MA100

Mathematical Methods

Solutions to summer 2015 Examination

1. (a) The derivative of $f : [-1, \infty) \to \mathbb{R}$ is given by

$$f' = (2x - 9)e^x + (x^2 - 9x + 19)e^x = (x^2 - 7x + 10)e^x = (x - 2)(x - 5)e^x,$$

so the stationary points of f are

$$x = 2 \quad \text{and} \quad x = 5.$$

The derivative satisfies

$$\begin{array}{ll} f'(x) > 0 & x \in [-1, 2) \\ f'(x) < 0 & x \in (2, 5) \\ f'(x) > 0 & x \in (5, \infty) \end{array}$$

so, clearly, a global minimum exists and there are only two candidates for it: the left endpoint $x = -1$ and the stationary point $x = 5$. We have

$$f(-1) = 29e^{-1} \quad \text{and} \quad f(5) = -e^5,$$

so the global minimum is $m = 5$ and $f(m) = -e^5$.

[9 marks]

(b) The derivative of g is given in terms of the partial derivatives of H and the derivative of f by

$$g'(x) = H_x(x, y) + H_y(x, y) f'(x),$$

where y should be replaced by $f(x)$. The formula for the Taylor polynomial P_1 about 0 is

$$P_1(x) = g(0) + g'(0)x.$$

Using the given information, we find

$$g(0) = H(0, f(0)) = H(0, 19) = 2,$$

$$g'(0) = \frac{\partial H}{\partial x}(0, f(0)) + \frac{\partial H}{\partial y}(0, f(0))f'(0) = \frac{\partial H}{\partial x}(0, 19) + \frac{\partial H}{\partial y}(0, 19)10 = 15,$$

and hence obtain the stated result

$$P_1(x) = 2 + 15x.$$

[8 marks] [not seen in this form]

(c) Using the definition of h we find that the left limit is

$$\lim_{\varepsilon \to 0^+} \frac{h(1) - h(1-\varepsilon)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{4 - (1-\varepsilon)^2 - 3(1-\varepsilon)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{5\varepsilon - \varepsilon^2}{\varepsilon} = \lim_{\varepsilon \to 0^+} (5-\varepsilon) = 5,$$

and the right limit is

$$\lim_{\varepsilon \to 0^+} \frac{h(1+\varepsilon) - h(1)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{2(1+\varepsilon) + 2 - 4}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{2\varepsilon}{\varepsilon} = \lim_{\varepsilon \to 0^+} (2) = 2.$$

Hence h is not differentiable at $x = 1$ since the left and right limits are not equal.

[8 marks]

2. (a) A set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ is a linearly independent set if the only solution of the homogeneous equation

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots \alpha_k\mathbf{v}_k = \mathbf{0}$$

is the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$. The inequality $k > n$ guarantees that the set S is not a linearly independent set of vectors in $\mathbb{R}^n$.

[4 marks]

(b) The linear span $\text{Lin}(S)$ of $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is the set of all linear combinations of the vectors in S. The inequality $k < n$ guarantees that the set S does not span $\mathbb{R}^n$.

[4 marks]

(c) (i) We place the vectors in X as columns in a matrix $\mathbf{A}$ and row reduce (full reduction is not necessary but is useful later):

$$\mathbf{A} = (\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3) = \begin{pmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 1 & 3 & 9 \\ 0 & 4 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 4 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\mathbf{A}$ does not have full column rank, the set X is not a linearly independent set.

[4 marks]

(ii) The (reduced) row echelon form of $\mathbf{A}$ reveals that the first two columns of $\mathbf{A}$ are linearly independent vectors and hence form a basis B for $\text{Lin}(X)$:

$$B = \{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \\ 4 \end{pmatrix} \right\}.$$

The reduced row echelon form of $\mathbf{A}$ also reveals that the vector $\begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$ belongs to the null space of $\mathbf{A}$. Hence the corresponding linear combination of the columns of $\mathbf{A}$ is equal to the zero vector, i.e.,

$$-3\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}.$$

It follows that the coordinates of the vectors in X with respect to the B basis are

$$(\mathbf{v}_1)_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (\mathbf{v}_2)_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (\mathbf{v}_3)_B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

The dimension of $\text{Lin}(X)$ is 2 since a basis for $\text{Lin}(X)$ consists of two vectors.

[**5 marks**]

(iii) For a Cartesian description, we can either eliminate the free parameters s, t from the vector parametric equation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 3 \\ 4 \end{pmatrix}$$

for $\text{Lin}(X)$ or, alternatively, use the fact that the column space of $\mathbf{A}$ is orthogonal to the null space of $\mathbf{A}^T$. Row reducing $\mathbf{A}^T$ we find that

$$\mathbf{A}^T = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 4 \\ 7 & 2 & 9 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 4 \\ 0 & 2 & 2 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so

$$N(\mathbf{A}^T) = \text{Lin}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Hence, a Cartesian description for $\text{Lin}(X)$ is given by the set of equations

$$\begin{pmatrix} -1 & -1 & 1 & 0 \\ 0 & -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In order for $\mathbf{w} = \begin{pmatrix} a \\ b \\ 6 \\ 8 \end{pmatrix}$ to belong to $\text{Lin}(X)$ we need

$$\begin{cases} -a-b+6 &= 0 \\ -4b+8 &= 0 \end{cases}$$

which implies that $b = 2$ and $a = 4$.

[8 marks][contains harder parts]

3. (a) The column space of $\mathbf{A}$, $CS(\mathbf{A})$, is the linear span of the columns of $\mathbf{A}$. The null space of $\mathbf{A}$, $N(\mathbf{A})$, is the set of solutions of the homogeneous system $\mathbf{Ax} = \mathbf{0}$. To show that $N(\mathbf{A})$ is a subspace of $\mathbb{R}^n$, we use the Subspace Criterion. First, $N(\mathbf{A})$ is non-empty because the zero vector $\mathbf{0} \in \mathbb{R}^n$ satisfies $\mathbf{A0} = \mathbf{0}$. Moreover, $N(\mathbf{A})$ is closed under vector addition and scalar multiplication. Indeed, given any $\mathbf{u} \in N(\mathbf{A})$ and $\mathbf{w} \in N(\mathbf{A})$, we have $\mathbf{Au} = \mathbf{0}$ and $\mathbf{Aw} = \mathbf{0}$. Hence the sum $\mathbf{u}+\mathbf{w}$ satisfies $\mathbf{A}(\mathbf{u}+\mathbf{w}) = \mathbf{Au}+\mathbf{Aw} = \mathbf{0}+\mathbf{0} = \mathbf{0}$, which implies that $\mathbf{u}+\mathbf{w}$ is in $N(\mathbf{A})$. Similarly, for any $\lambda \in \mathbb{R}$, the vector $\lambda\mathbf{u}$ satisfies $\mathbf{A}(\lambda\mathbf{u}) = \lambda\mathbf{Au} = \lambda\mathbf{0} = \mathbf{0}$, which implies that $\lambda\mathbf{u}$ is in $N(\mathbf{A})$.

   [8 marks]

   (b) We have

$$R(T) = CS(\mathbf{A}) \qquad \text{and} \qquad \ker(T) = N(\mathbf{A}).$$

   To obtain a basis B_1 for $CS(\mathbf{A})$ and a basis B_2 for $N(\mathbf{A})$ we find the reduced row echelon form of $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 7 & 11 & 13 \\ 4 & 7 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & -3 & 6 \\ 0 & -1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   We therefore learn that the first two columns of $\mathbf{A}$ are linearly independent and span $CS(\mathbf{A})$, so

$$B_1 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 7 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 11 \\ 7 \end{pmatrix} \right\}.$$

Moreover, $N(\mathbf{A}) = \text{Lin}\left\{\begin{pmatrix}-5\\2\\1\end{pmatrix}\right\}$, so

$$B_2 = \left\{\begin{pmatrix}-5\\2\\1\end{pmatrix}\right\}.$$

[6 marks]

(c) (i) The Cartesian equation $y - z = 0$ imposed on $\mathbb{R}^3$ implies two free parameters. We let $x = t$ and $y = s$, then $z = s$. So the general element $\mathbf{x} \in \ker(S)$ is written in the form

$$\mathbf{x} = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}t\\s\\s\end{pmatrix} = t\begin{pmatrix}1\\0\\0\end{pmatrix} + s\begin{pmatrix}0\\1\\1\end{pmatrix},$$

which implies that

$$B_3 = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}.$$

[4 marks]

(ii) Every vector in $N(\mathbf{B}) = \ker(S)$ gives a linear combination of the columns of $\mathbf{B} = (\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3)$ that is equal to the zero vector. Hence, considering the vectors in B_3, we see that

$$(\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3)\begin{pmatrix}1\\0\\0\end{pmatrix} = \mathbf{0} \quad \text{and} \quad (\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3)\begin{pmatrix}0\\1\\1\end{pmatrix} = \mathbf{0},$$

which means that $\mathbf{c}_1 = \mathbf{0}$ and $\mathbf{c}_2 + \mathbf{c}_3 = \mathbf{0}$.

The additional information that $\mathbf{B}\begin{pmatrix}5\\0\\2\end{pmatrix} = \begin{pmatrix}6\\0\\2\\4\end{pmatrix}$ implies that

$$5\mathbf{c}_1 + 2\mathbf{c}_3 = \begin{pmatrix} 6 \\ 0 \\ 2 \\ 4 \end{pmatrix}.$$

We have three independent linear equations for the three columns of $\mathbf{B}$. In particular, with $\mathbf{c}_1 = \mathbf{0}$, we see that $\mathbf{c}_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ and hence $\mathbf{c}_2 = \begin{pmatrix} -3 \\ 0 \\ -1 \\ -2 \end{pmatrix}$. Therefore,

$$\mathbf{B} = \begin{pmatrix} 0 & -3 & 3 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{pmatrix}.$$

[7 marks] [not seen in this form]

4. (a) The partial derivatives of f are

$$f_x = 8x + 6y^2 \qquad \text{and} \qquad f_y = 12xy.$$

At the point $(1,2)$, these become $f_x(1,2) = 32$ and $f_y(1,2) = 24$. Hence, the tangent plane Π_T is described by the Cartesian equation

$$z - 35 = 32(x-1) + 24(y-2).$$

A vector parametric equation for the vertical plane Π_V is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 35 \end{pmatrix} + s\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

since Π_V passes through $(1,2,35)$ and has $\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ as direction vectors.

A Cartesian description for Π_V can be obtained by finding a normal vector,

$$\mathbf{n} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$$

and using the position vector $\begin{pmatrix} 1 \\ 2 \\ 35 \end{pmatrix}$ in the formula

$$\left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 2 \\ 35 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} \right\rangle.$$

The resulting Cartesian equation is

$$4x - 3y = -2.$$

[11 marks]

(b) A Cartesian description in $\mathbb{R}^3$ for the line ℓ is given by the simultaneous system of equations

$$\begin{cases} z - 35 &= 32(x-1) + 24(y-2) \\ 4x - 3y &= -2. \end{cases}$$

There is a free parameter in this system. Letting $y = t \in \mathbb{R}$, we find that $x = -\frac{1}{2} + \frac{3}{4}t$ and hence $z = 35 + 32(-\frac{3}{2} + \frac{3}{4}t) + 24(t-2) = -61 + 48t$. Hence, a vector parametric equation for ℓ is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1/2 \\ 0 \\ -61 \end{pmatrix} + t \begin{pmatrix} 3/4 \\ 1 \\ 48 \end{pmatrix}.$$

Rescaling the direction vector for ℓ in two stages,

$$\begin{pmatrix} 3/4 \\ 1 \\ 48 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 \\ 4 \\ 192 \end{pmatrix} \longrightarrow \frac{1}{5}\begin{pmatrix} 3 \\ 4 \\ 192 \end{pmatrix} = \mathbf{d}$$

we obtain a direction vector $\mathbf{d}$ whose first two components d_1 and d_2 satisfy $d_1^2 + d_2^2 = 1$ and $d_1 > 0$, as instructed in the question.

The third component d_3 is the directional derivative of f at the point $(1, 2, 35)$ in the direction $\mathbf{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. This is equal to the ratio of the infinitesimal 'vertical' change $f(1 + 3\varepsilon, 2 + 4\varepsilon) - f(1, 2)$ to the corresponding 'horizontal' change $\left\| \begin{pmatrix} 1+3\varepsilon \\ 2+4\varepsilon \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 3\varepsilon \\ 4\varepsilon \end{pmatrix} \right\| = 5\varepsilon$; i.e.

$$d_3 = f_{\mathbf{u}}(1, 2) = \lim_{\varepsilon \to 0^+} \frac{f(1 + 3\varepsilon, 2 + 4\varepsilon) - f(1, 2)}{5\varepsilon}.$$

Hence, to first order in ε, we have that

$$f(1 + 3\varepsilon, 2 + 4\varepsilon) = f(1, 2) + 5d_3\varepsilon = 35 + 5d_3\varepsilon.$$

[14 marks][contains harder parts]

5. (a) The characteristic polynomial equation for $\mathbf{A}$ is

$$\left| \begin{pmatrix} 2-\lambda & -2 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} \right| = -(2-\lambda)^2\lambda = 0,$$

which implies that $\lambda_1 = 2$ (of algebraic multiplicity 2) and $\lambda_2 = 0$ (of algebraic multiplicity 1) are the eigenvalues of $\mathbf{A}$. The corresponding eigenspaces are, by inspection,

$$N(\mathbf{A} - 2\mathbf{I}) = N(\begin{pmatrix} 0 & -2 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}) = \text{Lin}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

and

$$N(\mathbf{A}-0\mathbf{I}) = N(\begin{pmatrix} 2 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}) = \text{Lin}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Hence, setting $\mathbf{P} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we express $\mathbf{A}$ in the form $\mathbf{A} = \mathbf{PDP}^{-1}$.

[**10 marks**]

(b) We are told that $\mathbf{C}$ is a symmetric 3×3 matrix, so we know that $\mathbf{C}$ is orthogonally diagonalisable. Since the eigenvalue α is repeated and the other eigenvalue β is not equal to α, we deduce that

$$\dim\bigl(N(\mathbf{C}-\alpha\mathbf{I})\bigr) = 2 \quad \text{and} \quad \dim\bigl(N(\mathbf{C}-\beta\mathbf{I})\bigr) = 1.$$

We are also given that $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is an eigenvector of the eigenvalue β, from which it follows that

$$N(\mathbf{C}-\beta\mathbf{I}) = \text{Lin}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

Now, since $\mathbf{C}$ is orthogonally diagonalisable, eigenspaces corresponding to distinct eigenvalues are orthogonal. Hence $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a normal vector for the 2-dimensional eigenspace $N(\mathbf{A}-\alpha\mathbf{I})$. This means that a Cartesian equation for $N(\mathbf{C}-\alpha\mathbf{I})$ is

$$x + 2y + 3z = 0.$$

A basis for $N(\mathbf{C}-\alpha\mathbf{I})$ can be obtained by solving the Cartesian equation $x+2y+3z = 0$. We let $y = s$, $z = t$, so $x = -2s - 3t$. The general vector $\mathbf{x} \in N(\mathbf{C}-\alpha\mathbf{I})$ is thus given by

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

so

$$N(\mathbf{C} - \alpha\mathbf{I}) = \text{Lin}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

[7 marks] [not seen in this form]

(c) An orthonormal basis $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ for $N(\mathbf{C}-\alpha\mathbf{I})$ can be found by the Gram-Schmidt process. We let

$$\mathbf{u}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

and

$$\mathbf{w}_2 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} - \left\langle \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\rangle \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} - \frac{6}{5} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -3 \\ -6 \\ 5 \end{pmatrix},$$

which leads to

$$\mathbf{u}_2 = \frac{1}{\sqrt{70}} \begin{pmatrix} -3 \\ -6 \\ 5 \end{pmatrix}.$$

Therefore, an orthonormal basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathbb{R}^3$ consisting of eigenvectors of $\mathbf{C}$ is

$$B = \left\{ \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{70}} \begin{pmatrix} -3 \\ -6 \\ 5 \end{pmatrix}, \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

The matrix describing T with respect to the basis B is the diagonal matrix

$$\mathbf{D} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}.$$

[8 marks]

6. (a) (i) Setting the partial derivatives of h to zero, we obtain the simultaneous system

$$\begin{cases} h_x &= 8x &= 0 \\ h_y &= 12y + 3z &= 0 \\ h_z &= 3z^2 + 3y &= 0. \end{cases}$$

The first equation implies that $x = 0$. Solving the second equation for z in terms of y and substituting the solution $z = -4y$ into the second equation, we find that $16y^2 + y = 0$, so $y = 0$ or $y = -\frac{1}{16}$. Using $z = -4y$ and $x = 0$, we obtain two stationary points, namely

$$(0,0,0) \quad \text{and} \quad \left(0, -\frac{1}{16}, \frac{1}{4}\right).$$

The second derivative of h is given by

$$h''(x,y,z) = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 12 & 3 \\ 0 & 3 & 6z \end{pmatrix},$$

so

$$h''(0,0,0) = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 12 & 3 \\ 0 & 3 & 0 \end{pmatrix} \quad \text{and} \quad h''\left(0, -\frac{1}{16}, \frac{1}{4}\right) = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 12 & 3 \\ 0 & 3 & 3/2 \end{pmatrix}.$$

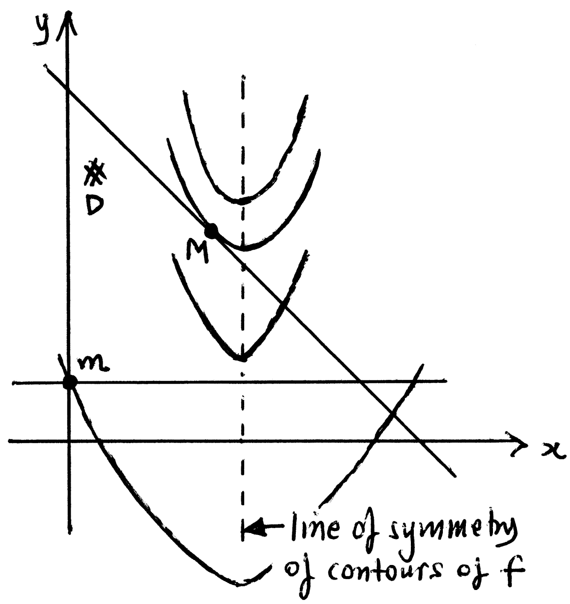
The principal minors of $h''(0,0,0)$ are $8, 96, -72$. Hence, $h''(0,0,0)$ is an indefinite matrix and $(0,0,0)$ is a saddle point. Regarding $h''\left(0, -\frac{1}{16}, \frac{1}{4}\right)$, its principal minors are $8, 96, 72$, so this is a positive definite matrix and the point $\left(0, -\frac{1}{16}, \frac{1}{4}\right)$ is a local minimum.

[11 marks]

(ii) We observe that $h(0,0,z) = z^3$ tends to $\pm\infty$ as z tends to $\pm\infty$, so no global extrema exist for h.

[2 marks]

(b) (i) The region D and some contours of f, i.e. curves of the form $y = (x-3)^2 + c$, are depicted below.



[4 marks]

(ii) A suitable Lagrangian for the maximisation problem is

$$L(x, y, \lambda) = y - (x-3)^2 + \lambda(6 - x - y)$$

since M is the point of tangency of one of the contours of f and the line $x+y=6$. Setting the partial derivatives of L equal to zero, we get the simultaneous system

$$\begin{cases} L_x &= -2(x-3) - \lambda &= 0 \\ L_y &= 1 - \lambda &= 0 \\ L_\lambda &= 6 - x - y &= 0. \end{cases}$$

The second equation implies that $\lambda = 1$. Substituting this value of λ in the first equation, we find that $x = 2.5$. Finally, the constraint implies that $y = 3.5$. Hence

$$M = (2.5, 3.5) \qquad \text{and} \qquad f(M) = 3.25.$$

[5 marks]

(iii) The point $m = (0, 1)$ is indicated on the graph above; it is not a point of tangency. The value of f at m is $f(0, 1) = -8$.

[3 marks][harder part]

7. (a) The characteristic polynomial equation for the matrix $\mathbf{A}$ is

$$|\mathbf{A} - \lambda\mathbf{I}| = \left| \begin{pmatrix} 5-\lambda & -3 \\ -3 & 5-\lambda \end{pmatrix} \right| = \lambda^2 - 10\lambda + 16 = (\lambda - 8)(\lambda - 2) = 0,$$

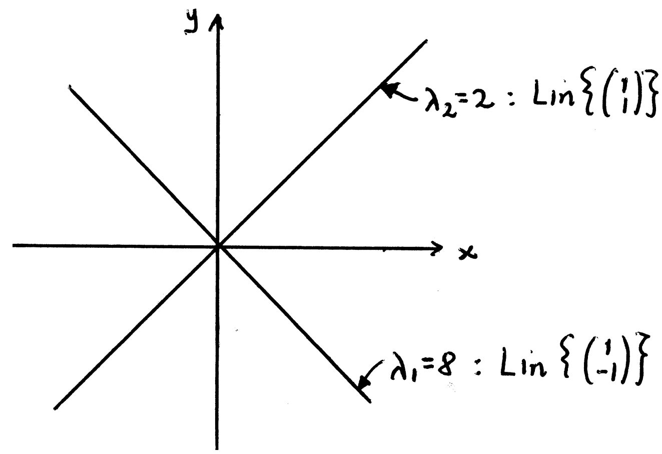
so the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 8$ and $\lambda_2 = 2$. The corresponding eigenspaces of $\mathbf{A}$ are

$$N(\mathbf{A} - \lambda_1\mathbf{I}) = N(\mathbf{A} - 8\mathbf{I}) = N(\begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix}) = \text{Lin}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

and

$$N(\mathbf{A} - \lambda_2\mathbf{I}) = N(\mathbf{A} - 2\mathbf{I}) = N(\begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}) = \text{Lin}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

The eigenspaces of $\mathbf{A}$ are sketched below:



[8 marks]

(b) The matrix $\mathbf{A}$ can be expessed in the form $\mathbf{A} = \mathbf{PDP}^{-1}$, where

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{D} = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}.$$

The general solution of the system of difference equation is given by

$$\mathbf{x}_t = \mathbf{A}^t \mathbf{x}_0.$$

Using $\mathbf{A} = \mathbf{PDP}^{-1}$ and the initial condition $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, we find the relevant particular solution

$$\begin{aligned} \mathbf{x}_t = \mathbf{PD}^t\mathbf{P}^{-1}\mathbf{x}_0 &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 8^t & 0 \\ 0 & 2^t \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 8^t & 2^t \\ -8^t & 2^t \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -8^t + 3(2^t) \\ 8^t + 3(2^t) \end{pmatrix}. \end{aligned}$$

[8 marks]

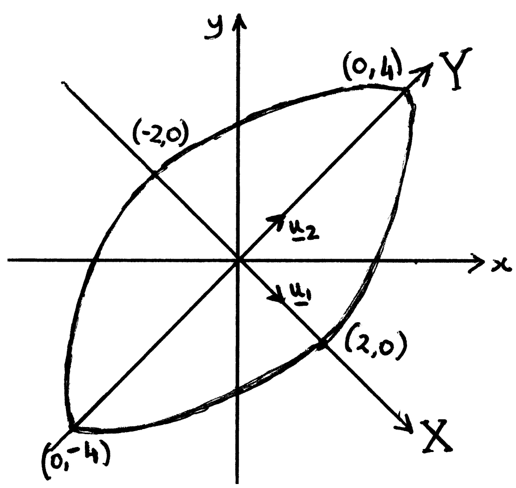
(c) An orthonormal basis B of eigenvectors of $\mathbf{A}$ is given by

$$B = \{\mathbf{u}_1, \mathbf{u}_2\} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Hence, with $\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$, we have $\mathbf{P}^T = \mathbf{P}^{-1}$ and $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T$. We now let $\mathbf{x} = \mathbf{P}\mathbf{z}$, where $\mathbf{z} = \begin{pmatrix} X \\ Y \end{pmatrix}$ are the coordinates of $\mathbf{x}$ with respect to the B basis. The equation for the conic section then becomes $\mathbf{z}^T\mathbf{D}\mathbf{z} = 32$, which simplifies to

$$4X^2 + Y^2 = 16.$$

The conic section is an ellipse, whose sketch is given below:



B-coordinates of points of intersection with B-axes are indicated on graph

[9 marks] [contains harder parts]

8. (a) We let $M = 2x\ \mathrm{sin}y - y\ \mathrm{sin}x$ and $N = x^2\ \mathrm{cos}y + \mathrm{cos}x$. We have

$$\frac{\partial M}{\partial y} = 2x\ \mathrm{cos}y - \mathrm{sin}x \qquad \text{and} \qquad \frac{\partial N}{\partial x} = 2x\ \mathrm{cos}y - \mathrm{sin}x,$$

so the equation is exact. To obtain its general solution, we solve the simultaneous partial differential equations

$$\left\{ \begin{array}{rcl} \dfrac{\partial F}{\partial x} & = & 2x\ \mathrm{sin}y - y\ \mathrm{sin}x \\ \\ \dfrac{\partial F}{\partial y} & = & x^2\ \mathrm{cos}y + \mathrm{cos}x \end{array} \right. \qquad \text{hence} \qquad \left\{ \begin{array}{rcl} F & = & x^2\ \mathrm{sin}y + y\ \mathrm{cos}x + f(y) \\ \\ F & = & x^2\ \mathrm{sin}y + y\ \mathrm{cos}x + g(x) \end{array} \right.$$

We see that $f(y) = g(y) = k$ for some constant k, so the general solution for $y(x)$ is given in implicit form by

$$x^2\ \mathrm{sin}y + y\ \mathrm{cos}x = C,$$

where C is an arbitrary constant.

[6 marks]

(b) The complementary function is

$$y(x) = Ae^{3x} + (B + Cx)e^{-x},$$

where A, B, C are arbitrary constants. A particular integral has the form $y(x) = a$ for some constant a to be determined. Substituting this expression in the differential equation, we find that $a = -4$, so the general solution of the differential equation is

$$y(x) = -4 + Ae^{3x} + (B + Cx)e^{-x}.$$

Since $\lim_{x\to\infty} y(x) = -4$, we infer that $A = 0$. Using this value of A, we find that

$$y'(x) = (C - B - Cx)e^{-x},$$

so the conditions $y(0) = 2$ and $y'(0) = 2$ result in the simultaneous system of equations

$$\begin{cases} -4 + B &= 2 \\ C - B &= 2 \end{cases}$$

which yields $B = 6$ and $C = 8$. Hence, the required particular solution is

$$y(x) = -4 + (6 + 8x)e^{-x}.$$

[13 marks]

(c) In order for this equation to be exact, we require that the following equation is satisfied identically in x and y:

$$\frac{\partial}{\partial y}\left(2x^a y^{b+3} + 8x^{a+2}y^b\right) = \frac{\partial}{\partial x}\left(3x^{a+1}y^{b+2}\right).$$

Performing the differentiations, we obtain

$$2(b+3)x^a y^{b+2} + 8bx^{a+2}y^{b-1} = 3(a+1)x^a y^{b+2}.$$

Comparing the coefficients of the various powers of x and y, we deduce that

$$\begin{cases} 2(b+3) &= 3(a+1) \\ 8b &= 0 \end{cases}$$

which implies that $a = 1$ and $b = 0$.

[6 marks] [unseen]