# THE TRIPARTITE RAMSEY NUMBER FOR TREES

JULIA BÖTTCHER, JAN HLADKÝ, AND DIANA PIGUET

Abstract. We prove that for all $\varepsilon > 0$ there are $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following holds. For any two-colouring of the edges of $K_{n,n,n}$ one colour contains copies of all trees $T$ of order $t \leq (3-\varepsilon)n/2$ and with maximum degree $\Delta(T) \leq n^{\alpha}$. This confirms a conjecture of Schelp.

## 1. Introduction and results

In this paper we address a Ramsey-type question posed by Schelp. For any finite family of graphs $\mathcal{F}$ we say that $K_m$ is *Ramsey* for $\mathcal{F}$ if in any edge-colouring of $K_m$ with green and red there are either copies of all members of $\mathcal{F}$ in green or in red. In this case we also write $K_m \to \mathcal{F}$. Define the *Ramsey number* $R(\mathcal{F})$ as the smallest integer $m$ such that $K_m \to \mathcal{F}$. The famous theorem of Ramsey [14] states that $R(\mathcal{F})$ always exists. Here we are interested in the case when $\mathcal{F}$ is a family of trees. Let $\mathcal{T}_t$ denote the class of trees of order $t$ and let $\mathcal{T}_t^{\Delta}$ be its restriction to trees of maximum degree at most $\Delta$. Ajtai, Komlós, Simonovits, and Szemerédi [1] announced a solution of the Erdős-Sós conjecture [1] which implies that $K_{2t-2} \to \mathcal{T}_t$ for large even $t$ and $K_{2t-3} \to \mathcal{T}_t$ for large odd $t$. This bound is best possible. For the case of odd $t$ this is also a consequence of a theorem by Zhao [18] concerning a conjecture of Loebl [4].

The graph $K_{R(\mathcal{F})}$ is obviously a Ramsey graph for $\mathcal{F}$ with as few vertices as possible. However, one may still ask, whether there exist graphs with fewer edges which are Ramsey for $\mathcal{F}$. This minimum number of edges is also called *size Ramsey number* and denoted by $R_s(\mathcal{F})$. Trivially $R_s(\mathcal{F}) \leq \binom{R(\mathcal{F})}{2}$, but it turns out that this inequality is often far from tight. The investigation of size Ramsey numbers recently received much attention. Trees are considered in [3, 9]. Progress on determining the size Ramsey number for classes of bounded degree graphs was made in [12].

A question of similar flavour is what happens when we do not confine ourselves to finding Ramsey graphs for $\mathcal{F}$ with fewer edges but require in addition that they are proper subgraphs of $K_m$ with $m$ very close to $R(\mathcal{F})$. This question has two aspects: a quantitative one (i.e., how many edges can be deleted from $K_m$ so that the remaining graph is still Ramsey) and a structural one (i.e., what is the structure of the edges that may be deleted). One of the most natural instances of the structural version of the problem is to consider *multipartite Ramsey graphs*, that is, to ask whether the complete spanning balanced $k$-partite subgraph of $K_{R(H)}$ is still Ramsey for $H$.

Questions of this type recently received some interest and were explored by Gyárfás, Sárközy, and Schelp [8] when $\mathcal{F}$ consists of an odd cycle and by Gyárfás, Ruszinkó, Sárközy, and Szemerédi [7] when $\mathcal{F}$ is a path. More specifically, the main result of [8] asserts that any colouring of the edges of the complete five-partite graph on $(2+o(1))t$ vertices with colour classes of the same order of magnitude contains a monochromatic copy of $C_t$. It is known (cf. [15], [5]) that $R(C_t) = 2t-1$ for every odd $t \geq 5$. Hence the main result of [8] shows that that the complete graph of order just a little bit larger than $2t$ remains Ramsey for $C_t$ even when one deletes edges so as to create five large holes in this graph. In a similar spirit, the main result of [7] states that one can find a monochromatic copy of a path $P_t$ in a two edge coloured graph $K_{n,n,n}$, where $3n = (\frac{1}{2}+o(1))3t$. This again asymptotically matches the bound on the Ramsey number $R(P_t) = \lfloor (3t-2)/2 \rfloor$ determined in [6].

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An analogous result for a tree $T_t$ on $t$ vertices should assert that for some $k$ the $k$-partite complete graph on $(1+o(1))R(T_t) = (2+o(1))t$ vertices is Ramsey for $T_t$. It is easy to see that we need $k > 2$. The following conjecture of Schelp [16] claims that we can choose $k = 3$ for trees with small maximum degree. Schelp conjectured that the tripartite graph $K_{n,n,n}$ is Ramsey for the class $\mathcal{T}_t^\Delta$ of trees on $t \leq 3n/2 - o(n)$ vertices with maximum degree at most $\Delta$ for constant $\Delta$. The conjecture thus asserts that we can delete three cliques of size $m/3$ from a graph $K_m$ with $m$ only slightly larger than $R(\mathcal{T}_t^\Delta)$ while maintaining the Ramsey property. In addition Schelp asked whether the same remains true when the constant maximum degree bound in the conjecture above is replaced by $\Delta \leq \frac{2}{3}t$ (which is easily seen to be best possible). Our main result is situated in-between these two cases, solving the problem for trees of maximum degree $n^\alpha$ for some small $\alpha$ and hence, in particular, answering the first conjecture above.

**Theorem 1.** *For all $\mu > 0$ there are $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have*

$$K_{n,n,n} \to \mathcal{T}_t^\Delta,$$

*with $\Delta \leq n^\alpha$ and $t \leq (3-\mu)n/2$.*

We use Szemerédi's regularity lemma [17] to establish this result. Due to the nature of the methods related to this lemma it follows that Theorem 1 remains true when $K_{n,n,n}$ is replaced by a much sparser graph: For any fixed $\mu \in (0, 1]$ a random subgraph of $K_{n,n,n}$ with edge probability $\mu$ allows for the same conclusion, as long as $n$ is sufficiently large (cf. Section 8).

The proof of Theorem 1 splits into a combinatorial part and a regularity based embedding part. Łuczak [13] first noted that a large joined[1] matching in a cluster graph obtained from an application of the regularity lemma is a suitable structure for embedding paths. In the present paper, we extend Łuczak's idea and use what we call "odd joined matchings" and "joined fork systems" in the cluster graph (a somewhat similar amplification of Łuczak's idea has appeared in [10], where the clusters inducing the joined matching were of different sizes). Here we show that in one colour we find either a large joined matching connected to an odd cycle, or both a large joined matching and a large joined fork system. This is the first main idea of the proof. We mention in addition that we shall distinguish two cases in order to obtain these odd joined matchings and fork systems: Either the edge colouring of $K_{n,n,n}$ under consideration possesses some nice properties which will allow us to infer the existence of the desired structures. Or this edge colouring is of a very special form, a so-called "extremal configuration", which we shall treat separately (see also Lemmas 9 and 10).

The second main idea concerns the embedding part, where we show that, given the above mentioned matchings and fork systems, we can embed each tree of $\mathcal{T}_t^\Delta$ into the subgraph of $K_{n,n,n}$ formed by the edges of the corresponding colour. For this purpose we formulate an embedding lemma (Lemma 13) that provides rather general conditions for the embedding of trees with growing maximum degree.

**Organisation.** In Section 2 we define a number of concepts which are crucial to the proof of Theorem 1 and we state the regularity lemma. For the subsequent presentation of the proof of Theorem 1 we choose a top-down approach: we first formulate the main lemmas needed for this proof, next show how these lemmas imply the theorem, and then we give the proofs of these lemmas. More precisely, we provide the main structural lemma (Lemma 8) that we need for the combinatorial part of the proof in Section 3. In Section 4 we formulate the above mentioned embedding lemma (Lemma 13) together with an auxiliary lemma (the so-called "assignment lemma" (Lemma 14), which prepares the application of the embedding lemma in the proof of Theorem 1. An outline and the details of this proof are also presented in Section 4. In Section 5 we prove the assignment lemma (Lemma 14) and in Section 6 the embedding lemma (Lemma 13). In Section 7 we prove the main structural lemma (Lemma 8). In Section 8 we close with some concluding remarks.

[1] The original name from [13] is "connected matching". In this paper we replace the word "connected" by the less overloaded "joined".

## 2. Definitions and Tools

In this section we first provide some basic definitions. Then, in Section 2.1, we define regularity and state the regularity lemma. Finally, in Section 2.2 we introduce a number of concepts that will turn out essential when dealing with edge coloured graphs in the proof of Theorem 1. In particular, we shall rigorously define the connected matchings and fork systems mentioned above along with the extremal configurations appearing in the proof of Theorem 1.

Let $G = (V, E)$ be a graph and $X, X', X'' \subseteq V$ be pairwise disjoint vertex sets. Then we define $E(X) := E \cap \binom{X}{2}$ and $E(X, X') := E \cap (X \times X')$ and write $G[X]$ for the graph with vertex set $X$ and edge set $E(X)$. Similarly, $G[X, X']$ is the bipartite graph with vertex set $X \dot\cup X'$ and edge set $E(X, X')$ and $G[X, X', X'']$ is the tripartite graph with vertex set $X \dot\cup X' \dot\cup X''$ and edge set $E(X, X') \dot\cup E(X', X'') \dot\cup E(X'', X)$. For convenience we frequently identify graphs $G$ with their edge set $E(G)$ and vice versa. We say that a subgraph $G'$ of $G$ covers a vertex $v$ of $G$ if $v$ is contained in some edge of $G'$. For a vertex set $D$ and an edge set $M$ we denote by $D \cap M$ the set of vertices from $D$ that appear in some edge of $M$. We write $\mathrm{N}(v)$ for the neighbourhood of a vertex $v$.

A *matching* $M$ in a graph $G = (V, E)$ is the union of vertex disjoint edges in $E$ and its size is the number of edges in $M$. For vertices $v$ covered by $M$ and vertex sets $U$ covered by $M$ we also write, abusing notation, $v \in M$ and $U \subseteq M$. Sometimes we also consider a matching as a bijection $M : V_M \to V_M$ where $V_M \subseteq V$ is the set of vertices covered by $M$. For $U \subseteq V_M$ we then denote by $M(U)$ the set of vertices $v \in V_M$ such that $uv \in M$ for some $u \in U$.

By $\mathcal{K}_n^\eta$, finally, we denote the class of all spanning subgraphs $K$ of $K_{n,n,n}$ with minimum degree $\delta(K) > (2 - \eta)n$. We also call the graphs in this class *$\eta$-complete tripartite graphs*.

2.1. **Regularity.** Let $G = (V, E)$ be a graph and $\varepsilon, d \in [0, 1]$. For disjoint nonempty vertex sets $U, W \subseteq V$ the *density* $d(U, W)$ of the pair $(U, W)$ is the number of edges that run between $U$ and $W$ divided by $|U||W|$. A pair $(U, W)$ is *$\varepsilon$-regular* if $|d(U', W') - d(U, W)| \le \varepsilon$ for all $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \ge \varepsilon|U|$ and $|W'| \ge \varepsilon|W|$. The following lemma states that in dense regular pairs most vertices have many neighbours. This is an immediate consequence of the definition of regular pairs.

**Lemma 2.** *Let $(U, U')$ be an $\varepsilon$-regular pair with density at least $d$ and $X \subseteq U$ with $|X| \ge \varepsilon|U|$. Then less than $\varepsilon|U'|$ vertices in $U'$ have less than $(d - \varepsilon)|X|$ neighbours in $X$.* □

In the rest of the paper we will say that all other vertices in $U'$ are $(\varepsilon, d)$-*typical* with respect to $X$ (or simply typical, when $\varepsilon$ and $d$ are clear from the context).

An *equipartition* of $G = (V, E)$ is a partition $V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_k$ of $V$ with $|V_i| = |V_j|$ for all $i, j \in [k]$. An *$\varepsilon$-regular partition* of $G = (V, E)$ with *reduced graph* $\mathbb{G} = ([k], E_{\mathbb{G}})$ is an equipartition $V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_k$ of $V$ with $|V_0| \le \varepsilon|V|$, such that $(V_i, V_j)$ is an $\varepsilon$-regular pair in $G$ whenever $ij \in E_{\mathbb{G}}$. If in addition $d(V_i, V_j) \ge d$ for every $ij \in E_{\mathbb{G}}$ we say that $G$ has $(\varepsilon, d)$-*reduced graph* $\mathbb{G}$. (Throughout this paper blackboard symbols such as $\mathbb{G}$ or $\mathbb{M}$ denote reduced graphs and their subgraphs.) The partition classes $V_i$ with $i \in [k]$ are also called *clusters* of $G$ and $V_0$ is the *bin set*. We also call a vertex $i$ of the reduced graph a cluster and identify it with its corresponding set $V_i$.

Suppose that $P$ is a partition of $V$. We then say that a partition $V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_s$ of $V$ *refines* $P$ if for every $i \in [s]$ there exists a member $A \in P$ such that $V_i \subseteq A$.

Now we state a version of Szemerédi's celebrated regularity lemma [17] for graphs $G$ with a given prepartition. This lemma asserts the existence of a regular partition of $G$ which refines the given prepartition.

**Lemma 3** (Regularity lemma). *For all $\varepsilon > 0$ and integers $k_0$ and $k_*$ there is an integer $k_1$ such that for all graphs $G = (V, E)$ on $n \ge k_1$ vertices the following holds. Let $G$ be given together with a partition $V = V_1^* \dot\cup \dots \dot\cup V_{k_*}^*$ of its vertices. Then there is $k \in \mathbb{N}$ with $k_0 \le k \le k_1$ such that $G$ has an $\varepsilon$-regular partition $V = V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_k$ refining $V_1^* \dot\cup \dots \dot\cup V_{k_*}^*$ with a reduced graph $\mathbb{G} = ([k], E_{\mathbb{G}})$, where $|E_{\mathbb{G}}| \ge \binom{k}{2} - \varepsilon k^2$.* □

We also say that $V = V_1^* \dot\cup \dots \dot\cup V_{k_*}^*$ is a *prepartition* of $G$.

2.2. **Coloured graphs.** A *coloured graph* $G$ is a graph $(V, E)$ together with a 2-colouring of its edges by red and green. For $\chi \in \{\text{red}, \text{green}\}$ we denote by $G(\chi)$ the subgraph of $G$ formed by exactly those edges with colour $\chi$. Two vertices are *$\chi$-joined* in $G$ if they lie in the same component of $G(\chi)$. Let $G$ be a coloured graph and $v$ be a vertex of $G$ and $\chi \in \{\text{red,green}\}$. Then a vertex $u$ is a *$\chi$-neighbour* of $v$ if $uv$ is an edge of colour $\chi$ in $G$. The *$\chi$-neighbourhood* of $v$ is the set of all $\chi$-neighbours of $v$.

**Definition 4** (joined, odd, even)**.** *Let $G'$ be a subgraph of an uncoloured graph $G$. Then we say that $G'$ is* joined *if any two vertices covered by $G'$ are in the same component of $G$. Further, the component of $G$ containing $G'$ is called the* continent of $G'$ *and is denoted by $G_{<G'>}$.*

*Similarly, if $G'$ is a $\chi$-monochromatic subgraph of a coloured graph $G$, we say that $G'$ is* joined *if any two vertices covered by $G'$ are $\chi$-joined in $G$. Again, the component of $G(\chi)$ containing $G'$ is called the* continent of $G'$ *and is denoted by $G_{<G'>}$.*

*Further, $G'$ is* odd *if there is an odd cycle in $G_{<G'>}$, otherwise $G'$ is* even.

For subgraphs containing edges of different colours the notion of being joined is not defined.

**Definition 5** (fork, fork system)**.** *An $r$-*fork *(or simply* fork*) is the complete bipartite graph $K_{1,r}$. We also say that an $r$-fork has $r$* prongs *and one* centre *by which we refer to the vertices in the two partition classes of $K_{1,r}$. A* fork system *$F$ in a graph $G$ is the union of pairwise vertex disjoint forks in $G$ (not necessarily having the same number of prongs). We say that $F$ has* ratio *$r$ if all its forks have at most $r$ prongs. Then we also call $F$ an $r$-fork system.*

Suppose $F$ is a joined fork system in $G$. Then the *size* $|F|$ of $F$ is defined as follows. We distinguish two cases. If $F$ is even, i. e., if the continent $G_{<F>}$ containing $F$ is bipartite, then the size $|F|$ of $F$ is the order of the bigger bipartition class of $G_{<F>}$. In particular, if all the centres belong to the same bipartition class of $G_{<F>}$, then $|F|$ is the total number of prongs in $F$. We stress, however, that in our proofs it need not be the case that all centres of $F$ belong to the same partition class of $G_{<F>}$.

If $F$ is odd, on the other hand then we would in principle like to use the same definition of $|F|$ as in the first case. For this purpose we first identify a suitable bipartite subgraph of $G_{<F>}$ and then define $|F|$ as the order of its bigger partition class. More precisely, the size $|F|$ of $F$ is the maximum number $f$ for which there exists a connected bipartite subgraph $G'$ of $G$ such that $F$ has size $f$ in $G'$. For a vertex set $D$ in $G$ we say that $F$ is *centred* in $D$ if the centres of the forks in $F$ all lie in $D$.

Next, we define two properties of coloured graphs that characterise structures (in a reduced graph) suitable for the embedding of trees as we shall see later (cf. Section 4). Roughly speaking, these properties guarantee the existence of large monochromatic joined matchings and fork systems.

**Definition 6** ($m$-odd, $(m, f, r)$-good)**.** *Let $G$ be a coloured graph on $n$ vertices. Then $G$ is called* $m$-odd *if $G$ contains a monochromatic odd joined matching of size at least $m$. We say that $G$ is* $(m, f, r)$-good *(in colour $\chi$) if $G$ contains a $\chi$-coloured joined matching $M$ of size at least $m$ as well as a $\chi$-coloured joined fork system $F$ of size at least $f$, and ratio at most $r$.*

We further need to define a set of special, so-called extremal, configurations of coloured graphs that will need special treatment in our proofs. To prepare their definition, let $K$ be a graph on $n$ vertices and $D, D'$ be disjoint vertex sets in $K$. We say that the bipartite graph $K[D, D']$ is *$\eta$-complete* if each vertex of $K[D, D']$ is incident to all but at most $\eta n$ vertices of the other bipartition class. If $K$ is a coloured graph then $K[D, D']$ is *$(\eta, \chi)$-complete* for some colour $\chi$ if it is $\eta$-complete and all edges in $K[D, D']$ are of colour $\chi$. We call a set $A$ *negligible* if $|A| < 2\eta n$. Otherwise, $A$ is *non-negligible*.

**Definition 7** (extremal)**.** *Let $K = (V, E)$ be a coloured graph of order $3n$. Suppose that $\eta > 0$ is given. We say that $K$ is a pyramid configuration with parameter $\eta$ if it satisfies* (E1) *below and a web configuration if it satisfies* (E2)*. In both cases we call $K$* extremal with parameter $\eta$ *or* $\eta$-extremal*. Otherwise we say that $K$ is* not $\eta$-extremal*.*

*We need the following auxiliary definition. We say that $K$ forms an $\eta$-*thread *if there is a colour $\chi$ and pairwise disjoint subsets $U^{(i)}, W^{(i)} \subseteq V$, $i \in [3]$ such that $|U^{(i)} \cup W^{(i)}| \geq (1-\eta)n$ and $K[U^{(i)}, W^{(j)}]$ is $(\eta, \chi)$-complete for all $i \neq j \in \{1,2,3\}$, the edges in all these bipartite graphs together form a connected bipartite subgraph $K_\chi$ of $K$ with (bi)partition classes $\bigcup U^{(i)}$ and $\bigcup W^{(i)}$ (see Figure 1).*

(E1) pyramid configurations: *There are (not necessarily distinct) colours $\chi, \chi'$ and pairwise disjoint subsets $D_1, D_2, D_1', D_2' \subseteq V$ of size at most $n$, with $|D_1|, |D_2| \geq (1-\eta)n$ and $|D_1'|+|D_2'| \geq (1-\eta)n$ where $D_1'$ and $D_2'$ are either empty or non-negligible. Further, $K[D_1, D_1']$ and $K[D_2, D_2']$ are $(\eta, \chi)$-complete and $K[D_1, D_2']$, $K[D_2, D_1']$, and $K[D_1, D_2]$ are $\eta$-complete.*

*In addition, either $K[D_1, D_2]$ is $(\eta, \chi')$-complete or both $K[D_1, D_2']$ and $K[D_1', D_2]$ are $(\eta, \chi')$-complete. In the first case we say the pyramid configuration has a $\chi'$-*tunnel*, and in the second case that it has a* crossing*. The pairs $(D_1, D_1')$ and $(D_2, D_2')$ are also called the* pyramids *of this configuration.*

(E2) web configuration: *$K$ forms an $\eta$-thread. Further there are sets $W_j^{(i)} \cup W_k^{(i)} = W^{(i)}$, $i \neq 3$, $j$ and $k$ distinct, and $W_1^{(3)} \cup W_2^{(3)} \cup W_3^{(3)} = W^{(3)}$, each of which is either empty or non-negligible, such that the following conditions are satisfied, for distinct $i, j, k$:*

1. $|U^{(1)}| \geq |U^{(2)}| \geq |U^{(3)} \cup W_3^{(3)}|$ *and* $|W_j^{(i)}| = |W_i^{(j)}| \leq n - |U^{(k)}|$,
2. *either* $W_3^{(3)} = \emptyset$ *or* $U_2^{(1)} = \emptyset$,
3. *either* $W^{(1)} = \emptyset$ *or* $|\bigcup W^{(i)} \setminus W_3^{(3)}| \leq (1-\eta)\frac{3}{2}n$,
4. *either* $U^{(3)} = \emptyset$ *or* $|\bigcup U^{(i)}| < (1-\eta)\frac{3}{2}n$ *or* $|U^{(2)} \cup U^{(3)}| \leq (1-\eta)\frac{3}{4}n$.

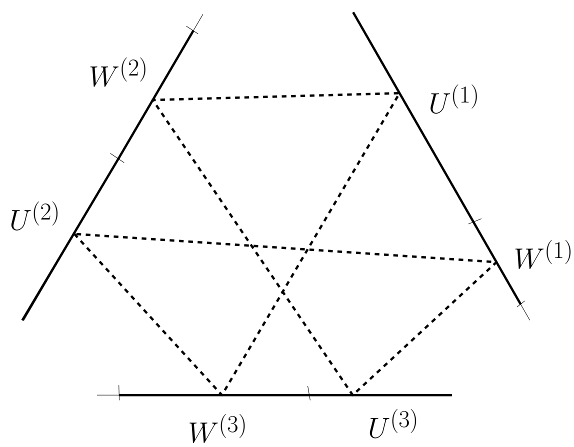


FIGURE 1. Bipartite subgraph of $K_\chi$. The dashed lines indicates between which sets there are $(\eta, \chi)$-complete graphs.

## 3. Joined matchings and fork systems

In this section we formulate the main structural lemma (Lemma 8) for the proof of Theorem 1. The proof of this lemma splits in two cases dealing with non-extremal configurations and extremal configurations, respectively. We formulate these cases in Lemma 9 and Lemma 10, respectively. Since the proofs of these lemmas are rather technical we defer them to Section 7 and only show here how these two lemmas imply Lemma 8.

In order to prove Theorem 1 we will use the following structural lemma about coloured graphs from $\mathcal{K}_n^\eta$. This lemma asserts that such graphs either contain large monochromatic odd joined matchings or appropriate joined fork systems. With the help of the regularity method we will then, in Section 4, use this result (on the reduced graph of a regular partition) to find monochromatic trees. The reason why odd joined matchings and joined fork systems are useful for this task is explained in Section 4.1.

**Lemma 8.** *For all $\eta' > 0$ there are $\eta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following holds. Every coloured graph $K \in \mathcal{K}_n^\eta$ is either $(1-\eta')\frac{3}{4}n$-odd or $\big((1-\eta')n, (1-\eta')\frac{3}{2}n, 3\big)$-good.*

We remark that the dependence of the constant $n_0$ and $\eta'$ is only linear, and in fact we can choose $n_0 = 200/\eta'$. Lemma 8 is a consequence of the following two lemmas. The first lemma analyses non-extremal members of $\mathcal{K}_n^\eta$.

**Lemma 9** (non-extremal configurations)**.** *For all $\eta' > 0$ there are $\eta \in (0, \eta')$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following holds. Let $K$ be a coloured graph from $\mathcal{K}_n^\eta$ that is not $\eta'$-extremal. Then $K$ is $(1-\eta')\frac{3}{4}n$-odd.*

The second lemma handles the extremal configurations.

**Lemma 10** (extremal configurations)**.** *For all $\eta' > 0$ there is $\eta \in (0, \eta')$ such that the following holds. Let $K$ be a coloured graph from $\mathcal{K}_n^\eta$ that is $\eta$-extremal. Then $K$ is $((1-\eta')n, (1-\eta')\frac{3}{2}n, 3)$-good.*

Proofs of Lemma 9 and Lemma 10 are provided in Section 7.2 on page 18 and Section 7.3 on page 24, respectively. We get Lemma 8 as an easy corollary.

*Proof of Lemma 8.* Given $\eta'$ let $\eta_{\mathrm{L10}} < \eta'$ be the constant provided by Lemma 10 for input $\eta'$ and let $\eta_{\mathrm{L9}}$ be the constant produced by Lemma 9 for input $\eta'_{\mathrm{L9}} := \eta_{\mathrm{L10}}$. Set $\eta := \eta_{\mathrm{L9}} = \min\{\eta_{\mathrm{L10}}, \eta_{\mathrm{L9}}\}$ and let $K \in \mathcal{K}_n^\eta$ be a given coloured graph. Then $K \in \mathcal{K}_n^{\eta_{\mathrm{L9}}}$ and by Lemma 9 the graph $K$ is either $(1-\eta'_{\mathrm{L9}})3n/4$-odd (and thus $(1-\eta')3n/4$-odd as oddness is monotone) or $\eta_{\mathrm{L10}}$-extremal. In the first case we are done and in the second case Lemma 10 implies that $K$ is $\big((1-\eta')n, (1-\eta')\frac{3}{2}n, 3\big)$-good (goodness is also monotone) and we are also done. □

## 4. Proof of theorem 1

The goal of this section is to prove Theorem 1. We will first provide a brief outline and the main ideas of this proof. Then we will state the two remaining lemmas which are necessary for this proof: our main embedding result (Lemma 13), and the so-called "assignment lemma" (Lemma 14) which provides sufficient conditions for the application of the embedding lemma. Since the proofs of these two lemmas are comparably long we do not provide them in this section but defer them to after the proof of Theorem 1, which we provide at the end of this section.

### 4.1. The idea of the proof.

We apply the Regularity Lemma on the coloured graph $K_{n,n,n}$ with prepartition as given by the partition classes of $K_{n,n,n}$. As a result we obtain a coloured reduced graph $\mathbb{K} \in \mathcal{K}_k^\eta$ where the colour of an edge in $\mathbb{K}$ corresponds to the majority colour in the underlying regular pair. Such a regular pair is well-known to possess almost as good embedding properties as a complete bipartite graph. We then apply our structural result Lemma 8 on this reduced graph and infer that $\mathbb{K}$ is either $(1-\eta')\frac{3}{4}k$-odd or $((1-\eta')k, (1-\eta')\frac{3}{2}k, 3)$-good, i.e., there is a colour, say green, such that $\mathbb{K}$ contains either an odd joined green matching $\mathbb{M}_o$ of size at least $(1-\eta')\frac{3}{4}k$, or it contains a joined green matching $\mathbb{M}$ of size at least $(1-\eta')k$ and a joined green 3-fork system $\mathbb{F}$ of size at least $(1-\eta')\frac{3}{2}k$. We shall show that using these structures we can embed any tree $T \in \mathcal{T}_t^\Delta$ into the green subgraph of $K_{n,n,n}$. As a preparatory step, we cut $T$ into small subtrees, called shrubs.

Now let us first consider the case when we have an odd matching $\mathbb{M}_o$. Our aim is to embed each shrub $S$ into a regular pair $(A, B)$ corresponding to an edge $e \in \mathbb{M}_o$. Shrubs are bipartite graphs. Therefore there are two ways of assigning the colour classes of $S$ to the clusters of $e$. This corresponds to two different *orientations* of $S$ for the embedding in $(A, B)$. Our strategy is to choose orientations for all shrubs (and hence assignments of their colour classes to clusters of edges in $\mathbb{M}_o$) in such a way that every cluster of $V(\mathbb{M}_o)$ receives roughly the same number of vertices of $T$. We will show (as part of the assignment lemma, Lemma 14) that this is possible without "over-filling" any cluster. It follows (as we show as part of the embedding lemma, Lemma 13) that we can embed all shrubs into regular pairs corresponding to edges of $\mathbb{M}_o$. The fact that $\mathbb{M}_o$ is joined and *odd* then implies that between any pair of edges in $\mathbb{M}_o$ there are walks of both even and odd length in the reduced graph $\mathbb{K}$. We will show (as part of Lemma 14; see also the "walk condition" in Definition 19) that this allows us to connect the shrubs and to obtain a copy of $T$ in the green subgraph of $K_{n,n,n}$.

For the second case, i.e., the case when we have a matching $\mathbb{M}$ as well as a 3-fork system $\mathbb{F}$ the basic strategy remains the same. We assign shrubs to edges of $\mathbb{M}$ or $\mathbb{F}$. In difference to the previous case, however, these substructures of the reduced graph are not odd. This means that we cannot choose the orientations of the shrubs as before. Rather, these orientations are determined by the connections between the shrubs. Therefore, we distinguish the following two situations when embedding the tree $T$. If the partition classes of $T$ are reasonably balanced, then we use the matching $\mathbb{M}$ for the embedding. If $T$ is unbalanced, on the other hand, we employ the fork system $\mathbb{F}$. Assume without loss of generality that the continent $\mathbb{K}_{\lhd\mathbb{F}\rhd}$ of $\mathbb{F}$ is bipartite (otherwise take a suitable bipartite subgraph instead) and let $\mathbb{V}_1$ be the bigger colour class of $\mathbb{K}_{\lhd\mathbb{F}\rhd}$ and $\mathbb{V}_2$ the smaller (i.e., $\mathbb{V}_1$ contains mostly, but not exclusively, the prongs of forks in $\mathbb{F}$, and $\mathbb{V}_2$ mostly the centres). We use $\mathbb{V}_1$ to accommodate the bigger partition class of $T$ and $\mathbb{V}_2$ for the smaller. (This is also part of the assignment lemma, Lemma 14.)

4.2. **The main embedding lemma.** As indicated, in the proof of the main theorem we will use the regularity lemma in conjunction with an embedding lemma (Lemma 13). This lemma states that a tree $T$ can be embedded into a graph given together with a regular partition if there is a homomorphism from $T$ to the reduced graph of the partition with suitable properties. In the following definition of a valid assignment we specify these properties. Roughly speaking, a valid assignment is a homomorphism $h$ from a tree $T$ to a (reduced) graph $\mathbb{G}$ such that no vertex of $\mathbb{G}$ receives too many vertices of $T$ and that $h$ does not "spread" in the tree too quickly in the following sense: for each vertex $x \in V(T)$ we require that the neighbours of $x$ occupy at most two vertices of $\mathbb{G}$.

**Definition 11** (valid assignment). *Let $\mathbb{G}$ be a graph on vertex set $[k]$, let $T$ be a tree and $L \in \mathbb{N}$. A mapping $h\colon V(T) \to [k]$ is a $L$-*valid assignment *of $T$ to $\mathbb{G}$ if*

1. *$h$ is a homomorphism from $T$ to $\mathbb{G}$,*
2. *$|h(\mathrm{N}_T(x))| \le 2$, for every $x \in V(T)$,*
3. *$|h^{-1}(i)| < L$, for every $i \in [k]$.*

For formulating the embedding lemma we need, in addition, the concept of a cut of a tree, which is a set of vertices that cuts the tree into small components which we call shrubs.

**Definition 12** (cut, shrubs). *Let $q \in \mathbb{N}$ and $T$ be a tree with vertex set $V(T)$. A set $C \subseteq V(T)$ is a $q$-*cut *(or simply* cut*) of $T$ if all components of $T - C$ are of size at most $q$. The components of $T - C$ are called the* shrubs *of $T$ corresponding to $C$.*

Now we can state the main embedding lemma.

**Lemma 13** (main embedding lemma). *Let $G$ be an $n$-vertex graph with an $(\varepsilon, d)$-reduced graph $\mathbb{G} = ([k], E(\mathbb{G}))$ and let $T$ be a tree with $\Delta(T) \le \Delta$ and a $q$-cut $C$. If $T$ has a $(1-\varrho)(1-\varepsilon)\frac{n}{k}$-valid assignment to $\mathbb{G}$ and $(\frac{1}{10}d\varrho - 10\varepsilon)\frac{n}{k} \ge |C| + q + \Delta$ then $T \subseteq G$.*

The proof of Lemma 13 is deferred to Section 6 on page 15. Before we can apply it for embedding a tree $T$ in the proof of Theorem 1 we need to construct a valid assignment for $T$. This is taken care of by the following lemma which states that this is possible if the reduced graph of some regular partition contains an odd joined matching or a suitable fork system. The proof of Lemma 14 is given in Section 5 on page 9.

**Lemma 14** (assignment lemma). *For all $\varepsilon, \mu > 0$ with $\varepsilon < \mu/10$ and for all $k \in \mathbb{N}$ there is $\alpha = \alpha(k) > 0$ and $n_0 = n_0(\mu, \varepsilon, k) \in \mathbb{N}$ such that for all $n \ge n_0$, all $r \in \mathbb{N}$, all graphs $\mathbb{G}$ of order $k$, and all trees $T$ with $\Delta(T) \le n^{\alpha}$ the following holds. Assume that either*

(M) *$\mathbb{G}$ contains an odd joined matching of size at least $m$ and that $t := |V(T)| \le (1-\mu)2m\frac{n}{k}$, or*

(F) *$\mathbb{G}$ contains a joined fork system with ratio $r$ and size at least $f$, and $T$ has colour class sizes $t_1$ and $t_2$ with $t_2 \le t_1 \le t'$ and $t_2 \le t'/r$, where $t' = (1-\mu)f\frac{n}{k}$.*

*Then there is an $(\varepsilon\frac{n}{k})$-cut $C$ of $T$ with $|C| \le \varepsilon\frac{n}{k}$ and a $(1-\frac{\mu}{2})(1-\varepsilon)\frac{n}{k}$-valid assignment of $T$ to $\mathbb{G}$.*

4.3. **The proof of the main theorem.**

*Proof of Theorem 1.* We start by defining the necessary constants. Given $\mu > 0$, set $\mu' := \eta'$ in such a way that

$$1 - \tfrac{\mu}{3} \leq (1-\eta')^2(1-\mu'). \tag{1}$$

Lemma 8 with input $\eta' > 0$ provides us with $\eta > 0$ and $k_0 \in \mathbb{N}$. The regularity lemma, Lemma 3, with input

$$\varepsilon := \min\{\tfrac{1}{100}\eta^2, \tfrac{1}{10}\eta'^2, 10^{-3}\mu'\} \tag{2}$$

and $k_0$ and $k_* := 3$ returns a constant $k_1$. Next we apply Lemma 14 with input $\frac{\varepsilon}{10}$ and $\mu'$ separately for each value $3k$ with $k_0 \leq 3k \leq k_1$ and get constants $\alpha(3k)$ and $n_0'(3k)$ for each of these applications. Set $\alpha := \min\{\alpha(3k)\colon\ k_0 \leq 3k \leq k_1\}$ and $n_0' := \max\{n_0'(3k)\colon\ k_0 \leq 3k \leq k_1\}$. Finally, choose

$$n_0 := \max\{n_0', k_1, (\tfrac{k_1}{\varepsilon})^{1/(1-\alpha)}\}\,. \tag{3}$$

We are given a complete tripartite graph $K_{n,n,n}$ with $n \geq n_0$ as input whose edges are coloured with green and red. Our goal is to select a colour and show that in this colour we can embed every member of $\mathcal{T}_t^\Delta$ with $\Delta \leq n^\alpha$ and $t \leq (3-\mu)n/2$.

We first select the colour. To this end let $G$ and $R$ be the subgraphs of $K_{n,n,n}$ formed by the green and red edges, respectively. We apply the regularity lemma, Lemma 3, with input $\frac{\varepsilon}{10}$ on the graph $G$ with prepartition $V_1^*\dot\cup V_2^*\dot\cup V_3^*$ as given by the three partition classes of $K_{n,n,n}$. We obtain an $\frac{\varepsilon}{10}$-regular partition $V = V_0\dot\cup V_1\dot\cup\dots\dot\cup V_{3k}$ refining this prepartition such that $k_0 \leq 3k \leq k_1$. Each cluster of this partition lies entirely in one of the partition classes of $K_{n,n,n}$. Let $\mathbb{K} = ([3k], E_\mathbb{K})$ be the graph that contains edges for all $\varepsilon$-regular cluster pairs that do not lie in the same partition class of $K_{n,n,n}$. Clearly, $\mathbb{K}$ is a tripartite graph. Furthermore, there are less than $\varepsilon k^2$ pairs $(V_i, V_j)$ in our regular partition that are not $\frac{\varepsilon}{10}$-regular in $G$. It follows that at most $2\sqrt{\varepsilon}k$ clusters $V_i$ are contained in more than $\sqrt{\varepsilon}k$ irregular pairs. We move all these clusters and possibly up to $6\sqrt{\varepsilon}k$ additional clusters to the bin set $V_0$. The additional clusters are chosen in such a way that we obtain in each partition class of $\mathbb{K}$ the same number of clusters. We call the resulting bin set $V_0'$ and denote the remaining clusters by $V_1'\dot\cup\dots\dot\cup V_{3k'}'$ and the corresponding subgraph of $\mathbb{K}$ by $\mathbb{K}'$. Observe that $k' \geq (1-3\sqrt{\varepsilon})k$. Because each remaining cluster forms an irregular pair with at most $\sqrt{\varepsilon}k \leq 2\sqrt{\varepsilon}k' \leq \eta' k'$ of the remaining clusters we conclude that $\mathbb{K}'$ is a graph from $\mathcal{K}_{k'}^\eta$. In addition, it easily follows from the definition of $\varepsilon$-regularity that each pair $(V_i, V_j)$ with $i, j \in [k']$ which is $\varepsilon$-regular in $G$ is also $\varepsilon$-regular in $R$. This motivates the following "majority" colouring of $\mathbb{K}'$: We colour the edges $ij$ of $\mathbb{K}'$ by green if the $\varepsilon$-regular pair $(V_i, V_j)$ has density at least $\frac{1}{2}$ and by red otherwise. In this way we obtain a coloured graph $\mathbb{K}'_\chi \in \mathcal{K}_{k'}^\eta$.

Now we are in a position to apply Lemma 8 to $\mathbb{K}'_\chi$. This lemma asserts that $\mathbb{K}'_\chi$ is either $(1-\eta')\frac{3}{4}k'$-odd or $\big((1-\eta')k', (1-\eta')\frac{3}{2}k', 3\big)$-good. By definition this means that in one of the colours of $\mathbb{K}'_\chi$, say in green, we

(O) either have an odd joined matching $\mathbb{M}_o$ of size $m_1 \geq (1-\eta')\frac{3}{4}k' \geq (1-\eta')(1-3\sqrt{\varepsilon})\frac{3}{4}k$,
(G) or a joined matching $\mathbb{M}$ of size $m_2 \geq (1-\eta')k' \geq (1-\eta')(1-3\sqrt{\varepsilon})k$ together with a joined fork system $\mathbb{F}$ of size $f \geq (1-\eta')\frac{3}{2}k' \geq (1-\eta')(1-3\sqrt{\varepsilon})\frac{3}{2}k$ and ratio 3.

In the following we use the matchings and fork systems we just obtained to show that we can embed all trees of $\mathcal{T}_t^\Delta$ in the corresponding system of regular pairs. For this purpose let $\mathbb{G}$ be the graph on vertex set $[3k]$ that contains precisely all green edges of $\mathbb{K}'_\chi$. Observe that $\mathbb{G}$ is an $(\varepsilon, 1/2)$-reduced graph for $G$.

Let $T \in \mathcal{T}_t^\Delta$ be a tree of order $t \leq (3-\mu)n/2$ and with maximum degree $\Delta(T) \leq n^\alpha$. Now we distinguish two cases, depending on whether we obtained configuration (O) or configuration (G) from Lemma 8. In both cases we plan to appeal to Lemma 14 to show that $T$ has

$$\text{an } (\varepsilon\tfrac{n}{k})\text{-cut } C \text{ with } |C| \leq \varepsilon\tfrac{n}{k} \text{ and a } (1-\tfrac{\mu'}{2})(1-\varepsilon)\tfrac{3n}{3k}\text{-valid assignment to } \mathbb{G}. \tag{4}$$

Recall that we fed constants $\varepsilon$, $\mu' > 0$ and $3k$ into this lemma. Assume first that we are in configuration (O). Because $m_1 \geq (1-\eta')(1-2\sqrt{\varepsilon})\frac{3}{4}k$ we have

$$t \leq (3-\mu)\frac{n}{2} \leq 3(1-\tfrac{\mu}{3})\frac{n}{2}\frac{m_1}{(1-\eta')(1-3\sqrt{\varepsilon})\frac{3}{4}k} \overset{(1),(2)}{\leq} (1-\mu')2m_1 \cdot \frac{3n}{3k}\,.$$

Hence by (M) of Lemma 14 applied with the matching $\mathbb{M}_o$ (with $n$ replaced by $\tilde{n} := 3n$ and $k$ replaced by $\tilde{k} := 3k$) we get (4) for $T$ in this case, as $\Delta(T) \leq n^\alpha \leq \tilde{n}^\alpha$.

If we are in configuration (G), on the other hand, then let $t_1 \geq t_2$ be the sizes of the two colour classes of $T$. We distinguish two cases, using the two different structures provided in (G). Assume first that $t_2 \leq \frac{t}{3}$. Then, we calculate similarly as above that

$$t_2 \leq \tfrac{1}{3}t \leq (1-\tfrac{1}{3}\mu)\tfrac{n}{2} \leq \tfrac{1}{3}(1-\mu')f\tfrac{3n}{3k}\,, \qquad \text{and} \qquad t_1 \leq t \leq (1-\mu')f\tfrac{3n}{3k}\,.$$

Otherwise, if $t_2 \geq \frac{t}{3}$ then, similarly,

$$t_2 \leq t_1 \leq \tfrac{2}{3}t \leq (1-\tfrac{\mu}{3})n \leq (1-\mu')m_2 \cdot \tfrac{3n}{3k}\,.$$

Consequently, in both cases we can appeal to (F) of Lemma 14, in the first case applied to $\mathbb{F}$ with $r = 3$ and in the second to $\mathbb{M}$ with $r = 1$. We obtain (4) for $T$ as desired.

We finish our proof with an application of the main embedding lemma, Lemma 13. As remarked earlier $\mathbb{G}$ is an $(\varepsilon, 1/2)$-reduced graph for $G$. We further have (4). For applying Lemma 13 it thus remains to check that $(\frac{1}{2}\cdot\frac{1}{10}\varrho - 10\varepsilon)\frac{n}{k} \geq q + |C| + \Delta$ with $\varrho = \frac{1}{2}\mu'$, $q = \varepsilon\frac{n}{k}$, $|C| \leq \varepsilon\frac{n}{k}$, and $\Delta \leq n^\alpha$. Indeed,

$$\left(\tfrac{1}{20}\varrho - 10\varepsilon\right)\tfrac{n}{k} = \left(\tfrac{1}{20}\cdot\tfrac{1}{2}\mu' - 10\varepsilon\right)\tfrac{n}{k} \overset{(2)}{\geq} 3\varepsilon\tfrac{n}{k} \overset{(3)}{\geq} \varepsilon\tfrac{n}{k} + \varepsilon\tfrac{n}{k} + n^\alpha\,.$$

So Lemma 13 ensures that $T \subseteq G$, i. e., there is an embedding of $T$ in the subgraph induced by the green edges in $K_{n,n,n}$. □

## 5. Valid Assignments

In this section we will provide a proof for Lemma 14. The idea is as follows. Given a tree $T$ and a graph $G$ with reduced graph $\mathbb{G}$ we first construct a cut of $T$ that provides us with a collection of small shrubs (see Lemma 15). Then we distribute these shrubs to edges of the given matching or fork system in $\mathbb{G}$ (see Lemmas 16 and 17). Finally, we slightly modify this assignment in order to obtain a homomorphism from $T$ to $\mathbb{G}$ that satisfies the conditions required for a valid assignment (see Lemma 20).

**Lemma 15.** *For every $q \in \mathbb{N}$ and for any tree $T$ there is a $q$-cut of $T$ that has size at most $\frac{|V(T)|}{q}$.*

*Proof.* To prove Lemma 15 we need the following fact.

**Fact 1.** *For any $q \in \mathbb{N}$ and any tree $T$ with $|V(T)| > q$, there is a vertex $x \in V(T)$ such that the following holds. If $F_x$ is the forest consisting of all components of $T - x$ with size at most $q$, then $|V(F_x)| + 1 > q$.*

To see this, root the tree $T$ at an arbitrary vertex $x_0$. If $x_0$ does not have the required property, it follows from $|V(T)| > q$ that there exists a component $T_1$ in $T - x_0$ with $|V(T_1)| > q$. Set $x_1 := \mathrm{N}(x_0) \cap V(T_1)$. Let $F(T_1 - x_1)$ be the forest consisting of the components of $T_1 - x_1$ that have size at most $q$. Observe that $F(T_1 - x_1)$ is a subgraph of $F_{x_1}$. So if $|F(T_1 - x_1)| + 1 > q$, then $x_1$ has the property required by Fact 1. Otherwise there exists a component $T_2$ in $T_1 - x_1$ of size larger than $q$. Observe that $T_2$ is also a component of $T - x_1$. Now repeat the procedure just described by setting $x_2 = \mathrm{N}(x_1) \cap V(T_2)$ and so on, i.e., more generally we obtain trees $T_i$ and vertices $x_i = \mathrm{N}(x_{i-1}) \cap V(T_i)$. As the size of $T_i$ decreases as $i$ increases, there must be an $x_i$ with the property required by Fact 1. □

Now we prove Lemma 15. Set $C = \emptyset$. Repeat the following process until it stops. Choose a component $T'$ of $T - C$ with size larger than $q$. Apply Fact 1 to $T'$ and obtain a cut vertex $x$ of $T'$ together with a forest $F_x$ consisting of components of $T' - x$ that have size at most $q$ and is such that $|V(F_x) \cup \{x\}| > q$. Add $x$ to $C$ and repeat unless there is no component of size larger than $q$ in $T-C$. As $|V(T-C)|$ decreases this process stops. Observe that then $C$ is a $q$-cut. By the choice of $C$ we obtain

$$|V(T)| = \sum_{x\in C} |V(F_x) \cup \{x\}| > |C| \cdot q,$$

which implies the required bound on the size of $C$. □

After Lemma 15 provided us with a cut and some corresponding shrubs we will distribute each of these shrubs $T_i$ to an edge $e$ of the odd matching or the fork system in the reduced graph by assigning one colour class of $T_i$ to one end of $e$ and the other colour class to the other end. Here our goal is to distribute the shrubs and their vertices in such a way that no cluster receives too many vertices. The next two lemmas guarantee that this can be done. Lemma 16 takes care of the distribution of the shrubs to the clusters of a matching $M$ and Lemma 17 to those of a fork system $F$. We provide Lemmas 16 and 17 with numbers $a_{i,1}$ and $a_{i,2}$ as input. These numbers represent the sizes of the colour classes $A_{i,1}$ and $A_{i,2}$ of the shrub $T_i$. Since we do not need any other information about the shrubs in these lemmas the shrubs $T_i$ do not explicitly appear in their statement. Both lemmas then produces a mapping $\phi$ representing the assignment of the colour classes $A_{i,1}$ and $A_{i,2}$ to the clusters of $M$ or $F$.

**Lemma 16.** *Let $\{a_{i,j}\}_{i\in[s],\, j\in[2]}$ be natural numbers with sum at most $t$ and $a_{i,1}+a_{i,2}\le q$ for all $i\in[s]$, and let $M$ be a matching on vertices $V(M)$. Then there is a mapping $\phi\colon [s]\times[2]\to V(M)$ such that $\phi(i,1)\phi(i,2)\in M$ for all $i\in[s]$ and*

$$\sum_{(i,j)\in\phi^{-1}(v)} a_{i,j} \le \frac{t}{2|M|} + 2q \qquad \textit{for all } v\in V(M). \tag{5}$$

*Proof.* A simple greedy construction gives the mapping $\phi$: We consider the numbers $a_{i,j}$ as weights that are distributed, first among the edges, and then among the vertices of $M$. For this purpose greedily assign pairs $(a_{i,1},a_{i,2})$ to the edges of $M$, in each step choosing an edge with minimum total weight. Then, clearly, no edge receives weight more than $q+t/|M|$. In a second round, do the following for each edge $vw$ of $M$. For the pairs $(a_{i,1},a_{i,2})$ that were assigned to $e$, greedily assign one of the weights of this pair to $v$ and the other one to $w$, such that the total weight on $v$ and on $w$ are as equal as possible. Hence each of these vertices receives weight at most $\frac{1}{2}(q+t/|M|)+q$ and so the mapping $\phi$ corresponding to this weight distribution satisfies the desired properties. □

For the following lemma recall the definition of $|F|$ for a joined fork system $F$, which was given below Definition 5. In this definition we associated with $F$ a bipartition $V_1(F)\dot\cup V_2(F)$ and denoted by $|F|$ the size of the bigger partition class.

**Lemma 17.** *Let $\{a_{i,1}\}_{i\in[s]}$ and $\{a_{i,2}\}_{i\in[s]}$ be natural numbers with sum at most $t_1$ and $t_2$, respectively. Let $q\le t_1+t_2=:t$ and assume that $a_{i,1}+a_{i,2}\le q$ for all $i\in[s]$. Let $F$ be a fork system with ratio at most $r$ and partition classes $V_1(F)$ and $V_2(F)$ where $|V_1(F)|\ge|V_2(F)|$. Then there is a mapping $\phi\colon[s]\times[2]\to V_1(F)\cup V_2(F)$ such that $\phi(i,1)\phi(i,2)\in F$ and $\phi(i,j)\in V_j(F)$ for all $i\in[s], j\in[2]$ satisfying that for all $v_1\in V_1(F)$, $v_2\in V_2(F)$ we have*

$$\sum_{(i,1)\in\phi^{-1}(v_1)} a_{i,1} \le \frac{t_1}{|F|} + \sqrt{12tq|F|} \qquad \textit{and} \qquad \sum_{(i,2)\in\phi^{-1}(v_2)} a_{i,2} \le \frac{rt_2}{|F|} + \sqrt{12tq|F|}\,. \tag{6}$$

In the proof of this lemma we will make use the so-called Hoeffding bound for sums of independent random variables (see, e.g., [2, Theorem A.1.16]).

**Theorem 18.** *Let $X_1,\dots,X_s$ be independent random variables with $\mathbb{E}X_i=0$ and $|X_i|\le 1$ for all $i\in[s]$ and let $X$ be their sum. Then $\mathbb{P}[X>a]\le\exp(-a^2/(2s))$.* □

*Proof of Lemma 17.* For showing this lemma we use a probabilistic argument and again consider the $a_{i,j}$ as weights which are distributed among the vertices of $F$.

Observe first that we can assume without loss of generality that for all but at most one $i\in[s]$ we have $\frac{1}{2}q\le a_{i,1}+a_{i,2}$ since otherwise we can group weights $a_{i,1}$ together, and also group the corresponding $a_{i,2}$ together, such that this condition is satisfied and continue with these grouped weights. This in turn implies, that $s \leq (2t/q) + 1 \leq 3t/q$.

We start by assigning weights $a_{i,1}$ to vertices of $V_1(F)$ by (randomly) constructing a mapping $\phi_1 \colon [s] \times [1] \to V_1(F)$. To this end, independently and uniformly at random choose for each $i \in [s]$ an image $\phi_1(i,1)$ in $V_1(F)$. Clearly, there is a unique way of extending such a mapping $\phi_1$ to a mapping $\phi \colon [s] \times [2] \to V_1(F) \cup V_2(F)$ satisfying $\phi(i,1)\phi(i,2) \in F$. We claim that the probability that $\phi_1$ gives rise to a mapping $\phi$ which satisfies the assertions of the lemma is positive.

Indeed, for any fixed vertex $v = v_1 \in V_1(F)$ or $v = v_2 \in V_2(F)$ let $\sigma(v)$ be the event that the mapping $\phi$ does not satisfy (6) for $v$. We will show that $\sigma(v)$ occurs with probability strictly less than $1/(2|F|)$, which clearly implies the claim above. We first consider the case $v = v_1 \in V_1(F)$. For each $i \in [s]$ let $\mathbb{1}_i$ be the indicator variable for the event $\phi(i,1) = v_1$ and define a random variable $X_i$ by setting

$$X_i = \left(\mathbb{1}_i - \tfrac{1}{|F|}\right) \tfrac{a_{i,1}}{q}.$$

Observe that these variables are independent, and satisfy $\mathbb{E}X_i = 0$ and $|X_i| \leq 1$ and so Theorem 18 applied with $a = \sqrt{12t|F|/q}$ asserts that

$$\mathbb{P}\Big[\sum_{i\in[s]} X_i > \sqrt{12t|F|/q}\Big] \leq \exp\left(-\frac{12t|F|}{q \cdot 2s}\right) \leq \exp\left(-2|F|\right) < \frac{1}{2|F|} \tag{7}$$

where we used $s \leq 3t/q$. Now, by definition we have

$$X := \sum_{i\in[s]} X_i = \frac{1}{q} \sum_{(i,1)\in\phi^{-1}(v_1)} a_{i,1} - \frac{t_1}{|F|q},$$

and so, if (6) did not hold for $v_1$, then we had $X > \sqrt{12t|F|/q}$, which by (7) occurs with probability less than $1/(2|F|)$.

For the case $v = v_2 \in V_2(F)$ we proceed similarly. Let $r' \leq r$ be the number of prongs of the fork that contains $v_2$. We define indicator variables $\mathbb{1}'_i$ for the events $\phi(i,2) = v_2$ for $i \in [s]$ and random variables

$$Y_i = \left(\mathbb{1}'_i - \tfrac{r'}{|F|}\right) \tfrac{a_{i,2}}{q}.$$

with $\mathbb{E}Y_i = 0$ and $|Y_i| \leq 1$. The rest of the argument showing that $\sigma(v_2)$ occurs with probability strictly less than $1/(2|F|)$ is completely analogous to the case $v = v_1$ above. With this we are done. $\square$

As explained earlier these two previous lemmas will allow us to assign the shrubs of a tree $T$ to edges of a reduced graph $\mathbb{G}$. By applying them we will obtain a mapping $\psi$ from the vertices of $T$ to those of $\mathbb{G}$ that is a homomorphism when restricted to the shrubs of $T$. The following lemma transforms such a $\psi$ to a homomorphism $h$ from the whole tree $T$ to $\mathbb{G}$ that "almost" coincides with $\psi$ provided the structures of $T$ and $\mathbb{G}$ are "compatible" with respect to $\psi$ in the sense of the following definition.

**Definition 19** (walk condition)**.** *Let $T$ be a tree and $C \subseteq V(T)$. A mapping $\psi \colon V(T) \setminus C \to \mathbb{G}$ satisfies the* walk condition *if for any $x, y \in V(T) \setminus C$ such that there is a path $P_{x,y}$ from $x$ to $y$ whose internal vertices are all in $C$ there is a walk $\mathbb{P}_{x,y}$ between $\psi(x)$ and $\psi(y)$ in $\mathbb{G}$ such that the length of $P_{x,y}$ and the length of $\mathbb{P}_{x,y}$ have the same parity.*

**Lemma 20.** *Let $T$ be a tree with maximum degree $\Delta$, let $C$ be a cut of $T$, and let $\mathbb{G}$ be a graph on $k$ vertices. Let $\psi \colon V(T) \setminus C \to V(\mathbb{G})$ be a homomorphism that maps each shrub of $T$ corresponding to $C$ to an edge of $\mathbb{G}$ and that satisfies the walk condition. Then there is a homomorphism $h \colon V(T) \to V(\mathbb{G})$ satisfying*

(h1) $|h(\mathrm{N}_T(x))| \leq 2$ *for all vertices $x \in V(T)$ and*
(h2) $|\{x \in V(T) : h(x) \neq \psi(x)\}| \leq 3|C|\Delta^{2k+1}$.

Observe that Property (h1) in this lemma asserts that images of neighbours of any vertex in $T$ occupy at most two vertices in $\mathbb{G}$. By assumption, this is clearly true for $\psi$ but we need to make sure that $h$ inherits this feature. Property (h2) on the other hand states that $h$ and $\psi$ do not differ much. The assumption that $\psi$ satisfies the walk condition is essential for the construction of the homomorphism $h$.

*Proof of Lemma 20.* We start with some definitions. Choose a non-empty shrub corresponding to $C$ in $T$ and call it *shrub* 1. Then choose a cut-vertex $x_0^* \in C$ adjacent to this shrub. We consider $x_0^*$ as the *root* of the tree $T$. This naturally induces the following partial order $\prec$ on the vertices $V(T)$ of $T$: For vertices $x, y \in V(T)$ we have $x \prec y$ iff $y$ is a descendant of $x$ in the tree $T$ with root $x_0^*$. Note that $x_0^*$ is the unique minimal element of $\prec$ and the leaves of $T$ are its maximal elements. Further, for $x \in C$ set $W_x := \{z \in V(T) : \mathrm{dist}_T(x,z) \le 2k+1 \;\&\; x \prec z\}$ and let $W = C \cup \bigcup_{x\in C} W_x$. Observe that the bound on the maximum degree of $T$ implies that $|W| \le 2\Delta^{2k+1}|C| + |C| \le 3\Delta^{2k+1}|C|$. For $x \in V(T) \setminus W$, we set $h(x) := \psi(x)$. This ensures that Condition (h2) is fulfilled. In addition the following fact holds because $\psi$ maps each shrub to an edge of $\mathbb{G}$.

**Fact 1.** *The mapping $h$ restricted to $V(T) \setminus W$ is a homomorphism. For all vertices $x \in V(T) \setminus W$ all children $y$ of $x$ that are not cut-vertices have the same $h(y)$.* □

We shall extend $h$ to the set $W$. Our strategy is roughly as follows: We start by defining $h(x_0^*)$ for the root cut-vertex $x_0^*$ in a suitable way. Recall that all children of $x_0^*$ are contained in $W$. Then, we let $h$ map all non-cut-vertex children $y \in \mathrm{N}_T(x_0^*) \setminus C$ of $x_0^*$ to a suitable neighbour of $h(x_0^*)$ in $\mathbb{G}$ and do the following for each of these $y$. Observe that $y$ is the root of some shrub, which we will call the *shrub of* $y$. Now, $h(y)$ and $\psi(y)$ might be different. However, we will argue that there is a walk in $\mathbb{G}$ of even length $m \le 2k$ between $h(y)$ and $\psi(y)$. Then we will define $h$ for all vertices $y' \in W_{x_0^*}$ contained in the shrub of $y$ and with distance at most $m$ from $y$. More precisely we will use the walk of length $m$ between $h(y)$ and $\psi(y)$ and let $h$ map all $y'$ with distance $i$ to $y$ to the $i$-th vertex of this walk. All vertices $z$ in the shrub of $y$ for which $h$ is still undefined after these steps are then mapped to $h(z) := \psi(z)$. Once this has been done for all $y \in \mathrm{N}_T(x_0^*) \setminus C$ we proceed in the same way with the next cut-vertex: We choose a cut-vertex $x^*$ with parent $x$ for which $h(x)$ is already defined and proceed similarly for $x^*$ as we did for $x_0^*$.

We now make the procedure for the extension of $h$ on $W$ precise. Throughout this procedure we will assert the following property for all non-cut vertices $y$ of $T$ such that $h(y)$ is defined.

$$\text{There is a walk of even length in } \mathbb{G} \text{ between } h(y) \text{ and } \psi(y). \tag{8}$$

Observe that (8) trivially holds for all $y \in V(T) \setminus W$.

As explained, we start our procedure with the root $x_0^*$ of the tree $T$. Let $x_1$ be the root of shrub 1. By definition $x_1$ is adjacent to $x_0^*$. Note that, while $\psi$ is not defined on $x_0^*$ it is defined on $x_1$. Hence we can legitimately set $h(y) = \psi(x_1)$ for all neighbours $y \notin C$ of $x_0^*$ in $T$ and choose $h(x_0^*)$ arbitrarily in $\mathrm{N}_{\mathbb{G}}(\psi(x_1))$. Observe that this is consistent with (8) because for any neighbour $y \notin C$ of $x_0^*$ we have $h(y) = \psi(x_1)$ and $\mathrm{dist}_T(y, x_1) \in \{0, 2\}$. By assumption $\psi$ satisfies the walk condition. Hence there is a walk in $\mathbb{G}$ with even length between $h(y) = \psi(x_1)$ and $\psi(y)$. Let $\mathbb{P}_y = v_0, v_1, \ldots, v_m$ be a walk in $\mathbb{G}$ of minimum even length with $v_0 = h(y)$ and $v_m = \psi(y)$. As $\mathbb{G}$ has $k$ vertices we have that $m \le 2k$. For all vertices $z \in W$ that are in the shrub of $y$ and satisfy $\mathrm{dist}_T(y,z) = j$ for some $j \le m$, we then define $h(z) := v_j$. For the remaining vertices $z \in W$ in the shrub of $y$ we set $h(z) := \psi(z)$. Observe that this is again consistent with (8) and in conjunction with Fact 1 implies the following condition (which we will also guarantee throughout the whole process of defining $h$).

**Fact 2.** *Let $x^* \in C$ and $y \notin C$ such that $h(x^*)$ and $h(y)$ are defined. Then the following holds:*

(i) *All children $y' \notin C$ of $x^*$ have the same $h(y')$ and $h(x^*)h(y') \in E(\mathbb{G})$.*
(ii) *All children $y' \notin C$ of $y$ have the same $h(y')$ and $h(y)h(y') \in E(\mathbb{G})$.*

□

In this way we have defined $h$ for all shrubs adjacent to the root $x_0^*$.

Next we consider any vertex $x^* \in C \cap \mathrm{N}_T(x_0^*)$ and set $h(x^*) := h(x_1)$, where $x_1$ is as defined above. We let $z^*$ be the parent of $x^*$, i.e., $z^* = x_0^*$. Then set $h(y) := h(z^*)$ for all children $y \notin C$ of $x^*$. This is consistent with Fact 2. Afterwards we have the following situation: $x^*$ and

$z^* = x_0^*$ are neighbouring cut-vertices and the vertex $x_1$ is a non-cut-vertex neighbour of $x_0^*$. Let $y \in \mathrm{N}_T(x^*) \setminus C$. Then we have $\mathrm{dist}_T(x_1, y) = 3$. Because $y$ and $x_1$ are both non-cut vertices the properties of $\psi$ imply as before that there is a walk in $\mathbb{G}$ of odd length between $\psi(x_1)$ and $\psi(y)$. By the walk condition and the facts that $h(x_1) = \psi(x_1)$ and $h(x_0^*) = h(y)$, we know that in $\mathbb{G}$ there is a walk $\mathbb{P}_y$ of even length $m \le 2k$ between $h(y)$ and $\psi(y)$. This verifies (8) for $y$. We thus can define $h$ for the vertices $z$ contained in the shrub of $y$ as above: if $\mathrm{dist}_T(y, z) \le m$ then we use this walk and set $h(z)$ according to $\mathrm{dist}_T(y, z)$ and otherwise we set $h(z) := \psi(z)$. With this we stay consistent with (8) and Fact 2. We then repeat the above procedure for all $x^* \in C \cap \mathrm{N}_T(x_0^*)$ which implies that the next fact holds true.

**Fact 3.** *All vertices $x \in \mathrm{N}_T(x_0^*)$ have the same $h(x)$.* □

Now we are in the following situation.

**Fact 4.** *Suppose that $h(x^*)$ is defined for a cut vertex $x^*$. Then $h$ is defined on all shrubs adjacent to $x^*$. Moreover, for each cut vertex $x^*$ for which $h(x^*)$ is undefined and which has a parent $z$ for which $h(z)$ is defined, this parent $z$ has itself a parent $z'$ with $h(z')$ defined and $h(z)h(z')$ is an edge of $\mathbb{G}$.* □

As long as $h$ is not defined for all $z \in V(T)$ we then repeat the following. We choose a cut vertex $x^*$ with $h(x^*)$ undefined that is minimal with respect to this property in $\prec$. Denote the parent of $x^*$ by $z$ and let $z'$ be the parent of $z$. Then, by Fact 4, the mapping $h$ has already been defined for $z'$ and $z$. Set $h(x^*) := h(z')$ and for all children $y \notin C$ of $x^*$ set $h(y) := h(z)$. Because $h(z')h(z)$ is an edge of $\mathbb{G}$ by Fact 4 this gives the following property for $x^*$ (which we, again, guarantee throughout the definition of $h$).

**Fact 5.** *For all cut vertices $x^* \in C$ with $h(x^*)$ defined we have that $h(x^*)h(z)$ is an edge of $\mathbb{G}$, where $z$ is the parent of $x^*$. Moreover if $x^* \notin \{x_0^*\} \cup (C \cap \mathrm{N}_T(x_0^*))$, we have that $h(x^*) = h(z')$, where $z'$ is the parent of $z$.* □

Because (8) holds for $z'$ and $dist_T(x^*, z') = 2$, the definition of $h(y)$ is consistent with (8), i.e. there is a walk of even length in $\mathbb{G}$ between $h(y)$ and $\psi(y)$ for all children $y \notin C$ of $x^*$. Accordingly we can again define $h$ for the vertices in the shrub of $y$ as before, using this walk.

This finishes the description of the definition of $h$. It remains to verify that $h$ is a homomorphism and satisfies Condition (h1). For the first part it suffices to check that for any $y \in V(T) \setminus \{x_0^*\}$ with parent $x$ we have $h(y) \in \mathrm{N}_{\mathbb{G}}(h(x))$. If $y$ is a vertex in some shrub then Facts 2(i) and 2(ii) imply that $h(x)h(y)$ is an edge of $\mathbb{G}$. If $y$ is a cut-vertex, on the other hand, Fact 5 implies that $h(x)h(y)$ is an edge of $\mathbb{G}$. So $h$ is a homomorphism.

Further, by Fact 2(i) and (ii) we get for all vertices $x$ of $T$ that all children $x' \notin C$ of $x$ have the same $h(x')$. By Fact 5, if $x \neq x_0^*$ then all children $x' \in C$ of $x$ and the parent $z$ of $x$ have the same $h(x') = h(z')$. Together wit Fact 3, this implies Property (h1). □

Now we are ready to prove Lemma 14.

*Proof of Lemma 14.* Given $\varepsilon, \mu > 0$ with $\varepsilon \le \mu/10$ and $k \in \mathbb{N}$ we set $\alpha$, $n_0$ and an auxiliary constant $\beta > 0$ such that

$$\alpha \cdot (2k+1) = \tfrac{1}{2}, \qquad \beta = \varepsilon\mu/(500k^3), \quad \text{and} \quad n_0 = (1500k/(\varepsilon\mu))^4. \tag{9}$$

Let $\mathbb{G}$ be a graph of order $k$ that has an odd joined matching $\mathbb{M}$ of size at least $m$ or a fork system $\mathbb{F}$ of size at least $f$ and ratio $r$. Let $T$ be a tree satisfying the respective conditions of Case (M) or (F) and let $V_1(T)$ and $V_2(T)$ denote the two partition classes of $T$ with $t_1 = |V_1(T)| \ge |V_2(T)| = t_2$. We first construct a $q$-cut $C$ for $T$ with $q := \beta n \le \varepsilon \frac{n}{k}$. Lemma 15 asserts that there is such a cut $C$ with

$$|C| \le \frac{|V(T)|}{q} \le \frac{(1-\mu)2k\frac{n}{k}}{\beta n} \overset{(9)}{\le} \frac{1000k^3}{\varepsilon\mu} \overset{(9)}{\le} \varepsilon \frac{n}{k}. \tag{10}$$

Let $T_1, \ldots, T_s$ be the shrubs of $T$ corresponding to the cut $C$. We now distinguish whether we are in Case (M) or (F) of the lemma. In both cases we will construct a mapping $\psi$ that is a homomorphism from $T - C$ to either $\mathbb{M}$ or $\mathbb{F}$ and satisfies the walk condition. After this case distinction the mapping $\psi$ will serve as input for Lemma 20 which we then use to finish this proof.

*Case* (M): In this case we apply Lemma 16 in order to obtain an assignment of the shrubs to matching edges of $\mathbb{M}$ as follows. Set $a_{i,j} := |V(T_i) \cap V_j(T)|$ for all $i \in [s]$, $j \in [2]$. This implies that $\sum_{i,j} a_{i,j} \leq |V(T)| \leq t = (1-\mu)2m\frac{n}{k}$ and, because $C$ is an $q$-cut, that $a_{i,1} + a_{i,2} \leq q$ for all $i \in [s]$. Accordingly Lemma 16 produces a mapping $\phi : [s] \times [2] \to V(\mathbb{M})$ satisfying $\phi(i,1)\phi(i,2) \in \mathbb{M}$ and (5).

We now use $\phi$ to construct a mapping $\psi \colon T \setminus C \to V(\mathbb{M})$. Set $\psi(v) := \phi(a_{i,j})$ for all $v \in V(T_i) \cap V_j$. Note that this definition together with (5) gives

$$|\psi^{-1}(\ell)| \leq \frac{t}{2m} + 2q \leq (1-\mu)\frac{n}{k} + 2\beta n \tag{11}$$

for all vertices $\ell$ of $\mathbb{M}$. Each edge of $T - C$ lies in some shrub $T_i$, $i \in [s]$ and as the mapping $\phi$ sends each shrub $T_i$ to an edge of $\mathbb{M}$, the mapping $\psi$ is a homomorphism from $T - C$ to $\mathbb{M}$. Moreover, as $\mathbb{M}$ is an odd joined matching, for any pairs of vertices $\ell, \ell' \in V(\mathbb{M})$ there are both an even and an odd walk in $\mathbb{G}$ between $\ell$ and $\ell'$. Thus $\psi$ satisfies the walk condition.

*Case* (F): In this case we apply Lemma 17 in order to obtain an assignment of the shrubs corresponding to $C$ to edges of the connected fork system $\mathbb{F}$. For this application we use parameters $t_1 = |V_1(T)|$, $t_2 = |V_2(T)|$ and $a_{i,j} := |V(T_i) \cap V_j(T)|$ for all $i \in [s]$, $j \in [2]$. It follows that $\sum_i a_{i,1} = t_1$ and $\sum_i a_{i,2} = t_2$. Because $C$ is a $q$-cut, we further have $a_{i,1} + a_{i,2} \leq q$ for all $i \in [s]$. Accordingly Lemma 17 produces a mapping $\phi : [s] \times [2] \to V(\mathbb{F})$ satisfying $\phi(i,1)\phi(i,2) \in \mathbb{F}$ and (6).

Again, we use $\phi$ to construct the mapping $\psi \colon T \setminus C \to V(\mathbb{F})$ by setting $\psi(v) := \phi(a_{i,j})$ for all $v \in V(T_i) \cap V_j(T)$. By assumption we have $t_1 \leq t' = (1-\mu)f\frac{n}{k}$ and $t_2 \leq \frac{t'}{r} = (1-\mu)f\frac{n}{rk}$ and hence $t_1 + t_2 \leq (1-\mu)f\frac{n}{k}(1+\frac{1}{r})$. We write $V_1(\mathbb{F})$ and $V_2(\mathbb{F})$ to refer to the intersection of $V(\mathbb{F})$ with the respective partition classes of the bipartite continent $\mathbb{G}'_{\langle\mathbb{F}\rangle}$, $\mathbb{G}' \subseteq \mathbb{G}$, in which $\mathbb{F}$ has size $f$. Together with (6) this implies for all vertices $\ell_1 \in V_1(\mathbb{F})$ and $\ell_2 \in V_2(\mathbb{F})$ that

$$\begin{aligned} |\psi^{-1}(\ell_1)| &\leq \frac{(1-\mu)f\frac{n}{k}}{f} + \sqrt{12(1-\mu)f\tfrac{n}{k}(1+\tfrac{1}{r})qf} \\ &\leq (1-\mu)\tfrac{n}{k} + 2fn\sqrt{6\beta/k}\,, \end{aligned} \tag{12}$$

and similarly

$$|\psi^{-1}(\ell_2)| \leq \frac{r(1-\mu)f\frac{n}{rk}}{f} + 2fn\sqrt{6\beta/k} \leq (1-\mu)\frac{n}{k} + 2fn\sqrt{6\beta/k}\,. \tag{13}$$

Putting (12) and (13) together, we conclude for any $\ell \in V(\mathbb{F})$ that

$$|\psi^{-1}(\ell)| \leq (1-\mu)\tfrac{n}{k} + 2fn\sqrt{6\beta/k} \leq (1-\mu)\tfrac{n}{k} + 2n\sqrt{6\beta k}\,. \tag{14}$$

As before it is easy to see that the mapping $\psi$ is a homomorphism from $T - C$ to $\mathbb{F}$. Moreover, as $\mathbb{F}$ is a fork system, $\mathbb{F}$ is joined and bipartite with partition classes $V_1(\mathbb{F})$ and $V_2(\mathbb{F})$. Hence there is an even path between any two vertices $\ell, \ell' \in V_1(\mathbb{F})$ and between any two vertices $\ell, \ell' \in V_2(\mathbb{F})$, and an odd path between any two vertices $\ell \in V_1(\mathbb{F})$ and $\ell' \in V_2(\mathbb{F})$. Because $\psi$ maps vertices of $V_1(T)$ to $V_1(\mathbb{F})$ and vertices of $V_2(T)$ to vertices of $V_2(\mathbb{F})$, the mapping $\psi$ also satisfies the walk condition in this case.

*Applying Lemma 20*: In both Cases (M) and (F) we now apply Lemma 20 in order to transform $\psi$ into a homomorphism from the whole tree $T$ to $\mathbb{G}$. With input $T$, $\Delta := n^\alpha$, $C$, $\mathbb{G}$, and $\psi$ this lemma produces a homomorphism $h : V(T) \to V(\mathbb{G})$ satisfying (h1) and (h2). We claim that $h$ is the desired $(1-\mu/2)(1-\varepsilon)\frac{n}{k}$-valid assignment.

Indeed, $h$ is a homomorphism and so we have Condition 1 of Definition 11. Condition 2 follows from (h1). To check Condition 3 let $\ell$ be any vertex of $\mathbb{G}$. We need to verify that $|h^{-1}(\ell)| \leq (1-\frac{1}{2}\mu)(1-\varepsilon)\frac{n}{k}$. By (h2) we have $|h^{-1}(\ell)| \leq |\psi^{-1}(\ell)| + 3|C|\Delta^{2k+1}$. Because $|C| \leq 1000k/(\varepsilon\mu)$

by (10) and $\Delta^{2k+1} = n^{\alpha\cdot(2k+1)} = \sqrt{n}$ by (9) we infer that

$$|h^{-1}(\ell)| \le |\psi^{-1}(\ell)| + \frac{3000k}{\varepsilon\mu}\sqrt{n} \overset{(9)}{\le} |\psi^{-1}(\ell)| + \beta n$$
$$\overset{(11),(14)}{\le} (1-\mu)\tfrac{n}{k} + \max\left\{2\beta n, 2n\sqrt{6\beta k}\right\} + \beta n \overset{(9)}{\le} (1-\tfrac{1}{2}\mu)(1-\varepsilon)\tfrac{n}{k},$$

where in the last inequality we use that $\varepsilon \le \mu/10$. □

## 6. Proof of the main embedding lemma

Our proof of Lemma 13 uses a greedy strategy for embedding the vertices of a tree with valid assignment into the given host graph.

*Proof of Lemma 13.* Let $V_0\dot\cup V_1\dot\cup\dots\dot\cup V_k$ be an $(\varepsilon, d)$-regular partition of $G$ with reduced graph $\mathbb{G}$ and let $T$ be a tree with $\Delta(T) \le \Delta$ and with a $(1-\varrho)(1-\varepsilon)\frac{n}{k}$-valid assignment $h$ to $\mathbb{G}$. Further, let $C$ be an $q$-cut of $T$, let $T_1, \dots, T_s$ be the shrubs of $T$ corresponding to $C$, and assume that

$$(\tfrac{1}{10}d\varrho - 10\varepsilon)\tfrac{n}{k} \ge |C| + q + \Delta\,. \tag{15}$$

As last preparation we arbitrarily divide each cluster $V_i = V_i'\dot\cup V_i^*$ into a set $V_i'$ of size $(1-\frac{1}{2}\varrho)|V_i|$, which we will call *embedding space*, and the set of remaining vertices $V_i^*$, the so-called *connecting space*. Next we will first specify the order in which we embed the vertices of $T$ into $G$, then describe the actual embedding procedure, and finally justify the correctness of this procedure.

Pick an arbitrary vertex $x_1^* \in C$ as root of $T$ and order the cut vertices $C = \{x_1^*, \dots, x_q^*\}$, $c = |C|$ in such a way that on each $x_1^* - x_i^*$-path in $T$ there are no $x_j^*$ with $j > i$. Similarly, for each $i \in [s]$ let $t(i)$ denote the number of vertices in the shrub $T_i$ and order the vertices $y_1, \dots, y_{t(i)}$ of $T_i$ such that all paths in $T_i$ starting at the root of $T_i$ have solely ascending labels. For embedding $T$ into $G$ we process the cut vertices and shrubs according to these orderings, more precisely we first embed $x_1^*$, then all shrubs $T_i$ that have $x_1^*$ as parent, one after the other. For embedding $T_i$ we embed its vertices in the order $y_1, \dots, y_{t(i)}$ defined above. Then we embed the next cut vertex $x_2^*$ (which is a child of one of the shrubs embedded already or of $x_1^*$), then all child shrubs of $x_2^*$, and so on. Let $x_1, \dots, x_{|V(T)|}$ be the corresponding ordering of $V(T)$.

Before turning to the embedding procedure itself, observe that Property 2 of Definition 11 asserts the following fact. For a vertex $x_j$ of $T$ and for $i \in [k]$ let $\mathrm{N}_i(x_j)$ be the set of neighbours $x_{j'}$ of $x_j$ in $T$ with $j' > j$ and $h(x_{j'}) = i$.

**Fact 1.** *For every vertex $x_j$ of $T$ at most two sets $\mathrm{N}_i(x_j)$ are non-empty.* □

The idea for embedding $T$ into $G$ is as follows. We equip each vertex $x \in V(T)$ with a *candidate set* $V(x) \subseteq V_{h(x)}$ and from which $x$ will choose its image in $G$. To start with, we set $V(x^*) := V^*_{h(x^*)}$ for all vertices $x^* \in C$ and $V(x) := V'_{h(x)}$ for all other vertices $x$. Cut vertices will be embedded to vertices in a connecting space and non-cut vertices to vertices in an embedding space. Then we will process the vertices of $T$ in the order $x_1, \dots, x_{|V(T)|}$ defined above and embed them one by one. Whenever we embed a cut vertex $x^*$ to a vertex $v$ in this procedure we will set up so-called *reservoir sets* $R_i \subseteq V_i \cap \mathrm{N}_G(v)$ for all (at most two) clusters $V_i$ such that some child $x$ of $x^*$ is assigned to $V_i$, i.e., $h(x) = i$. Reservoir sets will be used for embedding the children $x$ of cut vertices $x^*$. They are necessary because the embedding of $x^*$ will cause the candidate set $V(x)$ of $x$ to shrink to a size $s$ and it can happen that more than $s$ vertices are embedded before we attempt to embed $x$. The reservoir sets assert that these vertices do not eat up $V(x)$. More precisely, when embedding $x^*$ then for each child $x$ of $x^*$ we restrict $V(x)$ to some reservoir set. Then we (temporarily) remove the vertices in these reservoir sets from all other candidate sets but put them back after processing all child shrubs of $x^*$. This will ensure that we have enough space for embedding the children of $x^*$, even after possibly embedding $\Delta - 1$ child shrubs of $x^*$.

Now let us provide the details of the embedding procedure. Throughout, $x^*$ will denote the cut vertex whose child-shrubs are currently processed. The set $U$ will denote the vertices in $G$ used so far; thus initialise this set to $U := \emptyset$. As indicated above, initialise $V(x^*) := V^*_{h(x^*)}$ for all vertices $x^* \in C$ and $V(x) := V'_{h(x)}$ for $x \in V(T) \setminus C$, and set $R_i := \emptyset$ for all $i \in [k]$. Recall the definition of typical vertices with respect to a set of vertices in a regular pair (see Lemma 2 and the definition below this lemma). For constructing an embedding $f\colon V(T)\to V(G)$ of $T$ into $G$, repeat the following steps:

1. Pick the next vertex $x$ from $x_1,\dots,x_{|V(T)|}$.
2. Choose a vertex $v\in V(x)\setminus U$ that is typical with respect to $V(y)\setminus U$ for all unembedded $y\in \mathrm{N}_T(x)$, set $f(x)=v$, and $U:=U\cup\{v\}$.
3. For all unembedded $y\in \mathrm{N}_T(x)$ set $V(y):=(V(y)\setminus U)\cap \mathrm{N}_G(v)$.
4. If $x\in C$ then set $x^*:=x$. Further, for all $i$ with $\mathrm{N}_i(x)\setminus C\neq\emptyset$ arbitrarily choose a reservoir set $R_i\subseteq (V'_i\setminus U)\cap \mathrm{N}_G(v)$ of size $5\varepsilon\frac{n}{k}+\Delta$, set $V(y):=R_i$ for all $y\in \mathrm{N}_i(x)\setminus C$, and (temporarily) remove $R_i$ from all other candidate sets in $V'_i$, i.e., set $V(y'):=V(y')\setminus R_i$ for all $y'\in V(T)\setminus \mathrm{N}_i(x)$.
5. After the vertices of all child shrubs of $x^*$ are embedded put the vertices in $R_i$ back to all candidate sets in $V'_i$ for all $i\in[k]$, i.e., $V(y):=V(y)\cup R_i$ for all $y\in V(T)\setminus C$ with $h(y)=i$, and set $R_i:=\emptyset$.

Steps 3 and 4 of this procedure guarantee for each vertex $y$ with embedded parent $x$ that the candidate set $V(y)$ is contained in $N_G(f(x))$. Accordingly, if we can argue that in Step 2 we can always choose an image $v$ of $x$ in $V(x)$ (and that we can choose the reservoir sets in Step 4) we indeed obtain an embedding $f$ of $T$ into $G$. To show this we first collect some observations that will be useful in the following analysis. The order of $V(T)$ guarantees that all child shrubs of a cut vertex are embedded before the next cut vertex. Notice that this implies the following fact (cf. Step 4 and Step 5).

**Fact 2.** *For all $i\in[k]$, at any point in the procedure, the reservoir set $R_i$ satisfies $|R_i|=5\varepsilon\frac{n}{k}+\Delta$ if there is a neighbour $x$ of the current cut-vertex $x^*$ such that $h(x)=i$ and $|R_i|=0$ otherwise. In addition no reservoir set gets changed before all child shrubs of $x^*$ are embedded.* □

Further, since $h$ is a $(1-\varrho)(1-\varepsilon)\frac{n}{k}$-valid assignment and only cut-vertices are embedded into connecting spaces $V^*_i$, we always have

$$|V'_i\cap U|\le (1-\tfrac12\varrho)\tfrac{n}{k}\quad\text{and}\quad |V^*_i\cap U|\le |C|\quad\text{for all } i\in[k]\,. \tag{16}$$

Now we check that Steps 2 and 4 can always be performed. To this end consider any iteration of the embedding procedure and suppose we are processing vertex $x$. We distinguish three cases.

*Case 1:* Assume that $x$ is a cut-vertex. Then we had $V(x)=V^*_{h(x)}$ until the parent $x'$ of $x$ got embedded. In the iteration when $x'$ got embedded then the set $V(x)$ shrunk to a set of size at least $(d-\varepsilon)|V^*_{h(x)}\setminus U|$ in Step 3 because $f(x')$ is typical with respect to $V^*_{h(x)}\setminus U$. No vertices embedded between $x'$ and $x$ (except for possible vertices in $C$) alter $V(x)$, and so by (16) we have

$$|V(x)\setminus U|\ge (d-\varepsilon)|V^*_{h(x)}|-|C|\ge (d-\varepsilon)\tfrac12\varrho\tfrac{n}{k}-|C|\overset{(15)}{>}4\varepsilon\tfrac{n}{k}$$

when we are about to choose $f(x)$. By Fact 1 at most two of the sets $\mathrm{N}_i(x)$ are non-empty and each of these two sets can contain cut vertices $y^*$ and non-cut vertices $y'$. We clearly have $V(y^*)=V^*_i$ and $V(y')=V'_i$ and so there are at most 4 different sets $V(y)\setminus U$, each of size at least $\frac12\varrho\frac{n}{k}-|C|>\varepsilon\frac{n}{k}$ by (16) and (15), with respect to which we need to choose a typical $f(x)$. By Lemma 2 there are less than $4\varepsilon\frac{n}{k}$ vertices in $V(x)\setminus U$ (which is a subset of $V_i$) that do not fulfil this requirement. Hence we can choose $f(x)$ whenever $x\in C$. In addition, we can choose the reservoir sets in Step 4 of this iteration: Indeed, let $i$ be such that $\mathrm{N}_i(x)\setminus C\neq\emptyset$ and let $y\in \mathrm{N}_i(x)\setminus C$ be a neighbour of $x$ we wish to embed to $V_i$. In Step 2, when we choose $f(x)$, then $V(y)=V'_i$ and so $f(x)$ is typical with respect to $V'_i\setminus U$. By Lemma 2 and (16) we thus have in Step 3 of this iteration that

$$|(V'_i\setminus U)\cap \mathrm{N}_G(v)|\ge (d-\varepsilon)|V'_i\setminus U|\ge (d-\varepsilon)\tfrac12\varrho\tfrac{n}{k}\overset{(15)}{\ge}5\varepsilon\tfrac{n}{k}+\Delta.$$

Therefore we can choose $R_i$ in Step 4.

*Case 2:* Assume that $x$ is not in $C$ but the child of a cut vertex $x^*$. Then $V(x)=R_{h(x)}$ before $x$ gets embedded. Moreover, due to Step 4, $R_i$ has been removed from all candidate sets besides those of the at most $\Delta$ neighbours of $x^*$. By Fact 2 we have $|R_{h(x)}|=5\varepsilon\frac{n}{k}+\Delta$ and so we conclude that $|V(x)\setminus U| \geq 5\varepsilon\frac{n}{k} > 4\varepsilon\frac{n}{k}$. As in the previous case, there are at most four different sets $V(y)\setminus U$ for unembedded neighbours $y$ of $x$, each of size at least $\frac{1}{2}\varrho\frac{n}{k} - |R_{h(y)}| = \frac{1}{2}\varrho\frac{n}{k} - 5\varepsilon\frac{n}{k} - \Delta \geq \varepsilon\frac{n}{k}$ by (15) and (16). Thus Lemma 2 guarantees that there is $v \in V(x) \setminus U$ which is typical with respect to all these sets $V(y) \setminus U$ and hence we can choose $f(x)$ in this case.

*Case 3:* As third and last case, let $x$ be a vertex of some shrub $T_j$ which is the child of a (non-cut) vertex $x'$ of $T_j$. Until $x'$ got embedded we had $V(x) = V'_{h(x)} \setminus R_{h(x)}$ and so, $v' = f(x')$ was chosen typical with respect to $V'_{h(x)}\setminus(R_{h(x)}\cup U)$ where $U$ is the set of used vertices in $G$ at the time when $x'$ got embedded. In the corresponding iteration $V(x)$ shrunk to $(V'_{h(x)}\setminus(R_{h(x)}\cup U))\cap \mathrm{N}_G(v')$. This together with (16) implies that immediately after this shrinking we had

$$|V(x) \setminus U| \geq (d-\varepsilon)(\tfrac{1}{2}\varrho\tfrac{n}{k} - |R_{h(x)}|) \geq (d-\varepsilon)(\tfrac{1}{2}\varrho\tfrac{n}{k} - 5\varepsilon\tfrac{n}{k} - \Delta) \overset{(15)}{>} 4\varepsilon\tfrac{n}{k} + |T_j|.$$

By construction only vertices from $T_j$ come between $x'$ and $x$ in the order of $V(T)$ and so when we want to embed $x$ in the procedure above we still have $|V(x)\setminus U| > 4\varepsilon\frac{n}{k}$ where $U$ now is the set of vertices used until the embedding of $x$. Similarly as in the other two cases there are at most four different types of candidate sets for non-embedded neighbours of $x$, all of these have more than $\varepsilon\frac{n}{k}$ vertices and so Lemma 2 allows us to choose an $f(x) \in V(x) \setminus U$ typical with respect to these sets. This concludes the case distinction and hence the proof of correctness of our embedding procedure. □

## 7. Coloured tripartite graphs are either good or odd

In this section we provide the proofs of Lemma 9 (in subsection 7.2) and Lemma 10 (in subsection 7.3). Before turning to these proofs we first collect some simple but useful propositions (in subsection 7.1).

### 7.1. Some tools.

We start with two observations about matchings in $\eta$-complete graphs. The first one states that a bipartite $\eta$-complete coloured graph contains a reasonably big matching in one of the two colours.

**Proposition 21.** *Let $K$ be a coloured graph on $n$ vertices and let $D$ and $D'$ be vertex sets of size at least $m$ in $K$. If $K[D, D']$ is $\eta$-complete then $K[D, D']$ contains a matching $M$ either in red or in green of size at least $\frac{m}{2} - \eta n$.*

*Proof.* Assume without loss of generality that $|D| \leq |D'|$. Colour a vertex $v \in D$ with red if it has more red-neighbours than green-neighbours in $K[D, D']$ and with green otherwise. By the pigeon-hole principle there is a set $X \subseteq D$ of size at least $\frac{1}{2}|D|$ such that all vertices in $X$ have the same colour, say red. But then each vertex in $X$ has at least $\frac{1}{2}|D'| - \eta n \geq |X| - \eta n$ red-neighbours in $D'$. Accordingly we can greedily construct a red matching of size at least $|X| - \eta n \geq \frac{m}{2} - \eta n$ between $X$ and $D'$. □

The next proposition gives a sufficient condition for the existence of an almost perfect matching in a subgraph of $K \in \mathcal{K}_n^\eta$.

**Proposition 22.** *Let $K \in \mathcal{K}_n^\eta$ have partition classes $V^{(1)}$, $V^{(2)}$, and $V^{(3)}$ and let $U^{(1)} \subseteq V^{(1)}$, $U^{(2)} \subseteq V^{(2)}$, $U^{(3)} \subseteq V^{(3)}$ with $|U^{(1)}| \geq |U^{(2)}| \geq |U^{(3)}|$. If $|U^{(1)}| \leq |U^{(2)} \cup U^{(3)}|$ then there is a matching in $K[U^{(1)}, U^{(2)}, U^{(3)}]$ covering at least $|U^{(1)} \cup U^{(2)} \cup U^{(3)}| - 4\eta n - 1$ vertices.*

*Proof.* Let $x := |U^{(2)}| - |U^{(3)}|$ and $y := \lfloor \frac{1}{2}(|U^{(1)}| - x) \rfloor$. Observe that $x \leq |U^{(2)}| \leq |U^{(1)}|$. Hence $y \geq 0$, $x + y \leq \frac{1}{2}(|U^{(1)}| + x) \leq \frac{1}{2}(|U^{(2)} \cup U^{(3)}| + x) = |U^{(2)}|$, and $y \leq \frac{1}{2}(|U^{(1)}| - x) \leq \frac{1}{2}(|U^{(2)} \cup U^{(3)}| - x) = |U^{(3)}|$. Choose arbitrary subsets $W^{(2)} \subseteq U^{(2)}$ of size $x + y$, $W^{(3)} \subseteq U^{(3)}$ of size $y$, set $W := W^{(2)} \cup W^{(3)}$, $\bar{W}^{(2)} := U^{(2)} \setminus W^{(2)}$ and $\bar{W}^{(3)} := U^{(3)} \setminus W^{(3)}$. Clearly $|\bar{W}^{(3)}| = |U^{(3)}| - y = |U^{(2)}| - (x+y) = |\bar{W}^{(2)}|$ and $|U^{(1)}| - 1 \leq x + 2y = |W| \leq |U^{(1)}|$. Thus we can choose a subset $W'$ of $U^{(1)}$ of size $|W|$ that covers all but at most 1 vertex of $U^{(1)}$ and so that $K[W, W']$ and $K[\bar{W}^{(2)}, \bar{W}^{(3)}]$ are $\eta$-complete balanced bipartite subgraphs. A simple greedy algorithm allows us then to find matchings of size at least $|W| - \eta n$ and $|\bar{W}^{(2)}| - \eta n$ in $K[W, W']$ and $K[\bar{W}^{(2)}, \bar{W}^{(3)}]$, respectively. These matchings together form a matching in $K[U^{(1)}, U^{(2)}, U^{(3)}]$ covering at least $|W \cup W' \cup \bar{W}^{(2)} \cup \bar{W}^{(3)}| - 4\eta n \geq |U^{(1)} \cup U^{(2)} \cup U^{(3)}| - 4\eta n - 1$ vertices. □

The following proposition shows that induced subgraphs of $\eta$-complete tripartite graphs are connected provided that they are not too small. Moreover, subgraphs that substantially intersect all three partition classes contain a triangle.

**Proposition 23.** *Let $K \in \mathcal{K}_n^\eta$ be a graph with partition classes $V^{(1)}$, $V^{(2)}$, $V^{(3)}$, and let $U^{(1)} \subseteq V^{(1)}$, $U^{(2)} \subseteq V^{(2)}$, $U^{(3)} \subseteq V^{(3)}$.*

(a) *If $|U^{(1)}| > 2\eta n$ then every pair of vertices in $U^{(2)} \cup U^{(3)}$ has a common neighbour in $U^{(1)}$.*
(b) *If $|U^{(1)}|, |U^{(2)}| > 2\eta n$ then $K[U^{(1)}, U^{(2)}]$ is connected.*
(c) *If $|U^{(1)}|, |U^{(2)}|, |U^{(3)}| > 2\eta n$ then $K[U^{(1)}, U^{(2)}, U^{(3)}]$ contains a triangle.*

*Proof.* As $K \in \mathcal{K}_n^\eta$, each vertex in $U^{(2)} \cup U^{(3)}$ is adjacent to at least $|U^{(1)}| - \eta n > |U^{(1)}|/2$ vertices in $U^{(1)}$. Thus every pair of vertices in $U^{(2)}$ has a common neighbour in $U^{(1)}$ which gives (a). For the proof of (b) observe that by (a) every pair of vertices in $U^{(2)}$ has a common neighbour in $U^{(1)}$. Since the same holds for pairs of vertices in $U^{(1)}$ the graph $K[U^{(1)}, U^{(2)}]$ is connected. To see (c) we use (a) again and infer that every pair of vertices in $U^{(1)} \times U^{(2)}$ has a common neighbour in $U^{(3)}$. As $|U^{(1)}|, |U^{(2)}| > 2\eta n$ there is some edge in $U^{(1)} \times U^{(2)}$ and thus there is a triangle in $K[U^{(1)}, U^{(2)}, U^{(3)}]$. □

Similar in spirit to (c) of Proposition 23 we can enforce a copy of a cycle of length 5 in a system of $\eta$-complete graphs as we show in the next proposition.

**Proposition 24.** *Let $K$ be a coloured graph on $n$ vertices, let $\chi$ be a colour, $vw$ be a $\chi$-coloured edge of $K$, and let $D_1, D_2, D_3 \subseteq V(K)$ such that all graphs $K[v, D_1]$, $K[D_1, D_2]$, $K[D_2, D_3]$, and $K[D_3, w]$ are $(\eta, \chi)$-complete bipartite graphs. Set $D := \bigcup_{i \in [3]} D_i \cup \{v, w\}$. If $|D_i| > 2\eta n + 2$ for all $i \in [3]$ then $K[D]$ contains a $\chi$-coloured copy of $C_5$.*

*Proof.* By Proposition 23(a) every pair of vertices in $D_1 \cup D_3$ is connected by a path of colour $\chi$ and length 2 with centre in $D_2 \setminus \{v, w\}$. Moreover, $v$ has at least $|D_1| - \eta n \geq 1$ neighbours in $D_1$ and similarly $w$ has a neighbour in $D_3$. Hence there is a $\chi$-coloured $C_5$ in $K[D]$. □

7.2. **Non-extremal configurations.** In this subsection we prove Lemma 9. In the proof of Lemma 9 we will use that coloured graphs $K$ from $\mathcal{K}_n^\eta$ have the following property $P$. Either one colour of $K$ has a big odd joined matching or both colours have big joined matchings whose continents are bipartite. Analysing these bipartite configurations will then lead to a proof of Lemma 9. Property $P$ is a consequence of the next lemma, Lemma 25, which states that if all joined matchings in a colour of $K$ are small then the other colour has bigger odd joined matchings.

**Lemma 25** (improving lemma)**.** *For every $\eta' > 0$ there are $\eta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following holds. Suppose that a coloured graph $K \in \mathcal{K}_n^\eta$ is neither $\eta'$-extremal nor $\frac{3}{4}(1-\eta')n$-odd. Let $M$ be a maximum joined matching in $K$ of colour $\chi$. If $\eta' n < |M| < (1-\eta')n$ then $K$ has an odd joined matching $M'$ in the other colour satisfying $|M'| > |M|$.*

*Proof.* Given $\eta'$ define $\tilde{\eta} := \eta'/3$ and let $\eta$ be small enough and $n_0$ large enough so that $(\frac{1}{100}\eta' - 5\eta)n_0 > 1$ (and hence $\eta < \frac{1}{500}\eta'$). For $n \geq n_0$ let $K = (V^{(1)} \dot\cup V^{(2)} \dot\cup V^{(3)}, E)$ be a coloured graph from $\mathcal{K}_n^\eta$ with partition classes $V^{(1)}$, $V^{(2)}$, and $V^{(3)}$ that is neither $\eta'$-extremal nor $(1-\eta')3n/4$-odd. Suppose $\chi$ = green and hence that $K$ has a maximum green joined matching $M$ with $\eta' n < |M| < (1-\eta')n$. For $i \neq j \in [3]$ let $M_{ij} := M \cap (V^{(i)} \times V^{(j)})$. We call the sets $M_{ij}$ the *blocks* of $M$ and say that a block $M_{ij}$ is *substantial* if $|M_{ij}| \geq \tilde{\eta} n$. Let $R$ be the set of vertices in $K$ not covered by $M$. For $i \in [3]$ let $R_i := R \cap V^{(i)}$.

**Fact 1.** *We have $|R_1| - |M_{23}| = |R_2| - |M_{13}| = |R_3| - |M_{12}| > \eta' n$.*

Indeed, $|R_1| + |M_{12}| + |M_{13}| = |R_2| + |M_{12}| + |M_{23}|$ and hence $|R_1| - |M_{23}| = |R_2| - |M_{13}| = |R_2| - |M_{13}|$ which proves the first part of this fact. For the second part observe that $|R_1| + 2|M_{12}| +$

$|R_2|+2|M_{23}|+|R_3|+2|M_{13}| = 3n$. Hence we conclude from $|M| = |M_{12}|+|M_{23}|+|M_{13}| < (1-\eta')n$ that

$$\begin{aligned} 3(|R_1| - |M_{23}|) &= (|R_1| - |M_{23}|) + (|R_2| - |M_{13}|) + (|R_3| - |M_{12}|) \\ &= 3(n - |M_{12}| - |M_{23}| - |M_{13}|) > 3\eta' n. \end{aligned}$$

This finished the proof of Fact 1. □

In the remainder we assume without loss of generality that $|R_1| \geq |R_2| \geq |R_3|$. By Fact 1 this implies that $|M_{23}| \geq \frac{1}{3}\eta' n$ since $|M| > \eta' n$ and hence $M_{23}$ is substantial. Our next main goal is to find a joined matching in red that is bigger than $M$. For achieving this goal the following fact about red connections between vertices of $R$ will turn out useful.

**Fact 2.** *There is a vertex $u^* \in R_1$ such that $R - u^*$ is red joined.*

To see this, assume first that there is a vertex $u^* \in R_1$ that has more than $4\eta n$ green-neighbours in $M_{23}$. Then more than $2\eta n$ of these neighbours are in, say, $M_{23} \cap V^{(2)}$. Call this set of vertices $V_*^{(2)}$. Now let $u \neq u^*$ be any vertex in $R \setminus V^{(3)}$. By the maximality of $M$ the vertex $u$ has no green-neighbours in $M(V_*^{(2)})$. This implies that $u$ has at least $|M(V_*^{(2)})| - \eta n > |M(V_*^{(2)})|/2$ red-neighbours in $M(V_*^{(2)})$. Thus any two vertices in $R \setminus V^{(3)}$ have a common red-neighbour in $M(V_*^{(2)})$. A vertex $u \in R_3$ on the other hand has at least $|R_1| - \eta n \geq |M_{23}| + \eta' n - \eta n > 2\eta n + 1$ neighbours in $R_1$ where the first inequality follows from Fact 1. If at least 2 of these neighbours are red then $u$ is red connected to $R_1 - u^*$. Otherwise $u$ has a set $U$ of more than $2\eta n$ green-neighbours in $R_1 - u^*$. But then, by the maximality of $M$, the graph $K[U, M_{23} \cap V^{(2)}]$ is red. Since $|M_{23}| \geq \eta' n > \eta n$ the vertex $u$ has a neighbour $v$ in $M_{23} \cap V^{(2)}$. Since $u$ has a green-neighbour in $R_1$ it follows from the maximality of $M$ that $uv$ is red. Thus $u$ is red connected to $U$ and therefore to all vertices of $(R \setminus V^{(3)}) - u^*$.

If there is no vertex in $R_1$ with more than $4\eta n$ green-neighbours in $M_{23}$ on the other hand, then any two vertices in $R_1$ obviously have at least $|M_{23}| - 4\eta n - 2\eta n \geq \frac{1}{3}\eta' n - 6\eta n > 0$ common red-neighbours in $V^{(2)} \cap M_{23}$. Moreover, by the maximality of $M$, each vertex $v \in R_3 \cup R_2$ is either red connected to $R_1$ or it has only red-neighbours in $M_{23}$. Thus $v$ has a common red-neighbour with any vertex in $R_1$ which proves Fact 2 also in this case. □

**Fact 3.** *$K$ has a red joined matching $M'$ with $|M'| \geq |M| + \frac{1}{4}\eta' n$.*

Let $uv$ be an arbitrary edge in $M_{23}$. Then, by the maximality of $M$, one vertex of this edge, say $u$, has at most one green-neighbour in $R_1$. By Fact 1 we have $|R_1| \geq |M_{23}| + \eta' n$ and since $u$ has at most $\eta n < \eta' n$ non-neighbours in $R_1$ it follows that $u$ has at least $|M_{23}| + 1$ red-neighbours in $R_1$. Thus, a simple greedy method allows us to construct a red matching $M'_{23}$ of size $|M_{23}|$ between $R_1 - u^*$ and such vertices $u$ of matching edges in $M_{23}$. Let $R'_1$ be the set of vertices in $R_1$ not covered by $M'_{23}$. We repeat this process with $M_{13}$ and $M_{12}$, respectively, to obtain red matchings $M'_{13}$ and $M'_{12}$ and sets $R'_2$ and $R'_3$.

By maximality of $M$, for each vertex $w \in R'_1$ the following is true: either $w$ has no green-neighbour in $M_{23}$, or $w$ has no green-neighbour in $R'_2$. Moreover $w$ has at most $\eta n$ non-neighbours. Observe that $|R'_2|, |R'_1| > \eta' n$ by Fact 1 and the set $X$ of vertices in $M_{23}$ that are not covered by $M'_{23}$ has size at least $\frac{1}{3}\eta' n$ since $|M_{23}| = |M'_{23}| \geq \frac{1}{3}\eta' n$ and each edge of $M'_{23}$ uses exactly one vertex from $M_{23}$. This implies that we can again use a greedy method to construct a red matching $M'_R$ with edges from $(R'_1 - u^*) \times (R'_2 \cup X)$ of size at least $\frac{1}{3}\eta' n - \eta n - 1 \geq \frac{1}{4}\eta' n$. Hence we obtain a red matching $M' := M'_{23} \dot\cup M'_{13} \dot\cup M'_{12} \dot\cup M'_R$ of size at least $|M| + \frac{1}{4}\eta' n$. For establishing Fact 3 it remains to show that $M'$ is red joined. This follows from Fact 2 since each edge of $M'$ intersects $R - u^*$. □

If the matching $M'$ is odd then the proof of Lemma 25 is complete. Hence assume in the remainder that $M'$ is even. Since $M'$ intersects $R - u^*$ this together with Fact 2 immediately implies the next fact. For simplifying the statement as well as the following arguments we will first delete the vertex $u^*$ from $K$ (and let $K$ denote the resulting graph from now on).

**Fact 4.** *No odd red cycle in $K$ contains a vertex of $R$.* □

Fact 8 below uses this observation to conclude that $K$ is extremal, contradicting the hypothesis of Lemma 25. To prepare the proof of this fact we first need some auxiliary observations.

**Fact 5.** *For any choice of $i,j,k$ such that $\{i,j,k\}=\{1,2,3\}$, if $M_{ij}$ is a substantial block then there is a vertex $v^*\in R_k$ such that $K[M_{ij},R_k-v^*]$ is red and $K[M_{ij}]$ is green.*

We first establish the first part of the statement. We may assume that there are vertices $v^*\in R_k$ and $v\in M_{ij}$ such that $v^*v$ is green (otherwise we are done). Without loss of generality $v\in V^{(i)}$. Let $X=\mathrm{N}(v^*)$. Then, by the maximality of $M$, all edges between $v^*$ and $X\cap R$ are red. By Fact 4 this implies that all edges between $X\cap R_i$ and $X\cap R_j$ are green. Since $\min\{|X\cap R_i|,|X\cap R_j|\}>\eta n$, this set of edges is not empty. We use the maximality of $M$ to infer that all edges between $M_{ij}$ and $X\cap(R_i\cup R_j)$ are red. Using Fact 4 this in turn implies that edges between $Y:=M_{ij}\cap X$ and $v^*$ are green. By the maximality of $M$ all edges between $M(Y)$ and $R_k-v^*$ are consequently red. We claim that therefore $K[R_j\cap X,R_k-v^*]$ is green. Indeed, assume there was a red edge $ww'\in(R_j\cap X)\times(R_k-v^*)$. Then $w$ and $w'$ have at least $|M(Y)\cap V^{(i)}|-2\eta n\geq|M_{ij}|-3\eta n\geq\tilde{\eta}n-3\eta n>0$ common neighbours $w''$ in $M(Y)\cap V^{(i)}$. Since edges between $M(Y)$ and $R_k-v^*$ and edges between $M_{ij}$ and $X\cap R_j$ are red, so are the edges $ww''$ and $w'w''$ and thus we have a red triangle $ww'w''$ contradicting Fact 4.

By Fact 1 we have $|R_j\cap X|\geq\eta'n-\eta n>\eta n$ and so each vertex in $R_k-v^*$ is connected by a green edge to some vertex in $R_j\cap X$. The maximality of $M$ implies that $K[M_{ij},R_k-v^*]$ is red as required.

For the second part of Fact 5 observe that the fact that $K[M_{ij},R_k-v^*]$ is red and $|R_k|\geq\eta'n>2\eta n+1$ imply that each pair of vertices in $M_{ij}$ has a common red neighbour in $R_k-v^*$ and so by Fact 4 the graph $K[M_{ij}]$ is green. This establishes Fact 5. □

Now we also delete all (at most 3) vertices from $R$ that play the rôle of $v^*$ in Fact 5 (and again keep the names for the resulting sets).

**Fact 6.** *Suppose that $\{i,j,k\}=\{1,2,3\}$ and that $M_{ij}$ is a substantial block. Then for one of the sets $V^{(i)}$ and $V^{(j)}$, say for $V^{(i)}$, the graph $K[M_{ij},R_i]$ is red and $K[R_k,R_i]$ is green. For the other set $V^{(j)}$ the following is true. If $v\in R_j$ then $K[v,M_{ij}]$ and $K[v,R]$ are monochromatic, with distinct colours.*

We start with the first part of this fact and distinguish two cases. First, assume that there is a red edge $ww'$ with $w\in R_k$ and $w'\in R_j$. We will show that in this case $K[M_{ij},R_i]$ is red and $K[R_k,R_i]$ is green. Since $M_{ij}$ is substantial, edges between $w$ and $M_{ij}$ are red by Fact 5 and hence, owing to Fact 4, edges between $M_{ij}\cap\mathrm{N}(w)$ and $w'$ are green. Since $K[M_{ij}]$ is green by Fact 5, since $M$ is maximal, and since each vertex in $M_{ij}\cap V^{(j)}$ has some neighbour in $M_{ij}\cap\mathrm{N}(w')$ this implies that all edges between $M_{ij}$ and $R_i$ are red. Moreover, edges between $M_{ij}\cap V^{(j)}$ and $R_k$ are red by Fact 5 and hence we conclude from Fact 4 that $K[R_k,R_i]$ is green. If, on the other hand, there is no red edge between $R_k$ and $R_j$ then the first part of the fact is true with $i$ and $j$ interchanged: Clearly $K[R_k,R_j]$ is green and by maximality of $M$ we infer that $K[M_{ij},R_j]$ is red.

For the second part of the fact suppose, say, that $K[M_{ij},R_i]$ is red and $K[R_k,R_i]$ is green. Let $v\in R_j$ and assume first that $v$ has a green neighbour in $M_{ij}$. The maximality of $M$ then implies that $K[v,R]$ is red and since $K[R_k,M_{ij}]$ is also red (by Fact 5) we get that $K[v,M_{ij}]$ is green. Hence it remains to consider the case that $K[v,M_{ij}]$ is red. By Fact 5 the graph $K[R_k,M_{ij}]$ is red and so Fact 4 forces the graph $K[v,R_k]$ to be green. To show that also $K[v,R_i]$ is green assume to the contrary that there is a red edge $vw$ with $w\in R_i$. Recall that $K[v,M_{ij}\cap V^{(i)}]$, $K[M_{ij}\cap V^{(i)},R_k]$, $K[R_k,M_{ij}\cap V^{(j)}]$, and $K[M_{ij}\cap V^{(j)},w]$ are red (and clearly $\eta$-complete). Since $|M_{ij}\cap V^{(i)}|,|R_k|,|M_{ij}\cap V^{(j)}|\geq\tilde{\eta}n-1\geq 2\eta n+2$ we can apply Proposition 24 to infer that there is a red $C_5$ touching $R$ which contradicts Fact 4. □

**Fact 7.** *For any choice of $i,j,k$ with $\{i,j,k\}=\{1,2,3\}$), if $M_{ij}$ and $M_{jk}$ are substantial, then $K[M_{ij},M_{jk}]$ and $K[R_k,R_i]$ are green and $K[M_{ij}\cup M_{jk},R_k\cup R_i]$ is red. Moreover, if $v\in R_j$ then $K[v,M_{ij}\cup M_{jk}]$ and $K[v,R]$ are monochromatic, with distinct colours.*

By Fact 6 every vertex in $R_k\cup R_i$ sends some green edges to $R$ and hence the maximality of $M$ implies that $K[M_{ij}\cup M_{jk},R_k\cup R_i]$ is red. Since there is no red triangle touching $R$, the graphs $K[M_{ij} \cap V^{(i)}, M_{jk} \cap V^{(j)}]$, $K[M_{ij} \cap V^{(j)}, M_{jk} \cap V^{(k)}]$, and $K[R_k, R_i]$ are green. Using Proposition 24 we get similarly as before that also edges in $K[M_{ij} \cap V^{i)}, M_{jk} \cap V^{(k)}]$ are green, since otherwise there was a red $C_5$ touching $R$. It remains to show the second part of Fact 7. By Fact 6 the graph $K[v, R]$ is monochromatic. Moreover, applying Fact 6 once to $M_{ij}$ and once to $M_{jk}$, we obtain that $K[v, R]$ and $K[v, M_{ij} \cup M_{jk}]$ are monochromatic graphs of distinct colours.□

Now we have gathered enough structural information to show that $K$ is $\tilde{\eta}$-extremal, a contradiction.

**Fact 8.** *$K$ is in web configuration with parameter $\tilde{\eta}$.*

We first argue that we can assume without loss of generality that Fact 6 implies for any $\{i, j, k\} = \{1, 2, 3\}$ that

$$K[R_i, M_{i,j}] \text{ is red} \qquad \text{if } M_{ij} \text{ is substantial and } i \neq 3 \tag{$*$}$$

Indeed, since $M_{2,3}$ is substantial, Fact 7 implies $(*)$ in the case $\{j, k\} = \{2, 3\}$. It remains to consider the case $\{j, k\} = \{1, 3\}$. If in addition the block $M_{13}$ is substantial then Fact 7 also implies $(*)$ in this case. If, on the other hand, $M_{13}$ (and hence also $M_{12}$) is not substantial, then $j = 3$ and hence $k = 1$. If in addition $K[R_i, M_{ij}] = K[R_2, M_{23}]$ is not red then $V^{(i)} = V^{(2)}$ plays the rôle of $V^{(j)} = V^{(3)}$ in Fact 6. Then however we may delete at most $\tilde{\eta}n$ vertices from $R_2$ in order to guarantee $|R_2| \leq |R_3|$ and then the following argument still works with $V^{(2)}$ and $V^{(3)}$ interchanged. This implies $(*)$.

To obtain the web configuration set $U^{(i)} := R_i$, for $i = 1, 2$ and let $U^{(3)}$ be the set of those vertices $v \in R_3$ such that $K[v, M_{23}]$ is red, let $W_3^{(3)} := R_3 \setminus U^{(3)}$, and define $W_j^{(i)} := M_{ij} \cap V^{(i)}$ for all $i \neq j \in [3]$. If any of the sets we just defined has less than $\tilde{\eta}n$ vertices delete all vertices in this set. Finally, define $W^{(i)}$, $i \in [3]$ as in the definition of the web configuration (Definition 7). Observe that this together with Fact 6 implies that $K[W_3^{(3)}, M_{23}]$ is green and $K[W_3^{(3)}, R]$ is red.

Now let $\{i, j, k\} = \{1, 2, 3\}$ arbitrarily. Clearly we have $|U^{(i)} \cup W^{(i)}| \geq (1 - 3\tilde{\eta})n \geq (1 - \eta')n$. Moreover $K[U^{(i)}, W^{(j)}]$ is $\eta$-complete. We next verify that this graph is also red. We distinguish two cases. First assume that $j \neq 3$. In this case

$$U^{(i)} \subseteq R_i \quad \text{and} \quad W^{(j)} = W_i^{(j)} \cup W_k^{(j)} \subseteq (M_{ij} \cap V^{(j)}) \cup (M_{jk} \cap V^{(j)}) \,.$$

We have $W_k^{(j)} \neq \emptyset$ only if $M_{jk}$ is substantial and then Fact 5 implies that $K[R_i, M_{jk}]$ is red. Similarly $W_i^{(j)} \neq \emptyset$ only if $M_{ij}$ is substantial. By $(*)$ we have that $K[R_i, M_{ij}]$ is red if $i \neq 3$. By the definition of $U^{(3)}$ we also get that $K[U^{(i)}, M_{ij}]$ is red if $i = 3$. Thus all edges between $U^{(i)}$ and $W^{(j)}$ are red as desired. If $j = 3$ on the other hand then $U^{(1)} \subseteq R_i$ and $W^{(j)} = W_i^{(j)} \cup W_k^{(j)} \cup W_3^{(3)} \subseteq (M_{ij} \cap V^{(j)}) \cup (M_{ik} \cap V^{(j)}) \cup W_3^{(3)}$. Analogous to the argument in the first case the graphs $K[R_i, M_{jk}]$ and $K[R_i, M_{ij}]$ are red (since $i \neq 3$ and by $(*)$). As noted above in addition all edges between $R$ and $W_3^{(3)}$ are red and so $K[U^{(i)}, W^{(j)}]$ is also red in this case.

We finish the proof of Fact 8 (and hence Lemma 25) by checking that we have a web configuration with colour $\chi =$ red. Observe that the graph $K[\bigcup U^{(i)}, \bigcup W^{(j)}]$ is connected and bipartite. We now verify Conditions 1–4 of the web configuration. For Condition 2 assume that $W_3^{(3)} \neq \emptyset$. Fact 7 and the definition of $W^{(3)}$ imply then that $M_{12}$ is not substantial and hence $|W_2^{(1)}| = 0$. Moreover, since $|R_1| \geq |R_2| \geq |R_3|$ we get the first part of Condition 1, and $|W_j^{(i)}| = |W_i^{(j)}|$ is clearly true by definition. By Fact 1 we have $n - |M_{ik} \cup M_{jk}| = |R_k| > |M_{ij}|$ which implies $n - |W^{(k)}| > |W_j^{(i)}|$ unless $k = 3$ and $W_3^{(3)} \neq \emptyset$ (if $k \neq 3$ or $W_3^{(3)} = \emptyset$ then $|M_{ij}| = |W_j^{(i)}|$). And if $W_3^{(3)} \neq \emptyset$ Condition 2 implies $|W_j^{(i)}| = |U_2^{(1)}| = 0$ and thus we also get $n - |W^{(k)}| > |W_j^{(i)}|$ in this case. This establishes Condition 1. To see Condition 3, note that if $U^{(1)}$ is non-empty then either $M_{12}$ or $M_{13}$ are substantial. Since in addition $M_{23}$ is substantial by assumption we conclude from Fact 7 that there is a green triangle connected to $M_{23}$ and hence to the green matching $M$. As $K$ is not $\frac{3}{4}(1 - \eta')n$-odd this implies $\frac{1}{2}|\bigcup_{i,j} W_j^{(i)}| \leq |M| < \frac{3}{4}(1 - \eta')n$. It remains to verify Condition 4. Assume, for a contradiction, that $U^{(3)} \neq \emptyset$ and $|\bigcup U^{(i)}| \geq (1 - \eta)\frac{3}{2}n$ and $|U^{(2)} \cup U^{(3)}| > (1 - \eta)\frac{3}{4}n$. As $|R_1| \geq |R_2| \geq |R_3| \geq |U^{(3)}|$ and $U^{(3)} \neq \emptyset$ all these sets have size at least $\tilde{\eta}n$ and so $U^{(i)} = R_i$, for $i = 1, 2, 3$. By Fact 6 and the definition of $U^{(3)}$ the graph $K[U^{(1)}, U^{(2)}, U^{(3)}]$ is $(\eta, \text{green})$-complete and thus contains a green triangle by Proposition 23(c) and is connected by (b) of the same proposition. Observe that this implies that any matching in $K[U^{(1)}, U^{(2)}, U^{(3)}]$ is joined and odd. We will show that $K[U^{(1)}, U^{(2)}, U^{(3)}]$ contains a green matching of size at least $\frac{3}{4}(1-\eta')n$ contradicting the fact that $K$ is not $\frac{3}{4}(1-\eta')n$-odd. We distinguish two cases. If $|U^{(1)}| \geq |U^{(2)} \cup U^{(3)}|$ an easy greedy algorithm guarantees a green matching of size $|U^{(2)} \cup U^{(3)}| - \eta n > (1-3\eta)\frac{3}{4}n \geq \frac{3}{4}(1-\eta')n$ in $K[U^{(1)}, U^{(2)} \cup U^{(3)}]$. If $|U^{(1)}| \leq |U^{(2)} \cup U^{(3)}|$, on the other hand, there is a green matching covering at least $|U^{(1)} \cup U^{(2)} \cup U^{(3)}| - 4\eta n - 1 > (1-4\eta)\frac{3}{2}n - 1 \geq \frac{3}{2}(1-\eta')n$ vertices in $K[U^{(1)}, U^{(2)}, U^{(3)}]$ by Proposition 22. □

□

We will now use Lemma 25 to prove Lemma 9.

*Proof of Lemma 9.* Let $\eta'$ be given and set $\tilde{\eta} := \eta'/15$. Let $\eta_{\mathrm{L25}}$ and $n_0$ be provided by Lemma 25 for input $\eta'_{\mathrm{L25}} = \tilde{\eta}$ and set $\eta := \min\{\eta_{\mathrm{L25}}, \tilde{\eta}/5\}$. Let $K = (V^{(1)} \dot\cup V^{(2)} \dot\cup V^{(3)}, E)$ be a non-extremal coloured graph in $\mathcal{K}_n^\eta$ with partition classes $V^{(1)}$, $V^{(2)}$, $V^{(3)}$ and assume for a contradiction that $K$ is not $(1-\eta')3n/4$-odd.

Our first step is to show that $K$ has big green and red joined matchings.

**Fact 1.** *$K$ has even joined matchings $M_r$ and $M_g$ in red and green, respectively, with $|M_r|, |M_g| \geq (1-\tilde{\eta})n$.*

Assume for a contradiction that a maximum matching $M$ in red has size less than $(1-\tilde{\eta})n$. By Lemma 25 applied with $\tilde{\eta}$ we conclude that there is an odd joined matching $M'$ with $|M'| > |M|$. On the other hand $K$ is not $\big((1-\eta')3n/4\big)$-odd, hence $|M'| < (1-\eta')3n/4$. Another application of Lemma 25 with $\tilde{\eta} \leq \eta'$ thus provides us with a red joined matching of size bigger than $|M'|$ which contradicts the maximality of $M$. We conclude that there is a red joined matching $M_r$, and by symmetry also a green joined matching $M_g$, of size at least $(1-\tilde{\eta})n$. Clearly, $M_r$ and $M_g$ are even since $K$ is not $\big((1-\eta')3n/4\big)$-odd. □

Let $R$ be the continent of $M_r$ and $G$ be the continent of $M_g$ in $K$. Fact 1 states, that $R$ and $G$ are bipartite. We observe in the following fact that both $R$ and $G$ substantially intersect all three partition classes. For this purpose define $V_r^{(i)} := V^{(i)} \cap V(R)$ and $V_g^{(i)} := V^{(i)} \cap V(G)$, and further $\bar{V}_r^{(i)} := V^{(i)} \setminus V_r^{(i)}$ and $\bar{V}_g^{(i)} := V^{(i)} \setminus V_g^{(i)}$ for all $i \in [3]$.

**Fact 2.** *For all $i \in [3]$ we have $|V_r^{(i)}|, |V_g^{(i)}| \geq 2\tilde{\eta}n$.*

Indeed, assume that $|V_r^{(i)}| < 2\tilde{\eta}n$ which implies $|\bar{V}_r^{(i)}| > (1-2\tilde{\eta})n$. As $|M_r| \geq (1-\tilde{\eta})n$ it follows that $|V_r^{(\ell)}| > (1-3\tilde{\eta})n$ for $\ell \neq i$. By definition all edges between $\bar{V}_r^{(i)}$ and $V_r^{(j)} \cup V_r^{(k)}$ are green and thus $K$ is in pyramid configuration with tunnel, pyramids $(V_r^{(j)}, \bar{V}_r^{(i)})$ and $(V_r^{(k)}, \emptyset)$, and parameter $3\tilde{\eta} < \eta'$, which is a contradiction. □

Next we strengthen the last fact by showing that at most one of the sets $\{\bar{V}_c^{(i)}, \bar{V}_g^{(i)} : i \in [3]\}$ is significant.

**Fact 3.** *For at most one element $D \in \{\bar{V}_r^{(i)}, \bar{V}_g^{(i)} : i \in [3]\}$ we have that $|D| \geq \tilde{\eta}n$. If such a $D$ exists we assume, without loss of generality, $D = \bar{V}_r^{(1)}$.*

Assume that $\bar{V}_r^{(1)} \geq \tilde{\eta}n$. First we show that then

$$|\bar{V}_r^{(2)}|, |\bar{V}_r^{(3)}| < \tfrac{\tilde{\eta}}{2}n\,. \tag{17}$$

Assume for a contradiction and without loss of generality that $|\bar{V}_r^{(2)}| \geq \frac{\tilde{\eta}}{2}n$. By definition, all edges in $E(\bar{V}_r^{(1)}, V_r^{(2)} \dot\cup V_r^{(3)})$ and $E(\bar{V}_r^{(2)}, V_r^{(1)} \dot\cup V_r^{(3)})$ are green. Since $|\bar{V}_r^{(1)}|, |\bar{V}_r^{(2)}| \geq \tilde{\eta}n > 2\eta n$ by assumption and $|V_r^{(1)}|, |V_r^{(2)}| \geq 2\tilde{\eta}n/2 > 2\eta n$ (by Fact 2) we can apply Proposition 23(b) to infer that the graph with edges $E(\bar{V}_r^{(1)}, V_r^{(2)} \dot\cup V_r^{(3)})$ and $E(\bar{V}_r^{(2)}, V_r^{(1)} \dot\cup V_r^{(3)})$ is connected. As $M_r$ is even we conclude that all edges in $E(V_r^{(i)}, V_r^{(j)})$, are red for all $i \neq j \in [3]$. Since by Fact 2 $|V_r^{(i)}| \geq \tilde{\eta}n > 2\eta n$, for $i \in [3]$, we infer from Proposition 23(c) that the graph $K[V_r^{(1)}, V_r^{(2)}, V_r^{(3)}] \subseteq R$ contains a red triangle which contradicts the fact that $M_r$ is even.

Thus it remains to show that $|\bar{V}_g^{(i)}| < \tilde{\eta}n$ for all $i \in [3]$. By (17) and Fact 2 we have for $i = 2, 3$ that $|V_r^{(i)} \cap V_g^{(i)}| > \frac{\tilde{\eta}}{2}n > \eta n$ which implies that there is an edge in $E(V_r^{(2)} \cap V_g^{(2)}, V_r^{(3)} \cap V_g^{(3)})$. By assumption we also have $|\bar{V}_r^{(1)}| \geq \tilde{\eta}n > 2\eta n$ and thus each pair of vertices in $V_r^{(2)} \dot\cup V_r^{(3)}$ has a common neighbour in $\bar{V}_r^{(1)}$ by (a) of Proposition 23. By definition of $\bar{V}_r^{(1)}$ all edges in $E(\bar{V}_r^{(1)}, V_r^{(2)} \dot\cup V_r^{(3)})$ are green, and therefore we conclude that all edges in $E(V_r^{(2)} \cap V_g^{(2)}, V_r^{(3)} \cap V_g^{(3)})$ are red since otherwise there would be a green triangle connected to $M_g$. Accordingly $|\bar{V}_g^{(1)}| \leq 2\eta n < \tilde{\eta}n/2$ since otherwise we could equally argue that all edges in $E(V_r^{(2)} \cap V_g^{(2)}, V_r^{(3)} \cap V_g^{(3)})$ are green, a contradiction. Therefore $|V_g^{(1)}| \geq (1 - \tilde{\eta}/2)n$. As $|\bar{V}_r^{(1)}| \geq \tilde{\eta}n$ this implies $|V_g^{(1)} \cap \bar{V}_r^{(1)}| \geq \tilde{\eta}/2n > \eta n$ and from (17) we also get $|V_r^{(2)} \cap \bar{V}_g^{(2)}| \geq \tilde{\eta}/2n > \eta n$. Thus there is an edge in $E(V_g^{(1)} \cap \bar{V}_r^{(1)}, V_r^{(2)} \cap \bar{V}_g^{(2)})$. However, this edge can neither be red since it connects $\bar{V}_r^{(1)}$ and $V_r^{(2)}$, nor green since it connects $\bar{V}_g^{(2)}$ and $V_g^{(1)}$, a contradiction. Therefore $|\bar{V}_g^{(2)}| < \tilde{\eta}n$ and by symmetry also $|\bar{V}_g^{(3)}| < \tilde{\eta}n$ which finishes the proof of Fact 3. □

Now, we label the vertices in each of the bipartite graphs $R$ and $G$ according to their bipartition class by 1 and 2 (where the choice of bipartition class 1 and bipartition class 2 is arbitrary). In the remaining part of the proof we examine the distribution of these bipartition classes over the partition classes of $K$. Let $F_{ij}$ denote the set of vertices in $V(R) \cap V(G)$ with label $i$ in $R$ and label $j$ in $G$ for $i, j \in [2]$. Let further $F_{0j}$ be the set of vertices in $\bar{V}_r^{(1)} \cap V(G)$ that have label $j$ in $G$ for $j \in [2]$. Next we observe that each of the sets $F_{ij}$ with $i, j \in [2]$ is essentially contained in one partition class of $K$.

**Fact 4.** *For all $i, j \in [2]$ there is at most one partition class $\ell \in [3]$ of $K$ with $|F_{ij} \cap V^{(\ell)}| \geq \tilde{\eta}n$. Moreover $E(F_{0j}, F_{ij}) = \emptyset$.*

To prove the first part of Fact 4 assume for a contradiction that $|F_{ij} \cap V^{(\ell)}|, |F_{ij} \cap V^{(\ell')}| \geq \tilde{\eta}n$ for $\ell \neq \ell'$. Then there would be an edge in $K[V^{(\ell)} \cap F_{ij}, V^{(\ell')} \cap F_{ij}]$ since $\tilde{\eta} > \eta$. This contradicts the fact that $F_{ij}$ is independent by definition. For the second part observe that an edge in $E(F_{0j}, F_{ij})$ can neither be red as such an edge would connect vertices from $\bar{V}_r^{(1)}$ to $R$ nor green since $F_{0j} \cup F_{ij}$ lies in one bipartition class $j$ of $G$. □

**Fact 5.** *There are indices $k, \ell \in [3]$ with $k \neq \ell$ and indices $b, b', c, c' \in [2]$ with $bb' \neq cc'$ such that $|F_{bb'} \cap V^{(k)}|, |F_{cc'} \cap V^{(\ell)}| \geq (1 - 5\tilde{\eta})n$ and $|F_{0b'}|, |F_{0c'}| \leq \tilde{\eta}n$.*

We divide the proof of this fact into three cases: The first case deals with $\bar{V}_r^{(1)} \neq \emptyset$, the second one with $\bar{V}_r^{(1)} = \emptyset$ and the additional assumption that there are $k \in [3]$ and $ij \neq i'j' \in [2]$ such that $|V^{(k)} \cap F_{ij}|, |V^{(k)} \cap F_{i'j'}| \geq \tilde{\eta}n$. The third and remaining case treats the situation when $\bar{V}_r^{(1)} = \emptyset$ and for each $k \in [3]$ there is at most one index pair $(i, j)$, where $i = i(k)$ and $j = j(k)$, with $|V^{(k)} \cap F_{ij}| \geq \tilde{\eta}n$.

For the first case, let $j \in [2]$ be such that $F_{0j} \neq \emptyset$. Observe that then the second part of Fact 4 implies that $|F_{1j} \cap (V^{(2)} \cup V^{(3)})|, |F_{2j} \cap (V^{(2)} \cup V^{(3)})| < \eta n$. Let $c' = b' \in [2]$ with $c' \neq j$. Then, because Fact 3 implies that $|\bar{V}_r^{(i)}|, |\bar{V}_g^{(i)}| < \tilde{\eta}n$ for $i = 2, 3$, we have that $|V^{(2)} \cap (F_{1b'} \cup F_{2b'})| \geq (1 - 4\tilde{\eta})n$ and $|V^{(3)} \cap (F_{1c'} \cup F_{2c'})| \geq (1 - 4\tilde{\eta})n$. Thus there is a $b \in [2]$ such that $|V^{(2)} \cap F_{bb'}| \geq \tilde{\eta}n$. Let $c' \in [2]$ with $c' \neq b'$. The first part of Fact 4 implies that $|V^{(3)} \cap F_{bc'}| < \tilde{\eta}n$, thus $|V^{(3)} \cap F_{cc'}| \geq (1 - 5\tilde{\eta})n \geq \tilde{\eta}n$. By symmetry we also get $|V^{(2)} \cap F_{bb'}| \geq (1 - 5\tilde{\eta})n$. This proves the first part of the statement for the first case. To see the second part, observe that if $F_{0b'} \neq \emptyset$, then $|F_{1b'} \cap (V^{(2)} \cup V^{(3)})|, |F_{2b'} \cap (V^{(1)} \cup V^{(3)})| < \eta n$ by Fact 4, a contradiction.

The second part of the second and third cases is straightforward as $F_{01}, F_{02} \subseteq \bar{V}_r^{(1)} = \emptyset$. To see the first part of the second case let $k \in [3]$ and $ij \neq i'j' \in [2]$ with $|V^{(k)} \cap F_{ij}|, |V^{(k)} \cap F_{i'j'}| \geq \tilde{\eta}n$, and let $\ell, \ell' \in [3]$ be such that $\{\ell, \ell', k\} = \{1, 2, 3\}$. The first part of Fact 4 implies that $|F_{ij} \cap V^{(\ell)}|, |F_{i'j'} \cap V^{(\ell)}|, |F_{ij} \cap V^{(\ell')}|, |F_{i'j'} \cap V^{(\ell')}| < \tilde{\eta}n$. Thus $|(F_{ij'} \cup F_{i'j}) \cap V^{(\ell)}| \geq (1 - 2\tilde{\eta})n - 2\tilde{\eta}n$, as $|\bar{V}_r^{(\ell)}|, |\bar{V}_g^{(\ell)}| < \tilde{\eta}n$. Without loss of generality, let $ij'$ be such that $|V^{(\ell)} \cap F_{ij'}| \geq \tilde{\eta}n$. We set $b := i$, $b' := j'$, $c = i'$ and $c' := j$. The rest of the proof is similar to the first case, proving that then $|V^{(\ell')} \cap F_{cc'}| \geq (1 - 5\tilde{\eta})n$ and by symmetry that $|V^{(\ell)} \cap F_{bb'}| \geq (1 - 5\tilde{\eta})n$.

It remains to prove the first part of the third case. For this observe that for all $k \in [3]$ we have that $|V^{(k)} \cap \bigcup_{(i'j') \neq (i,j)} F_{i'j'}| < 3\tilde{\eta}n$, where $i = i(k)$ and $j = j(k)$ are as specified in the definition of the third case. Observe also that $|\bar{V}_r^{(k)}|, |\bar{V}_g^{(k)}| < \tilde{\eta}n$ for $k \in \{2,3\}$ by Fact 3. This implies $|V^{(k)} \cap F_{i'j'}| \geq (1-5\tilde{\eta})n$ for $k \in \{2,3\}$ and $i' \in [3] \setminus \{i(k)\}$, $j' \in [3] \setminus \{j(k)\}$, as desired. Hence, for $k = 2$ and $\ell = 3$ we obtain indices $b, b', c, c'$ such that $|V^{(k)} \cap F_{bb'}|, |V^{(\ell)} \cap F_{cc'}| \geq (1-5\tilde{\eta})n$, with $bb' \neq cc'$ as required in Fact 5. □

This brings us to the last step, which shows that $K$ is extremal, a contradiction.

**Fact 6.** *$K$ is in pyramid configuration with parameter $\eta'$.*

Let $k, \ell \in [3]$ and $b, b', c, c' \in [2]$ be as in Fact 5. Let $\ell' \in [3] \setminus \{k, \ell\}$. Assume without loss of generality that $b = b' = 1$. Thus Fact 5 states that $|F_{11} \cap V^{(k)}| \geq (1-5\tilde{\eta})n$ and $|F_{01}| \leq \tilde{\eta}n$. We distinguish two cases. First, assume that $c' = 2$ and set $\bar{c} := 3 - c$. By Fact 5 this implies $|F_{c2} \cap V^{(\ell)}| \geq (1-5\tilde{\eta})n$ and $|F_{02}| \leq \tilde{\eta}n$ and thus $|(F_{\bar{c}2} \cup F_{21}) \cap V^{(\ell')}| \geq (1-5\tilde{\eta})n$ by Fact 4. Moreover $E(F_{11} \cap V^{(k)}, F_{21} \cap V^{(\ell')}))$ forms an $\eta$-complete red bipartite graph since $F_{11} \cup F_{21}$ is an independent set in $G$. Similarly $E(F_{c2} \cap V^{(\ell)}, F_{\bar{c}2} \cap V^{(\ell')}))$ forms an $\eta$-complete red bipartite graph. Further, if $c = 2$ then $E(F_{c2} \cap V^{(\ell)}, F_{21} \cap V^{(\ell')})$ and $E(F_{11} \cap V^{(k)}, F_{\bar{c}2} \cap V^{(\ell')})$ form $\eta$-complete green bipartite graphs (leading to crossings) and if $c = 1$ then $E(F_{11} \cap V^{(k)}, F_{c2} \cap V^{(\ell)})$ forms an $\eta$-complete green bipartite graph (leading to a tunnel). Therefore, in both subcases, $K$ is in pyramid configuration with parameter $5\tilde{\eta} \leq \eta'$ and pyramids $(F_{11} \cap V^{(k)}, F_{21} \cap V^{(\ell')})$ and $(F_{c2} \cap V^{(\ell)}, F_{\bar{c}2} \cap V^{(\ell')})$, unless one of the sets $F_{21} \cap V^{(\ell')}$ and $F_{\bar{c}2} \cap V^{(\ell')}$ has size at most $10\tilde{\eta}n$. In this case, however, we can simply replace this set by the empty set and still obtain a pyramid configuration with parameter at most $15\tilde{\eta} \leq \eta'$.

In the case $c' = 1$ we have $c = 2$. Fact 5 guarantees that $|F_{21} \cap V^{(\ell)}| \geq (1-5\tilde{\eta})n$. Since $|F_{01}| \leq \tilde{\eta}n$ we conclude from Fact 4 that $|(F_{12} \cup F_{22} \cup F_{02}) \cap V^{(\ell')}| \geq (1-5\tilde{\eta})n$. Similarly as before $E(F_{11} \cap V^{(1)}, (F_{12} \cup F_{02}) \cap V^{(\ell')})$ and $E(F_{21} \cap V^{(\ell)}, F_{22} \cap V^{(\ell')})$ form $\eta$-complete green bipartite graphs and $E(F_{11} \cap V^{(k)}, F_{21} \cap V^{(\ell)})$ forms an $\eta$-complete red bipartite graph. Accordingly we also get a pyramid configuration with parameter $5\tilde{\eta} \leq \eta'$ in this case, where the pyramids are $(F_{11} \cap V^{(k)}, (F_{12} \cup F_{02}) \cap V^{(\ell')})$ and $(F_{21} \cap V^{(\ell)}, F_{22} \cap V^{(\ell')})$ unless, again, $(F_{12} \cup F_{02}) \cap V^{(\ell')}$ or $F_{22} \cap V^{(\ell')}$ are too small in which case we proceed as above. □

□

7.3. **Extremal configurations.** Our aim in this section is to provide a proof of Lemma 10. This proof naturally splits into two cases concerning pyramid and web configurations, respectively. The former is covered by Proposition 26, the latter by Proposition 27.

**Proposition 26.** *Lemma 10 is true for pyramid configurations.*

*Proof.* Given $\eta'$ set $\eta = \eta'/3$. Let $K$ be a coloured graph from $\mathcal{K}_n^\eta$ that is in pyramid configuration with parameter $\eta$ and pyramids $(D_1, D_1')$ and $(D_2, D_2')$ such that the requirements of (E1) in Definition 7 are met for colours $\chi$ and $\chi'$.

**Fact 1.** *If the pyramid configuration has crossings then $K$ is $\big((1-\eta')n, (1-\eta')\frac{3}{2}n, 2\big)$-good.*

Indeed, by Proposition 21 there is a matching $M$ of colour either $\chi$ or $\chi'$ and size at least $(1-2\eta)\frac{1}{2}n$ in $K[D_1, D_2]$. Note further, that the pyramid configuration with crossings is symmetric with respect to the colours $\chi$ and $\chi'$ and hence we may suppose, without loss of generality, that $M$ is of colour $\chi$ and that $|D_1'| \geq (1-\eta)\frac{1}{2}n$. As $K[D_1, D_1']$ and $K[D_2, D_2']$ are $(\eta, \chi)$-complete, there are $\chi$-coloured matchings $M_1$ and $M_2$ in $K[D_1, D_1']$ and $K[D_2 \setminus M, D_2']$, respectively, of size at least $\min\{|D_1'|, |D_1|\} - \eta n$ and $\min\{|D_2'|, |D_2 \setminus M|\} - \eta n$, respectively. This implies

$$|M| + |M_1| + |M_2| \geq (1-3\eta)\frac{3}{2}n = (1-\eta')\frac{3}{2}n.$$

Observe that, depending on the size of $M$, either $M \cup M_2$ or $M_1 \cup M_2$ is a matching of size at least $(1-3\eta)n = (1-\eta')n$. Now, the union of $M$, $M_1$, and $M_2$ forms a 2-fork system $F$ and since $K[D_1, D_1']$ and $K[D_2, D_2']$ are $(\eta, \chi)$-complete the bipartite graph formed by these two graphs and $M$ is connected and has partition classes $D_1 \cup D_2'$ and $D_2 \cup D_1'$. It follows that $F$ has size $|M| + |M_1| + |M_2| \geq (1 - \eta')\frac{3}{2}n$. $\square$

**Fact 2.** *If the pyramid configuration has a $\chi'$-tunnel and if there is a matching $M$ of colour $\chi'$ and size at least $(1-\eta')\frac{1}{2}n$ in $K[D_1, D_1' \cup D_2']$ or in $K[D_2, D_1' \cup D_2']$ then $K$ is $\big((1-\eta')n, (1-\eta')\frac{3}{2}n, 2\big)$-good in colour $\chi'$.*

As $K$ has a $\chi'$-tunnel, there is a joined matching $M'$ of colour $\chi'$ and size at least $|D_1| - \eta n \geq (1 - \eta')n$ in $K[D_1, D_2]$. We will extend the matching $M'$ (which is a 1-fork system) to a 2-fork system. Without loss of generality assume that the matching $M$ promised by the assumptions is in $K[D_1, D_1' \cup D_2']$. As $M \cap D_1$ and $D_2$ are non-negligible the bipartite graph $K[M \cap D_1, D_2]$ is connected by Proposition 23(b) and thus $M$ is joined. Hence $M \cup M'$ forms a joined 2-fork system centred in $D_1$ and of size $|M'| + |M| \geq (1 - \eta')\frac{3}{2}n$. $\square$

**Fact 3.** *If the pyramid configuration has a $\chi'$-tunnel but no crossings and there is no matching of colour $\chi'$ and size at least $(1 - \eta')\frac{1}{2}n$ in $K[D_1, D_1' \cup D_2']$ or in $K[D_2, D_1' \cup D_2']$ then $K$ is $((1 - \eta')n, (1 - \eta')\frac{3}{2}n, 3)$-good in colour $\chi$.*

To obtain the 3-fork system note that Proposition 21 implies that there are matchings $M_1$ and $M_2$ of colour $\chi$ and sizes at least $(1-\eta')\frac{1}{2}n$ in $K[D_1, D_1' \cup D_2']$ and $K[D_2, D_1' \cup D_2']$, respectively. The union of $M_1$ and $M_2$ forms a 2-fork system $F$ centred in $D_1' \cup D_2'$ covering at least $(1 - \eta')\frac{1}{2}n$ vertices in $D_1$ and at least $(1 - \eta')\frac{1}{2}n$ vertices in $D_2$. We can assume without loss of generality that $|D_1'| \geq (1 - \eta)\frac{1}{2}n \geq (1 - \eta')\frac{1}{2}n$. As $K[D_1, D_1']$ is $(\eta, \chi)$-complete and $|D_1 \setminus F| \leq (1 - \eta')n - (1 - \eta')\frac{1}{2}n = (1 - \eta')\frac{1}{2}n$ we can greedily find a matching between $D_1'$ and $D_1 \setminus F$ covering all but at most $\eta n$ vertices of $D_1 \setminus F$. Its union with $F$ forms a 3-fork system $F'$ centred in $D_1' \cup D_2'$ covering at least $(1 - \eta')n$ vertices in $D_1$ and at least $(1 - \eta')\frac{1}{2}n$ vertices in $D_2$, implying that $F'$ has size at least $(1 - \eta')\frac{3}{2}n$. The graph $K[D_1, D_1'] \cup K[D_2, D_2']$ clearly contains a matching $M$ of size at least $|D_1' \cup D_2'| - \eta n \geq (1 - \eta')n$ in colour $\chi$.

Since the pyramid configuration has no crossings there are edges of colour $\chi$ in $K[D_1, D_2'] \cup K[D_2, D_1']$. Together with the fact that $D_1$, $D_2$, $D_1'$, and $D_2'$ are non-negligible, we obtain that the bipartite graphs $K[D_1, D_1' \cup D_2']$ and $K[D_2, D_1' \cup D_2']$ are connected by (b) of Proposition 23. Thus the matching $M$ and the fork system $F'$ are both joined. $\square$

$\square$

**Proposition 27.** *Lemma 10 is true for web configurations.*

*Proof.* Given $\eta'$ set $\eta = \eta'/5$ and let $K$ be a coloured graph from $\mathcal{K}_n^\eta$ that is in web configuration with parameter $\eta$, i. e., it satisfies (E2) of Definition 7. In this proof we construct only matchings and fork systems of colour $\chi$. Observe that these are joined by definition. We distinguish two cases.

*Case 1:* First assume that $|\bigcup U^{(i)}| < (1-\eta)\frac{3}{2}n$. We will show that in this case our configuration contains both a joined matching of size at least $(1 - \eta')n$ and a joined 3-fork system of size at least $(1 - \eta')\frac{3}{2}n$. We need the following auxiliary observation.

**Fact 1.** *If $|\bigcup U^{(i)}| < (1 - \eta)\frac{3}{2}n$ then $W_2^{(1)} = W_1^{(2)} = \emptyset$. Moreover $|U^{(1)}| + \eta n \geq |W_3^{(2)}| = |W_2^{(3)}|$ and $|U^{(2)}| + \eta n \geq |W_3^{(1)}| = |W_1^{(3)}|$.*

Indeed, by Condition 3 of (E2) either $W^{(1)} = \emptyset$ and hence $W_2^{(1)} \subseteq W^{(1)}$ is empty or $|W^{(1)} \cup W^{(2)} \cup (W^{(3)} \setminus W_3^{(3)})| \leq (1-\eta)\frac{3}{2}n$. In the second case we conclude from $|\bigcup U^{(i)}| < (1-\eta)\frac{3}{2}n$ that

$$|U^{(1)} \cup W^{(1)}| + |U^{(2)} \cup W^{(2)}| + |U^{(3)} \cup (W^{(3)} \setminus W_3^{(3)})| < (1 - \eta)3n.$$

As $|U^{(1)} \cup W^{(1)}|, |U^{(2)} \cup W^{(2)}|, |U^{(3)} \cup W^{(3)}| \geq (1-\eta)n$ it follows that $|U^{(3)} \cup (W^{(3)} \setminus W_3^{(3)})| < (1-\eta)n$ and thus $W_3^{(3)} \neq \emptyset$. By Condition 2 of (E2) we get $W_2^{(1)} = \emptyset$. For the second part of the fact observe that Condition 1 of (E2) states that $n - |W^{(1)}| \geq |W_3^{(2)}| = |W_2^{(3)}|$ and thus we conclude $|U^{(1)}| \geq (1 - \eta)n - |W^{(1)}| \geq |W_3^{(2)}| - \eta n = |W_2^{(3)}| - \eta n$. The inequality $|U^{(2)}| \geq |W_3^{(1)}| - \eta n$ is established in the same way. $\square$

**Fact 2.** *If $|\bigcup U^{(i)}| < (1-\eta)\frac{3}{2}n$ then $K$ is $\big((1-\eta')n, (1-\eta')\frac{3}{2}n, 3\big)$-good.*

From Condition 1 of (E2) we infer that $|W_3^{(1)}| < n - |W^{(2)}| \le |U^{(2)}| + \eta n$ and Fact 1 implies that $|U^{(1)}| + |W_3^{(1)}| = |U^{(1)}| + |W^{(1)}| \ge (1-\eta)n$ and $|U^{(3)}| + |W^{(3)}| \ge (1-\eta)n$. We thus conclude from $|\bigcup U^{(i)}| < (1-\eta)\frac{3}{2}n$ that

$$|W_3^{(1)}| - \eta n < |U^{(2)}| < (1-\eta)\tfrac{3}{2}n - |U^{(1)} \cup U^{(3)}| < |W_3^{(1)}| + |W^{(3)}| - \eta n.$$

This (together with the fact that $K[U^{(2)}, W_3^{(1)}]$ and $K[U^{(2)}, W^{(3)}]$ are $(\eta,\chi)$-complete) justifies that there is a $\chi$-coloured matching $M_1$ in $K[U^{(2)}, W_3^{(1)} \cup W^{(3)}]$ covering all vertices of $U^{(2)}$ and all but at most $\eta n$ vertices of $W_3^{(1)}$. Further, by Fact 1 we know that $|W_3^{(2)}| \le |U^{(1)}| + \eta n$ and hence we can find a matching $M_2$ of colour $\chi$ in (the $(\eta,\chi)$-complete graph) $K[W_3^{(2)}, U^{(1)}]$ covering all but at most $\eta n$ vertices of $W_3^{(2)}$. The matching $M := M_1 \cup M_2$ satisfies

$$|M| \ge |U^{(2)}| + |W_3^{(2)}| - \eta n = |U^{(2)}| + |W^{(2)}| - \eta n \ge (1-\eta)n - \eta n \ge (1-\eta')n,$$

where the equality follows from Fact 1. Next, we extend the matching $M$ to a joined 3-fork system of colour $\chi$ and size at least $(1-\eta')\frac{3}{2}n$ in the following way. Consider maximal matchings $M_3$, $M_4$, and $M_5$ in $K[U^{(2)}, W_1^{(3)} \setminus M_1]$, $K[U^{(1)}, W_2^{(3)} \setminus M_1]$ and $K[U^{(1)}, W_3^{(3)} \setminus M_1]$, respectively. By Fact 1 we infer that $M_3$ and $M_4$ each cover all but at most $\eta n$ vertices of $W_1^{(3)} \setminus M_1$ and $W_2^{(3)} \setminus M_1$, respectively. As $|W_3^{(3)}| \le |W_3^{(3)} \cup U^{(3)}| \le |U^{(1)}|$ by Condition 1 of (E2) the matching $M_5$ covers all but at most $\eta n$ vertices of $W_3^{(3)}$.

Then the union $M \cup M_3 \cup M_4 \cup M_5$ is a 3-fork system $F$ centred in $U^{(1)} \cup U^{(2)}$ and covering all but at most $5\eta n$ vertices of $W_3^{(1)} \cup W_3^{(2)} \cup W_1^{(3)} \cup W_2^{(3)} \cup W_3^{(3)} = W^{(1)} \cup W^{(2)} \cup W^{(3)}$. Thus $F$ has size at least $(1-\eta)3n - |U^{(1)} \cup U^{(2)} \cup U^{(3)}| - 5\eta n \ge (1-\eta')\frac{3}{2}n$. □

*Case 2:* Now we turn to the case $|\bigcup U^{(i)}| \ge (1-\eta)\frac{3}{2}n$. We further divide this case into two subcases, treating $U^{(3)} = \emptyset$ and $U^{(3)} \neq \emptyset$, respectively.

**Fact 3.** *If $|\bigcup U^{(i)}| \ge (1-\eta)\frac{3}{2}n$ and $U^{(3)} = \emptyset$ then $K$ is $\big((1-\eta')n, (1-\eta')\frac{3}{2}n, 2\big)$-good.*

By definition $|W^{(3)}| \ge (1-\eta)n - |U^{(3)}| = (1-\eta)n$ in this case. Therefore, using the fact that $K[U^{(1)}, W^{(3)}]$ and $K[U^{(2)}, W^{(3)}]$ are $(\eta,\chi)$-complete, we can greedily construct a maximal matching $M_A$ in $K[U^{(1)}, W^{(3)}]$ and a maximal matching $M'_B$ in $K[U^{(2)}, W^{(3)} \setminus M_A]$ such that the matching $M := M_A \cup M'_B$ covers $W^{(3)}$ (as $|U^{(1)} \cup U^{(2)}| = |\bigcup U^{(i)}| > |W^{(3)}| + \eta n$) and thus has size at least $(1-\eta)n$. Then we extend $M'_B$ to a maximal matching $M_B$ in $K[U^{(2)}, W^{(3)}]$. Observe that $M_A$ and $M_B$ cover all but at most $\eta n$ vertices of $U^{(1)}$ and $U^{(2)}$, respectively. Thus the 2-fork system $F := M_A \cup M_B$ has size at least $|U^{(1)} \cup U^{(2)}| - 2\eta n = |U^{(1)} \cup U^{(2)} \cup U^{(3)}| - 2\eta n \ge (1-\eta')\frac{3}{2}n$.
□

Now consider the subcase when $U^{(3)} \neq \emptyset$.

**Fact 4.** *If $|\bigcup U^{(i)}| \ge (1-\eta)\frac{3}{2}n$ and $U^{(3)} \neq \emptyset$ then $|U^{(2)} \cup U^{(3)}| \le (1-\eta)\frac{3}{4}n$ and we have $|W^{(3)}| \ge (1-\eta)\frac{1}{4}n$ and $|U^{(3)}| \le |W^{(2)}| - \eta n$.*

The first inequality follows from Condition 4 of (E2). Accordingly $|W^{(3)}| \ge (1-\eta)n - |U^{(3)}| \ge (1-\eta)\frac{1}{4}n$ which establishes the second inequality. For the third inequality we use that $|U^{(2)} \cup W^{(2)}| \ge (1-\eta)n$ by definition and so

$$|U^{(3)}| \le (1-\eta)\tfrac{3}{4}n - |U^{(2)}| \le (1-\eta)\tfrac{3}{4}n - (1-\eta)n + |W^{(2)}| \le |W^{(2)}| - \eta n.$$

□

**Fact 5.** *If $|\bigcup U^{(i)}| \ge (1-\eta)\frac{3}{2}n$ and $U^{(3)} \neq \emptyset$ then there is a matching $M$ of size at least $(1-\eta)n$ and colour $\chi$ covering $U^{(3)}$.*

Let $M_1$ be a maximal matching in $K[U^{(3)}, W^{(2)}]$. We conclude from Fact 4 that $M_1$ covers $U^{(3)}$. Let $M_2$ be a maximal matching in $K[W^{(3)}, U^{(1)} \cup U^{(2)}]$. As $|W^{(3)}| \le n - |U^{(3)}| \le |U^{(1)} \cup U^{(2)}| - \eta n$ the matching $M_2$ covers $W^{(3)}$. Setting $M := M_1 \cup M_2$, we obtain a matching of size $|M| = |U^{(3)}| + |W^{(3)}| \ge (1-\eta)n$ as required. □

**Fact 6.** *If* $|\bigcup U^{(i)}| \geq (1-\eta)\frac{3}{2}n$ *and* $U^{(3)} \neq \emptyset$, *then there is a* 3*-fork system of colour* $\chi$ *and of size at least* $(1-\eta')\frac{3}{2}n$.

Let $M$ be the matching from Fact 5. Clearly, we can greedily construct a 2-fork system $F'$ in the $(\eta,\chi)$-complete graph $K[W^{(3)}, (U^{(1)} \cup U^{(2)}) \setminus M]$ which either is of size $2|W^{(3)}|$ or covers all but at most $\eta n$ vertices of $|(U^{(1)} \cup U^{(2)}) \setminus M|$. Then $F := M \cup F'$ forms a 3-fork system. If the former case occurs we infer from Fact 4 that $F$ is of size at least $(1-\eta)n + 2|W^{(3)}| \geq (1-\eta)\frac{3}{2}n$. In the latter case $F$ covers all but at most $\eta n$ vertices of $\bigcup U^{(i)}$ and thus has size at least $(1-\eta)\frac{3}{2}n - \eta n \geq (1-2\eta)\frac{3}{2}n$. We conclude that $K$ is $((1-\eta')n, (1-\eta')\frac{3}{2}n, 3)$-good also in the subcase $|\bigcup U^{(i)}| \geq (1-\eta)\frac{3}{2}n$ and $U^{(3)} \neq \emptyset$. □

□

## 8. Concluding remarks

8.1. **Extensions to random graphs.** As noted earlier our proof of Theorem 1 applies to suitably chosen (sparser) subgraphs of $K_{n,n,n}$ as well. More precisely, for any fixed $p \in (0,1)$ the same method can be used to show that asymptotically almost surely $\mathcal{G}_p(n,n,n) \to \mathcal{T}_t^{\Delta}$, where $\mathcal{G}_p(n,n,n)$ is a random tripartite graph with edge probability $p$ and partition classes of size $n$, and where $t \leq (1-\mu)n/2$ and $\Delta \leq n^{\alpha}$ for a small positive $\alpha = \alpha(\mu, p)$. Indeed, standard methods can be used to show that the following holds asymptotically almost surely for $G = \mathcal{G}_p(n,n,n)$ with partition classes $V_1 \dot\cup V_2 \dot\cup V_3$ and for any $\zeta > 0$:

- $G$ has at most $4pn^2$ edges.
- $e(U,W) \geq p|U||W|/2$ for all $U \subseteq V_i$ and $W \subseteq V_j$, $i \neq j$, with $\min\{|U|,|W|\} > \zeta n$.

The first property guarantees that we obtain a graph with few edges compared to $K_n$. We claim further that these two properties imply that $G \to \mathcal{T}_t^{\Delta}$. To see this we proceed as in the proof of Theorem 1 and apply the regularity lemma on the coloured graph $G$. We then colour an edge in the reduced graph $\mathbb{G}$ by green or red, respectively, if the corresponding cluster pair is regular and has density at least $p/4$ in green or red. Using the two properties from above it is not difficult to verify that $\mathbb{G}$ is a coloured tripartite graph that is $\eta$-complete. Hence, from this point on, we can use the strategy described in the proof of Theorem 1, apply our structural lemma, Lemma 8, the assignment lemma, Lemma 14, and the embedding lemma, Lemma 13.

One may ask whether this approach can be pushed even further and consider random tripartite graphs $\mathcal{G}_p(n,n,n)$ with edge probabilities $p(n)$ that tend to zero as $n$ goes to infinity. It is likely that similar methods can be used in this case in conjunction with the regularity method for sparse graphs (see, e.g., [11]).

8.2. **Sharp version.** One can ask whether our asymptotic result in Theorem 1 can be strengthened to an exact one. In particular, it is reasonable to believe that

$$K_{n,n,n} \to \mathcal{T}_t^{\Delta}\ , \tag{18}$$

where $\Delta = n$ and $t = \lceil 3n/2 \rceil + 2$. It seems likely that imposing stricter bounds on $\Delta$ allows one to increase $t$. In particular, when $\Delta$ is a fixed constant and $n$ is large, (18) is likely to hold even for $t = cn$, where $c > 3/2$. This intuition is supported by the known bounds on the Ramsey number of somewhat balanced trees (cf. [10]).

It seems likely that some results in the directions above could be obtained using the Stability Method of Simonovits. The Stability Method is a widely used technique in the area which usually carries tedious technical difficulties.

8.3. **A question of Matoušek.** We close with an extension of Schelp's conjecture that was suggested to us by Jiří Matoušek.

**Question 28.** *Is it true that for all* $\Delta \in \mathbb{N}$ *and* $\mu > 0$ *there is an* $n_0 \in \mathbb{N}$ *such that the following holds for all* $n \geq n_0$? *If* $t \leq (1-\mu)\frac{1}{2}n$ *and* $G$ *is a graph on* $n$ *vertices with minimum degree* $\delta(G) \geq (\frac{2}{3}-\mu)n$ *then* $G \to \mathcal{T}_t^{\Delta}$.

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## References

1. M. Ajtai, J. Komlós, M. Simonovits, and E. Szemerédi, *Erdős-Sós conjecture*, In preparation.
2. N. Alon and J. H. Spencer, *The probabilistic method*, John Wiley & Sons, 2000.
3. J. Beck, *On size Ramsey number of paths, trees and circuits. II*, Mathematics of Ramsey theory, Algorithms Combin., vol. 5, Springer, 1990, pp. 34–45.
4. P. Erdős, Z. Füredi, M. Loebl, and V. T. Sós, *Discrepancy of trees*, Studia Sci. Math. Hungar. **30** (1995), no. 1-2, 47–57. MR MR1341566 (96e:05113)
5. R. J. Faudree and R. H. Schelp, *All Ramsey numbers for cycles in graphs*, Discrete Math. **8** (1974), 313–329. MR MR0345866 (49 #10596)
6. L. Gerencsér and A. Gyárfás, *On Ramsey-type problems*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **10** (1967), 167–170. MR MR0239997 (39 #1351)
7. A. Gyárfás, M. Ruszinkó, G. N. Sárközy, and E. Szemerédi, *Tripartite Ramsey numbers for paths*, J. Graph Theory **55** (2007), no. 2, 164–174.
8. András Gyárfás, Gábor N. Sárközy, and Richard H. Schelp, *Multipartite Ramsey numbers for odd cycles*, J. Graph Theory **61** (2009), no. 1, 12–21.
9. P. E. Haxell and Y. Kohayakawa, *The size-Ramsey number of trees*, Israel J. Math. **89** (1995), no. 1-3, 261–274.
10. P. E. Haxell, T. Łuczak, and P. W. Tingley, *Ramsey numbers for trees of small maximum degree*, Combinatorica **22** (2002), no. 2, 287–320, Special issue: Paul Erdős and his mathematics. MR MR1909088 (2003d:05141)
11. Y. Kohayakawa and V. Rödl, *Regular pairs in sparse random graphs. I*, Random Structures Algorithms **22** (2003), no. 4, 359–434.
12. Y. Kohayakawa, V. Rödl, M. Schacht, and E. Szemerédi, *Sparse partition universal graphs for graphs of bounded degree*, Preprint.
13. T. Łuczak, $R(C_n, C_n, C_n) \leq (4 + o(1))n$, J. Combin. Theory Ser. B **75** (1999), no. 2, 174–187.
14. F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. **30** (1930), no. 2, 264–286.
15. Vera Rosta, *On a Ramsey-type problem of J. A. Bondy and P. Erdős. I, II*, J. Combinatorial Theory Ser. B **15** (1973), 94–104; ibid. 15 (1973), 105–120.
16. R.H. Schelp, personal communication (Research Seminar, Alfréd Renyi Institute, Budapest), 2008.
17. E. Szemerédi, *Regular partitions of graphs*, Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), Colloq. Internat. CNRS, vol. 260, CNRS, Paris, 1978, pp. 399–401.
18. Y. Zhao, *Proof of the ($n/2$ - $n/2$ - $n/2$) Conjecture for large $n$*, Preprint.

Zentrum Mathematik, Technische Universität München, Boltzmannstrasse 3, D-85747 Garching bei München, Germany
Current address: Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, 05508–090 São Paulo, Brazil
*E-mail address*: julia@ime.usp.br

Department of Applied Mathematics, Faculty of mathematics and Physics, Charles University, Malostranské náměstí 25, 118 00, Prague, Czech Republic and Centre for Discrete Mathematics and its Applications (DIMAP), University of Warwick, Coventry CV4 7AL , United Kingdom
*E-mail address*: honzahladky@googlemail.com

Centre for Discrete Mathematics and its Applications (DIMAP), University of Warwick, Coventry CV4 7AL , United Kingdom
*E-mail address*: D.Piguet@warwick.ac.uk