AN IMPROVED ERROR TERM FOR MINIMUM H-DECOMPOSITIONS OF GRAPHS

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Abstract. We consider partitions of the edge set of a graph G into copies of a fixed graph H and single edges. Let $\phi_H(n)$ denote the minimum number p such that any n-vertex G admits such a partition with at most p parts. We show that $\phi_H(n) = \mathrm{ex}(n, K_r) + \Theta\big(\mathrm{biex}(n,H)\big)$ for $\chi(H) \geq 3$, where $\mathrm{biex}(n,H)$ is the extremal number of the decomposition family of H. Since $\mathrm{biex}(n,H) = O(n^{2-\gamma})$ for some $\gamma > 0$ this improves on the bound $\phi_H(n) = \mathrm{ex}(n,H) + o(n^2)$ by Pikhurko and Sousa [J. Combin. Theory Ser. B 97 (2007), 1041–1055]. In addition it extends a result of Özkahya and Person [J. Combin. Theory Ser. B, to appear].

1. Introduction

We study edge decompositions of a graph G into disjoint copies of another graph H and single edges. More formally, an *H-decomposition* of G is a decomposition $E(G) = \dot\bigcup_{i\in[t]} E(G_i)$ of its edge set, such that for all $i \in [t]$ either $|E(G_i)| = 1$ or G_i is isomorphic to H. Let $\phi_H(G)$ denote the minimum t such there is a decomposition $E(G) = \dot\bigcup_{i\in[t]} E(G_i)$ of this form, and let $\phi_H(n) := \max_{v(G)=n} \phi_H(G)$.

The function $\phi_H(n)$ was first studied in the seventies by Erdős, Goodman and Pósa [3], who showed that the minimal number $k(n)$ such that every n-vertex graph admits an edge decomposition into $k(n)$ cliques equals $\phi_{K_3}(n)$. They also proved that $\phi_{K_3}(n) = \mathrm{ex}(n, K_3)$, where $\mathrm{ex}(n,H)$ is the maximum number of edges in an H-free graph on n vertices. A decade later this result was extended to K_r for arbitrary r by Bollobás [1] who showed that $\phi_{K_r}(n) = \mathrm{ex}(n, K_r)$ for all $n \geq r \geq 3$.

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General graphs H were considered only recently by Pikhurko and Sousa [6], who proved the following upper bound for $\phi_H(n)$.

Theorem 1 (Theorem 1.1 from [6])**.** *If* $\chi(H) = r \geq 3$ *then*

$$\phi_H(n) = \mathrm{ex}(n, K_r) + o(n^2).$$

Pikhurko and Sousa also conjectured that if $\chi(H) \geq 3$ and if n is sufficiently large, then the correct value is the extremal number of H.

Conjecture 2. *For any graph H with chromatic number at least 3, there is an $n_0 = n_0(H)$ such that $\phi_H(n) = \mathrm{ex}(n, H)$ for all $n \geq n_0$.*

We remark that the function $\mathrm{ex}(n, H)$ is known precisely only for some graphs H, which renders Conjecture 2 difficult. However, $\mathrm{ex}(n, H)$ is known for the family of *edge-critical* graphs H, that is, graphs with $\chi(H) > \chi(H - e)$ for some edge e. And in fact, after Sousa [9, 7, 8] proved Conjecture 2 for a few special edge-critical graphs, Özkahya and Person [5] verified it for all of them.

Our contribution is an extension of the result of Özkahya and Person to arbitrary graphs H, which also improves on Theorem 1. We need the following definition. Given a graph H with $\chi(H) = r$, the *decomposition family* $\mathcal{F}_H$ of H is the set of bipartite graphs which are obtained from H by deleting $r - 2$ colour classes in some r-colouring of H. Observe that $\mathcal{F}_H$ may contain graphs which are disconnected, or even have isolated vertices. Let $\mathcal{F}_H^*$ be a minimal subfamily of $\mathcal{F}_H$ such that for any $F \in \mathcal{F}_H$, there exists $F' \in \mathcal{F}_H^*$ with $F' \subseteq F$. We define

$$\mathrm{biex}(n, H) := \mathrm{ex}(n, \mathcal{F}_H) = \mathrm{ex}(n, \mathcal{F}_H^*)\,.$$

Our main result states that the $o(n^2)$ error term in Theorem 1 can be replaced by $O\big(\mathrm{biex}(n, H)\big)$, which is $O(n^{2-\gamma})$ for some $\gamma > 0$ by the result of Kövari, Turán and Sós [4]. Furthermore, we show that our error term is of the correct order of magnitude.

Theorem 3. *For every integer $r \geq 3$ and every graph H with $\chi(H) = r$ there are constants $c = c(H) > 0$ and $C = C(H)$ and an integer n_0 such that for all $n \geq n_0$ we have*

$$\mathrm{ex}(n, K_r) + c \cdot \mathrm{biex}(n, H) \leq \phi_H(n) \leq \mathrm{ex}(n, K_r) + C \cdot \mathrm{biex}(n, H)\,.$$

Since for every edge-critical H and every n we have $\mathrm{biex}(n, H) = 0$, this is indeed an extension of the result of Özkahya and Person.

2. Outline of the proof and auxiliary lemmas

The lower bound of Theorem 3 is obtained as follows. We let F be an n-vertex $\mathcal{F}_H^*$-free graph with $\mathrm{biex}(n, H)$ edges, and let $c = (r-1)^{-2}$. There is an $n/(r-1)$-vertex subgraph F' of F with at least $c \cdot e(F)$ edges. We let G be obtained from the complete balanced $(r-1)$-partite graph on n vertices by inserting F' into the largest part. Clearly, we have $e(G) \geq \mathrm{ex}(n, K_r) + c \cdot \mathrm{biex}(n, H)$, and by definition of $\mathcal{F}^*_H$, the graph G is H-free, and therefore satisfies $\phi_H(G) = e(G) \geq \mathrm{ex}(n, K_r) + c \cdot \mathrm{biex}(n, H)$.

The upper bound of Theorem 3 is an immediate consequence of the following result.

Theorem 4. *For every integer $r \geq 3$ and every graph H with $\chi(H) = r$ there is a constant $C = C(H)$ and an integer n_0 such that the following holds. Every graph G on $n \geq n_0$ vertices and with*

$$e(G) \geq \mathrm{ex}(n, K_r) + C \cdot \mathrm{biex}(n, H)$$

satisfies $\phi_H(G) \leq \mathrm{ex}(n, K_r)$.

The proof of this theorem (see Section 3) uses the auxiliary lemmas collected in this section and roughly proceeds as follows. We start with a graph $G = (V, E)$ on n vertices with $e(G) \geq \mathrm{ex}(n, K_r) + C \cdot \mathrm{biex}(n, H)$. For contradiction we assume that $\phi_H(G) > \mathrm{ex}(n, K_r)$. This allows us to use a stability-type result (Lemma 5), which supplies us with a partition $V = V_1, \ldots, V_{r-1}$ with parts of roughly the same size and with few edges inside each part. Since $e(G) \geq \mathrm{ex}(n, K_r) + C \cdot \mathrm{biex}(n, H)$ we also know that between two parts only few edges are missing. Next, in each part V_i we identify the (small) set X_i of those vertices with many edges to V_i and set $V_i' := V_i \setminus X_i$ and $X := \bigcup X_i$.

Then we consider the graph $G[V \setminus X]$, and identify a copy of some F in the decomposition family of H in any $G[V_i']$, which we then complete to a copy of H using the classes V_j' with $j \neq i$ (see Lemma 6). We delete this copy of H from G and repeat this process. We shall show that this is possible until the number of edges in all $V_i \setminus X_i$ drops below $\mathrm{biex}(n, H)$, and thus this gives many edge-disjoint copies of H in G.

Finally, we find edge-disjoint copies of H each of which has one of its colour classes in X and the other $(r-1)$ colour classes in V_1', ..., V_{r-1}' (see Lemma 7). It is possible to find many copies of H in this way, because every vertex in X has many neighbours in *every* V_i.

In total these steps will allow us to find enough H-copies to obtain a contradiction.

2.1. **Notation.** Let G be a graph and $V(G) = V_1 \dot{\cup} \cdots \dot{\cup} V_s$ a partition of its vertex set. We write $e(V_i)$ for the number of edges of G with both ends in V_i and $e(V_i, V_j)$ for the number of edges of G with one end in V_i and one end in V_j. Moreover, for $v \in V$ we let $\deg_G(v, V_i) = \deg(v, V_i)$ denote the number of neighbours of v in V_i. An edge of G is called *crossing* (for $V_1 \dot{\cup} \cdots \dot{\cup} V_s$) if its ends lie in different classes of this partition. A subgraph H is called crossing if all of its edges are crossing, and *non-crossing* if none of its edges is crossing. The *chromatic excess* $\sigma(H)$ of H denotes the smallest size of a colour class in a proper $\chi(H)$-colouring of H.

2.2. **Auxiliary lemmas.** The proof of Theorem 4 relies on the following three lemmas. Firstly, we use a stability-type result which was observed in [5].

Lemma 5 (stability lemma [5])**.** *For every $\gamma > 0$, every integer $r \geq 3$, and every graph $H \neq K_r$ with $\chi(H) = r$ there is an integer n_0 such that the following holds. If $G = (V, E)$ has $n \geq n_0$ vertices and satisfies $\phi_H(G) \geq \mathrm{ex}(n, K_r)$, then there is a partition $V = V_1 \dot\cup \cdots \dot\cup V_{r-1}$ such that*

(*a*) $\deg(v, V_i) \leq \deg(v, V_j)$ *for all $v \in V_i$ and all $i, j \in [r-1]$,*

(*b*) $\sum_i e(V_i) < \gamma n^2$, *and*

(*c*) $\frac{n}{r-1} - 2\sqrt{\gamma} n \leq |V_i| \leq \frac{n}{r-1} + 2r\sqrt{\gamma} n$. □

We remark that this lemma is stated in [5] only with assertion (*b*). However, we can certainly assume that the partition obtained is a maximal $(r-1)$-cut, which implies (*a*), and for (*c*) see Claim 8 in [5].

The following lemma allows us to find many H-copies in a graph G with a partition such that each vertex has few neighbours inside its own partition class.

Lemma 6. *For every integer $r \geq 3$, every graph H with $\chi(H) = r$ and every positive $\beta \leq 1/\big(100 e(H)^4\big)$ there is an integer n_0 such that the following holds. Let $G = (V, E)$ be a graph on $n \geq n_0$ vertices, with a partition $V = V_1 \dot\cup \cdots \dot\cup V_{r-1}$ such that for all $i, j \in [r-1]$ with $i \neq j$*

(*i*) $\deg(v, V_j) \geq \left(\frac{1}{r-1} - \beta\right) n$ *for every $v \in V_i$,*

(*ii*) $\sum_{i'=1}^{r-1} e(V_{i'}) \leq \beta^2 n^2 / e(H)$ *and $\Delta(V_i) \leq 2\beta n$.*

Then we can consecutively delete edge-disjoint copies of H from G, until $e(V_i) \leq \mathrm{biex}(n, H)$ for all $i \in [r-1]$. Moreover, these H-copies can be chosen such that each of them contains a non-crossing $F \in \mathcal{F}_H^*$ and all edges in $E(H) \setminus E(F)$ are crossing.

Proof. Let $G = (V, E)$ be a graph and $V = V_1 \dot\cup \cdots \dot\cup V_{r-1}$ be a partition satisfying the conditions of the lemma. We proceed by selecting copies of H in G and deleting them, one at a time, in the following way. First we find a copy of some $F \in \mathcal{F}_H^*$ in $G[V_i]$ for some partition class V_i. Then we extend this F to a copy of H, using only vertices v of G for $H \setminus F$ which have at least $\left(\frac{1}{r-1} - 2\beta\right) n$ neighbours in every partition class other than their own. We say that such vertices v are *β-active.*

We need to show that this deletion process can be performed until $e(V_i) \leq \mathrm{biex}(n, H)$ for all $i \in [r-1]$. Clearly, while $e(V_i) > \mathrm{biex}(n, H)$ for some i, we find some $F \in \mathcal{F}_H^*$ in $G[V_i]$. Let such a copy of F be fixed in the following and assume without loss of generality that $V(F) \subseteq V_{r-1}$. It remains to show that F can be extended to a copy of H.

By condition (*i*), at the beginning of the deletion process every vertex is β-active, and every vertex which gets inactive has lost at least

βn neighbours in some partition class other than its own. Further, by condition (ii) we can find at most $\beta^2 n^2/e(H)$ copies of H in this way. Hence we conclude that even after the very last deletion step, the number of vertices which are not β-active is at most

$$\frac{\beta^2 n^2}{e(H)} e(H) \cdot \frac{1}{\beta n} = \beta n\,.$$

In addition, by condition (ii) we have $\Delta(V_{r-1}) \leq 2\beta n$ at the beginning of the deletion process. Recall moreover that, in this process, we use inactive vertices only in copies of some graph in $\mathcal{F}_H^*$ (and not to complete such a copy to an H-copy). Hence, throughout the process, we have for all $j \in [r-1]$ and all $v \in V \setminus V_j$ that

$$\text{(1)}\quad \deg(v, V_j) \geq \big(\tfrac{1}{r-1} - 2\beta\big)n - e(H) - e(H)\cdot 2\beta n \geq \tfrac{n}{r-1} - 5e(H)\beta n\,.$$

By condition (i) each partition class V_j has size at least $(\frac{1}{r-1} - 2\beta)n$, and thus size at most $\frac{n}{r-1} + 2r\beta n$. Moreover, by (1) each vertex $v \in V_j$ has at most

$$\tfrac{n}{r-1} + 2r\beta n - \big(\tfrac{n}{r-1} - 6e(H)\beta n\big) \leq 8e(H)\beta n$$

non-neighbours in each $V_{j'}$ with $j \neq j'$. Hence, any set $S \subseteq V \setminus V_j$ with $|S| \leq r \cdot v(H)$ has at least $|V_j| - 8r \cdot v(H)e(H)\beta n$ common neighbours in V_j. In particular, S has at least

$$(\tfrac{1}{r-1} - 2\beta)n - \beta n - 8r\cdot v(H)e(H)\beta n \geq \tfrac{n}{r-1} - 11e(H)^3\beta n > \beta n \geq v(H)$$

common neighbours in V_j which are β-active, where we used the condition $\beta \leq 1/(100e(H)^4)$ in the second inequality, and in the last inequality that n is sufficiently large.

When F gets selected in the deletion process, we use the above observation to construct within the β-active common neighbours of F a copy of the complete $(r-2)$-partite graph with $v(H)$ vertices in each part, as follows. We inductively find sets $S_i \subseteq V_i$ of size $v(H)$ which form the parts of this complete $(r-2)$-partite graph. For each $1 \leq i \leq r-2$ in turn, we note that $v(F) + (i-1)v(H) \leq r \cdot v(H)$, and therefore the set $v(F) \cup S_1 \cup \cdots \cup S_{i-1}$ has at least $\beta n \geq v(H)$ common neighbours in V_i which are β-active. We let S_i be any set of $v(H)$ of these β-active common neighbours. Thus we can extend F to a copy of H in G. $\square$

With the help of the next lemma we will find H-copies using those vertices which have many neighbours in their own partition class.

Lemma 7. *For every integer $r \geq 3$, every graph H with $\chi(H) = r$, and every positive $\beta \leq 1/\big(2e(H)^2\big)$ there are integers K and n_0 such that the following holds. Let $G = (V, E)$ be a graph on $n \geq n_0$ vertices, with a partition $V = X \,\dot\cup\, V_1' \,\dot\cup \cdots \dot\cup\, V_{r-1}'$ such that*

(i) *$e(V_i', V_j') > |V_i'||V_j'| - \beta^6 n^2$ for each $i, j \in [r-1]$ with $i \neq j$,*

(ii) *$|X| \leq \beta^6 n$.*

Then we can consecutively delete edge-disjoint copies of H from G, until for all but at most $K\big(\sigma(H)-1\big)$ vertices $x \in X$ there is an $i \in [r-1]$ such that $\deg(x, V_i') \le \beta^2 n$. Moreover, these H-copies can be chosen such that they are crossing for the partition $X \,\dot\cup\, V_1' \,\dot\cup \cdots \dot\cup\, V_{r-1}'$ and each of them uses exactly exactly $\sigma(H)$ vertices of X.

Proof. Without loss of generality we assume that there are only crossing edges in G (otherwise delete the non-crossing edges). We proceed as follows. In the beginning we set $X' := X$. Then we identify $\sigma(H)$ vertices in X' which are completely joined to a complete $(r-1)$-partite graph $K_{r-1}\big(v(H)\big)$ in $V \setminus X$, with $v(H)$ vertices in each part. The subgraph of G identified in this way clearly contains a copy of H with the desired properties, whose edges we delete from G. Next we delete those vertices x from X' with $\deg(x, V_i') \le \beta^2 n$ for some $i \in [r-1]$. Then we continue with the next copy of H.

We need to show that this process can be repeated until $X' \le K\big(\sigma(H)-1\big)$. Indeed, assume that we still have $X' > K\big(\sigma(H)-1\big)$. Observe that since $\sum_{x\in X}\deg(x) < |X|n$ we can find less than $|X|n \le \beta^6 n^2$ copies of H with the desired properties in total, where we used condition (ii). Hence, throughout the process at most $e(H)\beta^6 n^2$ edges are deleted from G. In addition, for each $x \in X'$ we have by definition $\deg(x, V_i') > \beta^2 n$ for all $i \in [r-1]$. Hence we can choose for each i a set $S_i \subseteq N_{V_i'}(x)$ of size $\beta^2 n$. By condition (i) the graph $G[\dot\cup S_i]$ has density at least

$$\frac{\binom{r-1}{2}\big((\beta^2 n)^2 - \beta^6 n^2 - e(H)\beta^6 n^2\big)}{\binom{(r-1)\beta^2 n}{2}} \ge \frac{r-2}{r-1}\big(1-(e(H)+1)\beta^2\big) > \frac{r-3}{r-2},$$

where we used $\beta \le 1/\big(2e(H)^2\big)$ in the last inequality. Thus, since n is sufficiently large, we can apply the supersaturation theorem of Erdős and Simonovits [2], to conclude that the graph $G[\dot\cup S_i]$ contains at least $\delta n^{(r-1)v(H)}$ copies of $K_{r-1}\big(v(H)\big)$, where $\delta > 0$ depends only on β and $e(H)$. Choosing $K := 1/\delta$, we can then use the pigeonhole principle and the fact that $|X'| > K\left(\sigma(H)-1\right)$ to infer that there are $\sigma(H)$ vertices in X' which are all adjacent to the vertices of one specific copy of $K_{r-1}\big(v(H)\big)$ in $G[\dot\cup S_i]$ as desired. □

In addition we shall use the following easy fact about $\mathrm{biex}(n,H)$.

Fact 8. *Let H be an r-chromatic graph, $r \ge 3$. If $\mathrm{biex}(n,H) < n-1$ then $\sigma(H) = 1$.*

Proof. If $\sigma(H) \ge 2$, then each F from $\mathcal{F}_H^*$ contains a matching of size 2. Thus $\mathrm{biex}(n,H) \ge n-1$ since the star $K_{1,n-1}$ does not contain two disjoint edges. □

3. Proof of Theorem 4

In this section we show how Lemmas 5, 6 and 7 imply Theorem 4.

Proof of Theorem 4. Let r and H with $\chi(H) = r \geq 3$ be given. If $H = K_r$, then the result of Bollobás [1] applies, hence we can assume that $H \neq K_r$. We choose

$$\beta := \frac{1}{100e(H)^4} \qquad \text{and} \qquad \gamma := \frac{\beta^{12}}{1000e(H)^4}\,. \tag{2}$$

Let K be the constant from Lemma 7 and choose

$$C := K \cdot v(H)\beta^{-1}\,. \tag{3}$$

Finally let n_0 be sufficiently large for Lemmas 5, 6 and 7.

Now let G be a graph with $n \geq n_0$ vertices and

$$e(G) \geq \mathrm{ex}(n, K_r) + C \cdot \mathrm{biex}(n, H)\,, \tag{4}$$

and assume for contradiction that

$$\phi_H(G) \geq \mathrm{ex}(n, K_r)\,. \tag{5}$$

Observe first that we may assume without loss of generality that

$$\delta(G) \geq \delta\big(T_{r-1}(n)\big)\,. \tag{6}$$

Indeed, if this is not the case, we can consecutively delete vertices of minimum degree until we arrive at a graph G_{n^*} on n^* vertices with $\delta(G_{n^*}) \geq \delta\big(T_{r-1}(n^*)\big)$. Denote the sequence of graphs obtained in this way by $G_n := G$, G_{n-1}, $\dots$, G_{n^*}. We have

$$\mathrm{ex}(n, K_r) \leq \phi_H(G) \leq \phi_H(G_{n-1}) + \delta\big(T_{r-1}(n)\big) - 1$$

and thus $\phi_H(G_{n-1}) \geq \mathrm{ex}(n-1, K_r) + 1$. Similarly $\phi_H(G_{n-i}) \geq \mathrm{ex}(n-i, K_r) + i$. Since n is sufficiently large there is an i^* such that $n - i^* \geq n_0$ and $i^* \geq \binom{n-i^*}{2} + 1$. Hence $n^* > n - i^* \geq n_0$, since otherwise $\phi_H(G_{n^*}) \geq \mathrm{ex}(n^*, K_r) + \binom{n^*}{2} + 1$, a contradiction. Thus we may assume (6).

Next, by (5), we can apply Lemma 5, which provides us with a partition $V_1 \dot\cup \ldots \dot\cup V_{r-1}$ of $V(G)$ such that assertions (a), (b) and (c) in Lemma 5 are satisfied. Let $m := \sum_{i=1}^{r-1} e(V_i)$. Equation (4) and Lemma 5(b) imply

$$C \cdot \mathrm{biex}(n, H) \leq m \leq \gamma n^2 \overset{(2)}{\leq} \beta^2 n^2 / e(H)\,. \tag{7}$$

Further, by the definition of m we clearly have $e(G) \leq \mathrm{ex}(n, K_r) + m$. Hence it will suffice to find $\frac{m}{e(H)-1} + 1$ edge-disjoint copies of H in G, since this would imply

$$\phi_H(G) \leq \mathrm{ex}(n, K_r) + m - \left(\frac{m}{e(H)-1} + 1\right)\big(e(H) - 1\big) < \mathrm{ex}(n, K_r)\,,$$

contradicting (5). So this will be our goal in the following, which we shall achieve by first applying Lemma 6 and then Lemma 7.

We prepare these applications by identifying for every $i \in [r-1]$ the set X_i of vertices in V_i with high degree to its own class, that is,

$$X_i := \big\{ v \in V_i \colon \deg(v, V_i) \geq \tfrac{1}{2}\beta n \big\}\,.$$

Let $X := \dot\bigcup_{i\in[r-1]} X_i$. This implies

$$|X| \le \frac{2m}{\frac{1}{2}\beta n} \overset{(7)}{\le} \frac{2\gamma n^2}{\frac{1}{2}\beta n} \overset{(2)}{\le} \sqrt{\gamma} n \overset{(2)}{\le} \beta^6 n\,. \tag{8}$$

In addition we set $V_i' := V_i \setminus X_i$ for all $i \in [r-1]$, $n' := |V \setminus X|$, $m' := \sum_{i=1}^{r-1} e(V_i \setminus X_i)$ and $m_X := m - m' = e(X) + \sum_{i=1}^{r-1} e(X_i, V_i')$.

Step 1. We want to apply Lemma 6 to the graph $G[V \setminus X]$ and the partition $V_1' \dot\cup \cdots \dot\cup V_{r-1}'$. We first need to check that the conditions are satisfied. By Lemma 5(c) and (8) we have for each for each $i, j \in [r-1]$ with $i \neq j$ that $|V \setminus (V_i \cup V_j')| \le (\frac{r-3}{r-1} + 6\sqrt{\gamma})n$. Moreover, by (8) we clearly have $n' \ge n/2$. Hence, by the definition of X, for each $v \in V_i'$ we have

$$\begin{aligned} \deg(v, V_j') &\overset{(6)}{\ge} \delta\big(T_{r-1}(n)\big) - |V \setminus (V_i \cup V_j'|)| - \tfrac{1}{2}\beta n \\ &\ge \big(\tfrac{1}{r-1} - 7\sqrt{\gamma} - \tfrac{1}{2}\beta\big)n \overset{(2)}{\ge} \big(\tfrac{1}{r-1} - \beta\big)n' \end{aligned} \tag{9}$$

and thus condition (i) of Lemma 6 is satisfied. Condition (ii) of Lemma 6 holds by (7) and the definition of X. Therefore we can apply Lemma 6.

This lemma asserts that we can consecutively delete copies of H from $G[V \setminus X]$, each containing a non-crossing $F \in \mathcal{F}_H^*$ and crossing edges otherwise, until $e(V_i') \le \mathrm{biex}(n', H)$. Denote the graph obtained after these deletions by G_1.

We have $\max_{F\in\mathcal{F}_H^*} e(F) \le e(H) - 2$, since $\chi(H) \ge 3$. Hence each copy of H deleted in this way uses at most $e(H)-2$ non-crossing edges, and so this gives at least

$$\frac{m' - (r-1)\,\mathrm{biex}(n, H)}{e(H) - 2} \ge \frac{m' - r\,\mathrm{biex}(n, H)}{e(H) - 2} \tag{10}$$

edge-disjoint copies of H in $G[V \setminus X]$.

By assertion (b) of Lemma 5 and the assumption $e(G) \ge \mathrm{ex}(n, K_r)$, we have that $e_G(V_i, V_j) \ge |V_i||V_j| - \gamma n^2$, and thus $e_G(V_i', V_j') \ge |V_i'||V_j'| - \gamma n^2$. Again by assertion ($b$) of Lemma 5, in obtaining G_1 as described above we delete at most γn^2 copies of H, so we have

$$\begin{aligned} e_{G_1}(V_i', V_j') &\ge |V_i'||V_j'| - \gamma n^2 - (e(H) - 1)\gamma n^2 \\ &= |V_i'||V_j'| - e(H)\gamma n^2 \overset{(2)}{\ge} |V_i'||V_j'| - \beta^6 n^2\,. \end{aligned} \tag{11}$$

Step 2. Next we want to apply Lemma 7 to G_1 and the partition $X \dot\cup V_1' \dot\cup \cdots \dot\cup V_{r-1}'$. Note that condition ($i$) of Lemma 7 is satisfied by (11) and condition (ii) by (8). Hence Lemma 7 allows us to delete crossing copies of H from G until all vertices x of a subset $X_0 \subseteq X$ with $|X| - |X_0| \le K(\sigma(H) - 1) =: K'$ have $\deg(x, V_{i(x)}') \le \beta^2 n$ for some $i(x) \in [r-1]$. Denote the graph obtained after these deletions by G_2.

Now let $x \in X_j$ for some $j \in [r-1]$ be arbitrary. We set $m_x := \deg_G(x, V_j \setminus X)$. Since no edges adjacent to x were deleted in step 1, if $x \in X_0$ then the number of edges adjacent to x deleted in step 2 is at least $\deg_G(x, V_{i(x)} \setminus X) - \beta^2 n \geq m_x - 2\beta^2 n$, where we used assertion (a) of Lemma 5 and (8) in the inequality. Hence, since $m_X = \sum_{x \in X} m_x + e(X)$, in total at least

$$\begin{aligned} m_X - K'n - |X_0|\beta^2 n - e(X) &\geq m_X - K'n - 2\beta^2 n|X| \\ &\overset{(8)}{\geq} m_X - K'n - 8\beta m \end{aligned}$$

edges adjacent to X were deleted in step 2. By Fact 8 we have $K' = K(\sigma(H)-1) = 0$ if $\mathrm{biex}(n,H) < n-1$. If $\mathrm{biex} \geq n-1 \geq n/2$ on the other hand, then $m \geq Cn/2$ by (7) and thus $K'n \leq 2K'm/C$. Observe moreover that, because $H \neq K_3$, each H-copy deleted in this step uses at least 2 edges which are not adjacent to X. We conclude that at least

$$\frac{m_X - \frac{2K(\sigma(H)-1)}{C}m - 8\beta m}{e(H)-2} \overset{(3)}{\geq} \frac{m_X - 9\beta m}{e(H)-2} \tag{12}$$

edge-disjoint copies of H were deleted from G_1 in step 2.

Combining (10) and (12) reveals that G contains

$$\begin{aligned} \frac{m' - r\,\mathrm{biex}(n,H)}{e(H)-2} + \frac{m_X - 9\beta m}{e(H)-2} &\overset{(7)}{\geq} \frac{m - \frac{r}{C}m - 9\beta m}{e(H)-2} \\ &\overset{(3)}{\geq} \frac{m - 10\beta m}{e(H)-2} \overset{(2)}{\geq} \frac{m}{e(H)-1} + 1 \end{aligned}$$

edge-disjoint copies of H, which gives the desired contradiction. □

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