THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE

Summer 2016 examination

MA103

Solutions

Question 1

(a) We are dealing with three statements p, q, r, each of which can be true ("T") or false ("F"). Using the simple truth tables for $a \vee b$ and $a \Rightarrow b$, we get the following truth table, showing both $(p \Rightarrow r) \vee (q \Rightarrow r)$ and $(p \vee q) \Rightarrow r$:

p	q	r	$p \Rightarrow r$	$q \Rightarrow r$	$(p \Rightarrow r) \vee (q \Rightarrow r)$	$p \vee q$	$(p \vee q) \Rightarrow r$
T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F
T	F	T	T	T	T	T	T
T	F	F	F	T	T	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	T	T	F
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

We see that there are two lines in which the truth values for $(p \Rightarrow r) \vee (q \Rightarrow r)$ and $(p \vee q) \Rightarrow r$ differ, which means that the two statements are not logically equivalent.
[4 + 2 pts, Standard question]

(b) Since $S_1 = 1$ and $S_2 = 2$, the statement is true for $n = 1$ and $n = 2$.

Now suppose that the statement is true for all $n \leq k$, for some $k \geq 2$, and consider the number S_{k+1}. Since $k + 1 \geq 3$, we know that $S_{k+1} = 2S_k + S_{k-1} - 2$.
Now if $k+1$ is even, then k is odd and $k-1$ is even, and hence by the induction hypothesis we have that S_k is odd and S_{k-1} is even. This means that $2S_k + S_{k-1} - 2$ is even ("two times odd plus even minus even" is even).
And if $k+1$ is odd, then k is even and $k-1$ is odd, and hence by the induction hypothesis we have that S_k is even and S_{k-1} is odd. This means that $2S_k + S_{k-1} - 2$ is odd ("two times even plus odd minus even" is odd).
We have shown that the statement is true for $n = k + 1$.

By the Principle of Induction, we can can conclude that $P(n)$ is true for all $n \in \mathbb{N}$.
[8 pts, Similar to many questions, although more involved than most seen]

(c) (i) If $z = Re^{i\theta}$, then $z^2 = R^2 e^{2i\theta}$ and $2\overline{z} = 2Re^{-i\theta}$. So to have $z^2 = 2\overline{z}$ we must have $R^2 = 2R$ and $e^{2i\theta} = e^{-i\theta}$.

Since $R^2 = 2R$ is equivalent to $R(R - 2) = 0$, we have $R = 0$ or $R = 2$.

And to have $e^{2i\theta} = e^{-i\theta}$, we must have that 2θ and $-\theta$ differ by a multiple of 2π. So we have $2\theta = -\theta + 2k\pi$ for some integer k, while we also want that $0 \leq \theta < 2\pi$. This gives $3\theta = 2k\pi$. If $k = 0$, then we get $\theta = 0$; if $k = 1$, then we get $\theta = \frac{2}{3}\pi$; and if $k = 2$, then we get $\theta = \frac{4}{3}\pi$. For all other values of k, we don't find $0 \leq \theta < 2\pi$.

Combining it all, if $R = 0$, then we have the one solution $z = 0$. And if $R = 2$, then we have $z = 2e^{0i} = 2$, $z = 2e^{2i\pi/3}$ and $z = 2e^{4i\pi/3}$.
[8 pts, Unseen]

(ii) We can write $0 = 0 + 0i$ and $2 = 2 + 0i$. For the other two solutions we find

$$2e^{2i\pi/3} = 2\big(\cos(\tfrac{2}{3}\pi) + i\sin(\tfrac{2}{3}\pi)\big) = 2\big(-\tfrac{1}{2} + i\tfrac{1}{2}\sqrt{3}\big) = -1 + i\sqrt{3},$$
$$2e^{4i\pi/3} = 2\big(\cos(\tfrac{4}{3}\pi) + i\sin(\tfrac{4}{3}\pi)\big) = 2\big(-\tfrac{1}{2} - i\tfrac{1}{2}\sqrt{3}\big) = -1 - i\sqrt{3}.$$

[3 pts, Standard]

Question 2

(a) (i) We have that d *is a divisor of* m if there exists an integer k such that $m = k \cdot d$.

The *greatest common divisor* $\gcd(m,n)$ of two integers m,n, not both zero, is the largest integer d such that d is a divisor of both m and n.
[1 + 1 pts, Bookwork]

(ii) Every integer is a divisor of 0, since we have $0 = 0 \cdot d$ for every integer d. That means that if we would ask for common divisors of 0 and 0, then we would have the set of all integers. Hence there would be no largest common divisor.
[3 pts, Discussed in lectures]

(iii) We first note that $\gcd(-51, 141) = \gcd(141, 51)$, and then start taking the steps in Euclid's algorithm as follows.

$$\begin{aligned} 141 &= 2 \times 51 + 39; \\ 51 &= 1 \times 39 + 12; \\ 39 &= 3 \times 12 + 3; \\ 12 &= 4 \times 3 + 0. \end{aligned}$$

As the final line ends in 0, we have found the greatest common divisor: $\gcd(-51, 141) = \gcd(141, 51) = 3$.
[4 pts, Standard]

(b) (i) If we have $x = 0.0\overline{119}$, then $1000x = 11.9\overline{119}$. This means that $999x = 1000x - x = 11.9\overline{119} - 0.0\overline{119} = 11.9 = \dfrac{119}{10}$. And hence we have $x = \dfrac{119}{10 \cdot 999} = \dfrac{119}{9,990}$.
[3 pts, Bookwork]

(ii) We can write $x = 0.01191\overline{191}$. This shows immediately that $r = 0.01191 = \dfrac{1191}{100,000}$ satisfies $0.0119 < r < 0.0\overline{119}$.
[3 pts, Standard]

(iii) From a result in the course we know that $\sqrt{2}$ is irrational. We also know that $1 < \sqrt{2} < 2$. This means that $0 < \dfrac{\sqrt{2}}{200,000} < \dfrac{2}{200,000}$. Since $\sqrt{2}$ is irrational, also $z = \dfrac{119}{10,000} + \dfrac{1}{200,000}\sqrt{2}$ is irrational. Note that z satisfies $z > \dfrac{119}{10,000} = 0.0119$ and $z < \dfrac{119}{10,000} + \dfrac{2}{200,000} = 0.0119 + 0.00001 = 0.01191 < 0.0\overline{119}$. So z has indeed the desired properties.
[5 pts, Unseen]

(c) Let $x \in (A \cup B) \setminus C$. That means that $x \in A \cup B$ and $x \notin C$. And from $x \in A \cup B$ we know that $x \in A$ or $x \in B$. If $x \in A$, then together with $x \notin C$ we have $x \in A \setminus C$, and hence $x \in (A \setminus C) \cup (B \setminus C)$. While if $x \in B$, then together with $x \notin C$ we have $x \in B \setminus C$, giving $x \in (A \setminus C) \cup (B \setminus C)$ again. So we can conclude that $(A \cup B) \setminus C \subseteq (A \setminus C) \cup (B \setminus C)$.
[5 pts, Unseen]

Question 3

(a) (i) The *contrapositive* of the statement is "if p/q can not be expressed as an Egyptian fraction with $k+1$ terms, then p/q can not be expressed as an Egyptian fraction with k terms".

The *converse* of the statement is "if p/q can be expressed as an Egyptian fraction with $k+1$ terms, then p/q can be expressed as an Egyptian fraction with k terms".
[2 + 1 pts, Standard]

(ii) We need to show that we can write $1/a = 1/b + 1/c$, for some natural numbers b, c, $b \neq c$. We greedily take $1/b < 1/a$ as large as possible, hence we take $b = a+1$. We find that $\dfrac{1}{a} - \dfrac{1}{a+1} = \dfrac{1}{a(a+1)}$. Hence we have $\dfrac{1}{a} = \dfrac{1}{a+1} + \dfrac{1}{a(a+1)}$. And since $a \geq 2$, we have $a(a+1) \neq a+1$, as required.
[4 pts, Unseen]

(iii) Let p/q, $0 < p/q < 1$, be a rational number and suppose that we can express p/q as an Egyptian fraction with $k \geq 2$ terms. In other words we can write $\dfrac{p}{q} = \dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_k}$, where $a_1, a_2, \ldots, a_k$ are different natural numbers. So we can assume that $a_1 < a_2 < \ldots < a_k$. Now in part (ii) we have seen that we can write $\dfrac{1}{a_k} = \dfrac{1}{a_k+1} + \dfrac{1}{a_k(a_k+1)}$, with $a_k < a_k+1 < a_k(a_k+1)$ (since $a_k > a_1 \geq 1$). Putting it together gives $\dfrac{p}{q} = \dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_{k-1}} + \dfrac{1}{a_k+1} + \dfrac{1}{a_k(a_k+1)}$, which gives an expression of p/q as an Egyptian fraction with $k+1$ terms.
[6 pts, Unseen]

(iv) The *contrapositive* of a statement is logically equivalent to the statement itself. Since we proved in (iii) that P is always true, that means that the contrapositive of P is also always true.
[2 pts, Bookwork]

(b) (i) If $c = 1$, the system becomes $\begin{cases} 5x + 3y = 2, \\ x + 2y = 1. \end{cases}$ Multiplying the second equation by 5 gives $5x + 10y = 5$. Since $10 = 3$ in $\mathbb{Z}_7$, that equation is equivalent to $5x + 3y = 5$. But as the first equation is $5x + 3y = 2$, we get $5 = 2$, which is not valid in $\mathbb{Z}_7$.
[3 pts, Standard]

(ii) Multiplying the first equation by 2 gives $10x+6y = 4$, which is equivalent to $3x+6y = 4$ in $\mathbb{Z}_7$. Multiplying the second equation by 3 gives $3cx + 6y = 3$. Subtracting the new first equation from the new second one gives $(3c-3)x = -1 = 6$ in $\mathbb{Z}_7$. Since 7 is a prime number, every element $a \in \mathbb{Z}_7$, $a \neq 0$, has an inverse $a^{-1} \in \mathbb{Z}_7$. Since $3c - 3 \neq 0$ if $c \neq 1$, there is an inverse $(3c-3)^{-1}$. That means that $(3c-3)x = 6$ has the solution $x = 6(3c-3)^{-1}$.

Substituting this value for x in the first equation leads to $5 \cdot 6(3c-3)^{-1} + 3y = 2$, which gives $3y = 2 - 30(3c-3)^{-1} = 2 + 5(3c-3)^{-1}$ (since $-30 = -2 = 5$ in $\mathbb{Z}_7$). The inverse of 3 in $\mathbb{Z}_7$ is 5 (since $3 \cdot 5 = 15 = 1$ in $\mathbb{Z}_7$). So for y we find the solution $y = 5 \cdot \big(2 + 5(3c-3)^{-1}\big) = 10 + 25(3c-3)^{-1} = 3 + 4(3c-3)^{-1}$ in $\mathbb{Z}_7$.

So the solution for the case $c \neq 1$ is $x = 6(3c-3)^{-1}$ and $y = 3 + 4(3c-3)^{-1}$.
[7 pts, Unseen in this form]

Question 4

(a) (i) A function is *surjective* if for all $y \in Y$ there exists an $x \in X$ such that $f(x) = y$.

A function is *injective* if for all $x_1, x_2 \in X$ with $x_1 \neq x_2$ we have that $f(x_1) \neq f(x_2)$.

A function is *bijective* if it is both surjective and injective.
[$1 + 1 + 1$ pts, Bookwork]

(ii) Form 1: For all natural numbers $m, n \in \mathbb{N}$, if there is an injection from $\mathbb{N}_m$ to $\mathbb{N}_n$ (where $\mathbb{N}_m = \{1, 2, 3, \ldots, m\}$), then $m \leq n$.

Form 2: Let A, B be two finite sets, and let f be a function from A to B. If $|A| > |B|$, then there exist $a_1, a_2 \in A$, $a_1 \neq a_2$, such that $f(a_1) = f(a_2)$.
[2 pts, Bookwork]

(iii) Suppose $f : X \to X$ is injective, but not surjective. Let X' be the set of elements in X that appear as an image $f(x)$ for $x \in X$. Since f is not surjective, we have that $X' \neq X$. But since we also have $X' \subseteq X$, this means that $|X'| < |X|$. Since we can consider f as a function from X to X', by the Pigeonhole Principle there are $x_1, x_2 \in X$, $x_1 \neq x_2$, such that $f(x_1) = f(x_2)$. But that contradicts that f is injective. Hence f must be surjective.
[6 pts, Unseen, and quite hard]

(iv) Define the function $f : \mathbb{N} \to \mathbb{N}$ by $f(x) = x+1$. Then f is injective, but not surjective (since there is no element $x \in \mathbb{N}$ such that $f(x) = 1$).
[3 pts, Unseen]

(b) (i) R is reflexive on $\mathbb{N}$. For this, we use that $\gcd(a, a) = a$ (if $a \in \mathbb{N}$). And if $x \in \mathbb{N}$, then $x + 1 \geq 2$, hence $\gcd(x + 1, x + 1) = x + 1 \geq 2$. So we have that xRx for all $x \in \mathbb{N}$.
[2 pts, Unseen, though similar to many exercises]

(ii) R is symmetric on $\mathbb{N}$. For all $a, b \in \mathbb{N}$ we have $\gcd(a, b) = \gcd(b, a)$. This means that $\gcd(x + 1, y + 1) \geq 2$ if and only if $\gcd(y + 1, x + 1) \geq 2$. So we have that $xRy \Rightarrow yRx$ for all $x, y \in \mathbb{N}$.
[3 pts, Unseen, though similar to many exercises]

(iii) R is not transitive on $\mathbb{N}$. Take $x = 1$, $y = 5$ and $z = 2$. Then we have that $\gcd(x + 1, y + 1) = \gcd(2, 6) = 2 \geq 2$ and $\gcd(y + 1, z + 1) = \gcd(6, 3) = 3 \geq 2$, hence xRy and yRz hold. But $\gcd(x + 1, z + 1) = \gcd(2, 3) = 1 \not\geq 2$, hence xRz does not hold. So it is not the case that $(xRy \wedge yRz) \Rightarrow xRz$ for all $x, y, z \in \mathbb{N}$, and hence R is not transitive.
[4 pts, Unseen]

(iv) Since R is not transitive, it cannot be an equivalence relation.
[2 pts, Bookwork]

Question 5

(a) (i) s is an upper bound for A if $s \geq a$ for all $a \in A$. s is the supremum of A if s is the least upper bound of A, i.e., s is an upper bound for A, and $s \leq t$ whenever t is an upper bound for A.
[3pts, Bookwork]

(ii) To show that $\sup(A \cup B) \geq \sup(A)$, it suffices to show that $t = \sup(A \cup B)$ is an upper bound for A, since it then follows that $\sup(A) \leq t$. But this is immediate, since, for every $a \in A$, $a \in A \cup B$, and so $a \leq t$.
[2pts, Similar to exercise]

(iii) Suppose that A dominates B, let $s = \sup(A)$, and take any $c \in A \cup B$. Either (i) $c \in A$, in which case $c \leq s$ since s is an upper bound for A, or (ii) $c \in B$, in which case there is some $a \in A$ with $c \leq a \leq s$, since A dominates B and s is an upper bound for A. Thus s is an upper bound for $A \cup B$.

This implies that $\sup(A \cup B) \leq s = \sup(A)$, and combining this with the previous part gives $\sup(A \cup B) = \sup A$.
[5pts, Unseen]

(iv) This is false. Consider $A = (0, 1)$, $B = (0, 1]$. Then $\sup(A \cup B) = \sup(A) = 1$, but A does not dominate B since $1 \in B$ but there is no element $a \in A$ with $a \geq 1$.
[2pts, Unseen]

(v) This is true. Take any $b \in B$. As s is an upper bound for B, but $s \notin B$, we have $b < s$. Now, as s is the supremum of A, b is not an upper bound for A, and so there is some $a \in A$ with $a > b$. Hence A dominates B.
[4pts, Unseen]

(b) To show that there is at least one such value, we use the Intermediate Value Theorem: if $g : [a, b] \to \mathbb{R}$ is a continuous function, and $g(a) \leq c \leq g(b)$, then there is some $x \in [a, b]$ with $g(x) = c$.
[2pts]

We apply the Intermediate Value Theorem with $g(x) = \sqrt{x} - f(x)$, and $[a, b] = [0, 1]$. We know that g is continuous as it is the sum of the continuous functions $\sqrt{x}$ and $-f(x)$. Also $g(0) = -f(0) = -1$, and $g(1) = 1 - f(1) \geq 1 - f(0) = 0$, since f is decreasing. Hence $g(0) \leq 0 \leq g(1)$, and so there is some $x \in [0, 1]$ with $g(x) = 0$, i.e., $f(x) = \sqrt{x}$.
[5pts, Unseen but routine]

To see that there is at most one such x, note that $g(x)$ is strictly increasing. Explicitly, suppose there are two solutions x_1 and x_2 with $x_1 < x_2$. Then $f(x_1) = \sqrt{x_1} < \sqrt{x_2} = f(x_2)$, contradicting the assumption that f is decreasing.
[2pts, Unseen]

Question 6

(a) (i) To say that $(a_n)_{n\in N}$ is convergent, with limit 1, means that, for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that, for $n > N$, $|a_n - 1| < \varepsilon$.
[3pts, Bookwork]

(ii) Suppose that $a_n \to 1$. We show that $a_n^2 \to 1$.

Fix $\varepsilon > 0$. As $a_n \to 1$, there is some $N \in \mathbb{N}$ such that, for $n > N$, $|a_n - 1| < \min(1, \varepsilon/3)$. Now we have, for $n > N$, $a_n \leq 2$, and therefore

$$|a_n^2 - 1| = |a_n - 1|\,|a_n + 1| < 3|a_n - 1| < 3\frac{\varepsilon}{3} = \varepsilon.$$

Hence indeed $a_n^2 \to 1$.
[6pts, essentially Bookwork]

(iii) We now show that $b_n = \max(a_n, a_n^2) \to 1$.

Fix $\varepsilon > 0$. Take $N_1 \in \mathbb{N}$ such that, for $n > N_1$, $|a_n - 1| < \varepsilon$. Take also $N_2 \in \mathbb{N}$ such that, for $n > N_2$, $|a_n^2 - 1| < \varepsilon$. Now take $N = \max(N_1, N_2)$. For $n > N$, we have $a_n < 1 + \varepsilon$ and $a_n^2 < 1 + \varepsilon$, so $b_n < 1 + \varepsilon$. Also we have $b_n \geq a_n > 1 - \varepsilon$. So $|b_n - 1| < \varepsilon$. Hence indeed $b_n \to 1$.
[4pts, Unseen, but related to a recent past exam question]

(b) (i) We note that

$$\begin{aligned}\sqrt{n+1} - \sqrt{n-1} &= \frac{(\sqrt{n+1} - \sqrt{n-1})(\sqrt{n+1} + \sqrt{n-1})}{\sqrt{n+1} + \sqrt{n-1}} \\ &= \frac{(n+1) - (n-1)}{\sqrt{n+1} + \sqrt{n-1}} = \frac{2}{\sqrt{n+1} + \sqrt{n-1}},\end{aligned}$$

and hence $a_n = \dfrac{2\sqrt{n}}{\sqrt{n+1} + \sqrt{n-1}} = \dfrac{2}{\sqrt{1 + \frac{1}{n}} + \sqrt{1 - \frac{1}{n}}}$.
[5pts, Similar examples have been seen]

By the Algebra of Limits, we have

$$\begin{aligned}\lim_{n\to\infty} a_n &= \frac{2}{\lim_{n\to\infty} \sqrt{1 + \frac{1}{n}} + \lim_{n\to\infty} \sqrt{1 - \frac{1}{n}}} \\ &= \frac{2}{\sqrt{1 + \lim_{n\to\infty} \frac{1}{n}} + \sqrt{1 - \lim_{n\to\infty} \frac{1}{n}}} = \frac{2}{\sqrt{1+0} + \sqrt{1-0}} = 1.\end{aligned}$$

[3pts]

(ii) We proved in the course that $2^{1/n} \to 1$ as $n \to \infty$. Hence there is some $N \in \mathbb{N}$ such that $2^{1/n} > \frac{1}{2}$ for $n > N$. We see that $b_n > \frac{1}{2}$ for even $n > N$, and $b_n < -\frac{1}{2}$ for odd $n > N$. This implies that $(b_n)_{n\in N}$ does not converge. (One could write more, but I think this should suffice.)
[4pts, Unseen]

Question 7

(a) (i) A function $\phi : G \to G'$ is a *homomorphism* if, for every $a, b \in G$, $\phi(a*b) = \phi(a)*'\phi(b)$.
[2pts, Bookwork]

(ii) The *kernel* of ϕ is $\ker(\phi) = \{a \in G \mid \phi(a) = e'\}$, where e' is the identity element of $(G', *')$.
[2pts, Bookwork]

(iii) To see that $\ker(\phi)$ is a subgroup of $(G, *)$, we have three things to check:
1) If $a, b \in \ker(\phi)$, $\phi(a) = \phi(b) = e'$, so $\phi(a * b) = \phi(a) * \phi(b) = e' *' e' = e'$, so $a * b \in \ker(\phi)$.
2) We are given that $\phi(e) = e'$, so that $e \in \ker(\phi)$.
3) If $a \in \ker(\phi)$, then $\phi(a^{-1}) = (\phi(a))^{-1} = (e')^{-1} = e'$, so $a^{-1} \in \ker(\phi)$.

Hence indeed $\ker(\phi)$ is a subgroup.
[5pts, Bookwork]

(b) (i) We show first that $g * \ker(\phi) \subseteq S_h$. An element of $g * \ker(\phi)$ is of the form $g * a$, where $a \in \ker(\phi)$. Now $\phi(g*a) = \phi(g)*'\phi(a) = h*'e' = h$, so $g*a \in S_h$, as required.
[3pts, Unseen]

Now suppose that $f \in S_h$, so that $\phi(f) = h$. We note that $f = g * (g^{-1} * f)$, and we claim that $g^{-1} * f \in \ker(\phi)$. Indeed, $\phi(g^{-1} * f) = (\phi(g))^{-1} *' \phi(f) = h^{-1} *' h = e'$. Hence $f \in g * \ker(\phi)$, as required.
[3pts, Unseen]

(ii) For the next part, we know that all left cosets of $\ker(\phi)$ have size $|\ker(\phi)|$, and there is one coset for each element of $\mathrm{im}(\phi)$. As the cosets (or indeed the inverse images of elements of $\mathrm{im}(\phi)$) partition the group, we have that $|G|$ is equal to the number of cosets times the size of each coset, as given.
[2pts, Unseen]

(c) The function θ is a homomorphism iff we have $\theta(a * b) = \theta(a) * \theta(b)$ for all $a, b \in G$, i.e., $a*b*a*b = a*a*b*b$ for all $a, b \in G$. This certainly holds if $b*a = a*b$ for all $a, b \in G$, i.e., if G is Abelian. Conversely, if, for all $a, b \in G$, we have $a * b * a * b = a * a * b * b$, then we also have $a^{-1} * a * b * a * b * b^{-1} = a^{-1} * a * a * b * b * b^{-1}$, and so $b * a = a * b$ – hence G is Abelian.
[6pts, Unseen, though related to material in lectures/exercises]

(d) If G is Abelian, then the function θ is a homomorphism. Its kernel is $\{g \mid g * g = e\}$, and its image is $\{a \mid a = g * g \text{ for some } g \in G\}$. The result now follows from (b).
[2pts, Unseen]

Question 8

(a) (i) A *basis* of a vector space V is a set B of vectors in B such that (i) B is linearly independent, and (ii) B spans V.

The vector space V has dimension d if there is a basis of cardinality d.
[4pts, Bookwork]

(ii) We follow the hint and take bases $\{\mathbf{u}_1, \mathbf{u}_2\}$ of U and $\{\mathbf{w}_1, \mathbf{w}_2\}$ of W. Now consider $\mathbf{u}_1, \mathbf{u}_2, \mathbf{w}_1, \mathbf{w}_2$. As there are four vectors here (though not necessarily distinct) and V has dimension 3, they are linearly dependent. Thus there are real numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$, not all zero, with

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \beta_1\mathbf{w}_1 + \beta_2\mathbf{w}_2 = \mathbf{0}.$$

We can then rewrite this as

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 = -\beta_1\mathbf{w}_1 - \beta_2\mathbf{w}_2 := \mathbf{v}.$$

The vector $\mathbf{v}$ is in U, since it is a linear combination of the basis elements of U, and similarly it is in W. Suppose that $\mathbf{v} = \mathbf{0}$. As $\mathbf{0} = \mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2$, then as $\mathbf{u}_1, \mathbf{u}_2$ are linearly independent, we have $\alpha_1 = \alpha_2 = 0$. Similarly, as $\mathbf{0} = \mathbf{v} = -\beta_1\mathbf{w}_1 - \beta_2\mathbf{w}_2$, we have $\beta_1 = \beta_2 = 0$. But this contradicts the assumption that not all of $\alpha_1, \alpha_2, \beta_1, \beta_2$ are zero. Therefore the vector $\mathbf{v}$ is a non-zero vector in $U \cap W$.
[11pts, Unseen]

(b) We have three things to check:
(i) The set L is closed under addition. Suppose then that f and g are in L; there are constants K_f and K_g such that, for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| \le K_f|x-y|$, and $|g(x) - g(y)| \le K_g|x-y|$. So we have, for all $x, y \in \mathbb{R}$,

$$\begin{aligned}|(f+g)(x) - (f+g)(y)| &\le |f(x) - f(y)| + |g(x) - g(y)| \\ &\le K_f|x-y| + K_g|x-y| = (K_f + K_g)|x-y|.\end{aligned}$$

So the function $f+g$ is Lipschitz, with constant $K_f + K_g$.

(ii) The zero function is in L: this is clear: we can take $K_0 = 0$.

(iii) The set L is closed under scalar multiplication. Indeed, for f in L with Lipschitz constant K_f, and $\alpha \in \mathbb{R}$, we have

$$|\alpha f(x) - \alpha f(y)| = |\alpha|\,|f(x) - f(y)| \le |\alpha| K_f |x-y|,$$

for all $x, y \in \mathbb{R}$, so the function αf is Lipschitz, with constant $|\alpha|K_f$.

Thus indeed L is a subspace of X.
[10pts, Unseen]

END OF PAPER