains **8** questions. You may attempt as many questions as you wish, but only your **BEST 6** answers will count towards the final mark. All questions carry equal numbers of marks.

Answers should be justified by showing working.

You are supplied with: Mathematics Answer Booklet.

Calculators are **not** allowed in this examination.

1 (a) Show that the function $f : [-1, \infty) \to \mathbb{R}$ defined by

$$f(x) = (x^2 - 9x + 19)e^x$$

attains a global minimum at a point $m \in [-1, \infty)$ and find $f(m)$.

(b) Consider the function $g : [-1, \infty) \to \mathbb{R}$ defined by

$$g(x) = H(x, f(x)),$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function and $f : [-1, \infty) \to \mathbb{R}$ is the function in part (a). Given that

$$H(0, 19) = 2, \qquad H_x(0, 19) = 5, \qquad H_y(0, 19) = 1,$$

show that the Taylor polynomial for g of degree 1 about the point 0 is given by

$$P_1(x) = 2 + 15x.$$

(c) Consider the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} x^2 + 3x & x \leq 1 \\ 2x + 2 & x > 1. \end{cases}$$

Calculate the limits

$$\lim_{\varepsilon \to 0^+} \frac{h(1) - h(1 - \varepsilon)}{\varepsilon} \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \frac{h(1 + \varepsilon) - h(1)}{\varepsilon}$$

and hence determine whether or not h is differentiable at $x = 1$.

2 (a) Give the definition of a *linearly independent* set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$. State an inequality involving k and n which, when satisfied, guarantees that S is *not* a linearly independent set.

(b) Give the definition of the *linear span* $\text{Lin}(S)$ of a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$. State an inequality involving k and n which, when satisfied, guarantees that S does *not* span $\mathbb{R}^n$.

(c) Consider the set of vectors $X = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 4 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 7 \\ 2 \\ 9 \\ 8 \end{pmatrix}.$$

(i) Is X a linearly independent set? You need to justify your answer.

(ii) Find a basis B for $\text{Lin}(X)$ and obtain the coordinates $(\mathbf{v}_1)_B, (\mathbf{v}_2)_B, (\mathbf{v}_3)_B$ of the vectors in X with respect to B. What is the dimension of $\text{Lin}(X)$?

(iii) Obtain a Cartesian description in $\mathbb{R}^4$ for $\text{Lin}(X)$. Hence, or otherwise, find the values of a and b for which the vector

$$\mathbf{w} = \begin{pmatrix} a \\ b \\ 6 \\ 8 \end{pmatrix}$$

belongs to $\text{Lin}(X)$.

3 (a) Define the *column space* $CS(\mathbf{A})$ and the *null space* $N(\mathbf{A})$ of a matrix $\mathbf{A}$. Prove that the null space $N(\mathbf{A})$ of an $m \times n$ matrix $\mathbf{A}$ is a *subspace* of $\mathbb{R}^n$.

(b) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ is defined by $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{and} \qquad \mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 7 & 11 & 13 \\ 4 & 7 & 6 \end{pmatrix}.$$

State how the range $R(T)$ and the kernel $\ker(T)$ of T are related to $CS(\mathbf{A})$ and $N(\mathbf{A})$ and find a basis B_1 for $CS(\mathbf{A})$ and a basis B_2 for $N(\mathbf{A})$.

(c) A linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^4$ has the property that

$$\ker(S) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; y - z = 0 \right\}.$$

(i) Find a basis B_3 for $\ker(S)$.

(ii) Let $\mathbf{B}$ be the matrix representing S by $S(\mathbf{x}) = \mathbf{B}\mathbf{x}$. Let $\mathbf{c}_1, \mathbf{c}_2$ and $\mathbf{c}_3$ be the columns of $\mathbf{B}$; that is, $\mathbf{B} = (\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3)$. Explain why $\mathbf{c}_1 = \mathbf{0}$ and $\mathbf{c}_2 + \mathbf{c}_3 = \mathbf{0}$. Given the additional information that

$$\mathbf{B} \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 2 \\ 4 \end{pmatrix},$$

find the matrix $\mathbf{B}$.

4 Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = 4x^2 + 6xy^2 + 7.$$

(a) Find a Cartesian equation in $\mathbb{R}^3$ for the *tangent plane* Π_T to the surface $z = f(x, y)$ at the point $(x, y, z) = (1, 2, 35)$. Also find a Cartesian equation in $\mathbb{R}^3$ for the *vertical plane* Π_V which contains the point $(1, 2, 35)$ and has the horizontal vector $\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ as one of its direction vectors.

(b) Obtain a Cartesian description in $\mathbb{R}^3$ for the line ℓ of intersection of the planes Π_T and Π_V and hence find a vector parametric equation in $\mathbb{R}^3$ for ℓ. Rescale your direction vector for the line ℓ so that it becomes

$$\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \qquad \text{with} \qquad d_1^2 + d_2^2 = 1 \quad \text{and} \quad d_1 > 0.$$

Interpret d_3 geometrically and hence explain why

$$f(1 + 3\varepsilon, 2 + 4\varepsilon) = 35 + 5d_3\varepsilon$$

to *first-order* in ε.

5 (a) Find an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

is expressed as $\mathbf{A} = \mathbf{PDP}^{-1}$. You **do not** need to find $\mathbf{P}^{-1}$.

(b) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$T(\mathbf{x}) = \mathbf{C}\mathbf{x},$$

where $\mathbf{C}$ is a *symmetric* matrix. It is known that $\mathbf{C}$ has a *repeated* eigenvalue α and another eigenvalue β, where $\alpha \neq \beta$. State the dimensions of the eigenspaces $N(\mathbf{C} - \alpha\mathbf{I})$ and $N(\mathbf{C} - \beta\mathbf{I})$. It is also known that

$$\mathbf{C} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \beta \\ 2\beta \\ 3\beta \end{pmatrix}.$$

Explain why $N(\mathbf{C} - \alpha\mathbf{I})$ is described by the Cartesian equation

$$x + 2y + 3z = 0$$

and then find a basis for $N(\mathbf{C} - \alpha\mathbf{I})$.

(c) Construct an *orthonormal* basis for $\mathbb{R}^3$ consisting of eigenvectors of $\mathbf{C}$ and write down the matrix that describes the transformation T with respect to this basis.

6 (a) Consider the function $h : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$h(x, y, z) = 4x^2 + 6y^2 + z^3 + 3yz.$$

(i) Find the two stationary points of h and classify each of them as a local maximum, a local minimum or a saddle point, stating the criteria you are using.

(ii) Show that h has no *global* extrema on $\mathbb{R}^3$.

(b) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = y - (x - 3)^2$$

and let the region $D \subset \mathbb{R}^2$ be given by

$$D = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x + y \leq 6,\ x \geq 0,\ y \geq 1\right\}.$$

(i) Sketch the region D *accurately* and draw some contours of f in order to show that f attains a constrained *maximum* at a point located on the boundary of D. Indicate this point on your graph by the letter M.

(ii) Use a suitable Lagrangian $L(x, y, \lambda)$ to *maximise* f on D. You need to state the coordinates of the point M and find the value of f at M.

(iii) Use your graph to show that f attains a constrained *minimum* at a point $m \in D$ which is *not* a point of tangency. Indicate m on your graph and find the value of f at m.

7 Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}.$$

(a) Sketch the eigenspaces of $\mathbf{A}$ on the xy-plane and label them with their corresponding eigenvalues.

(b) Diagonalise $\mathbf{A}$ and hence solve the linear system of difference equations

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t, \qquad \mathbf{x}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

subject to the initial conditions $x_0 = 1, y_0 = 2$.

(c) *Orthogonally* diagonalise $\mathbf{A}$ and hence sketch the conic section

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = 32, \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

You need to indicate on your sketch the orthonormal basis B of eigenvectors of $\mathbf{A}$ that you are using and also state the B-coordinates of the points of intersection of the conic section with the B-coordinate axes.

8 (a) Show that the ordinary differential equation

$$(2x\ \mathrm{sin}y - y\ \mathrm{sin}x)dx + (x^2\ \mathrm{cos}y + \mathrm{cos}x)dy = 0$$

is *exact*. Hence obtain its general solution in the form

$$F(x, y) = C$$

where C is an arbitrary constant.

(b) Find the particular solution of the ordinary differential equation

$$(D-3)(D+1)^2 y = 12$$

subject to the conditions

$$\lim_{x\to\infty} y(x) = -4, \quad y(0) = 2 \quad \text{and} \quad y'(0) = 2,$$

where $D = \dfrac{d}{dx}$.

(c) Find the values of the integers a and b that make the ordinary differential equation

$$(2x^a y^{b+3} + 8x^{a+2} y^b)dx + (3x^{a+1} y^{b+2})dy = 0$$

exact. You **do not** need to solve the resulting differential equation.