Embedding spanning subgraphs of small bandwidth

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(joint work with Mathias Schacht & Anusch Taraz)

Sufficient degree conditions for $H \subseteq G$

Theorem (Turán 1941)

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Question: What about spanning subgraphs?

Which minimum degree $\delta(G)$ ensures that every graph G contains a given spanning graph H?

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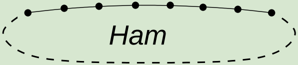
Theorem (Dirac 1952)

$\delta(G) \geq \frac{1}{2}n \quad \Rightarrow$ 

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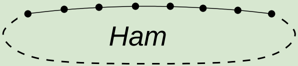
$\delta(G) \geq \frac{1}{2}n \quad \Rightarrow$ 

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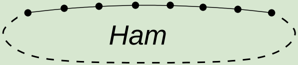
$\delta(G) \geq \frac{1}{2}n \quad \Rightarrow$ 

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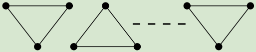
$\delta(G) \geq \frac{2}{3}n \quad \Rightarrow$  $\subseteq G$

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Conjecture of Bollobás and Komlós

Theorem

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- $\chi(H) = k$

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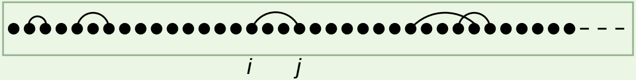
Bandwidth:

- $\mathrm{bw}(H) \leq b$ if there is a labelling of $V(H)$ by $1, \ldots, n$ s.t. for all $\{i, j\} \in E(H)$ we have $|i - j| \leq b$.



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Examples for H:

- Hamiltonian cycles (bandwidth 2)

- bounded degree planar graphs (bandwidth $O(n/\log_\Delta n)$)
- bounded degree graphs of sublinear tree width

The bandwidth condition is necessary

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- $\chi(H) = 2$, $\Delta(H) \leq \Delta$
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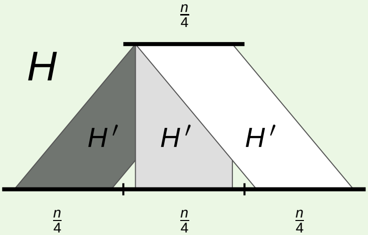
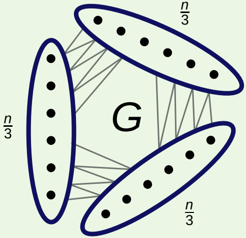
Counterexample:

- H' : random bipartite graph on $\frac{n}{4} + \frac{n}{4}$ vertices with $\Delta(H) \leq \frac{\Delta}{3}$.





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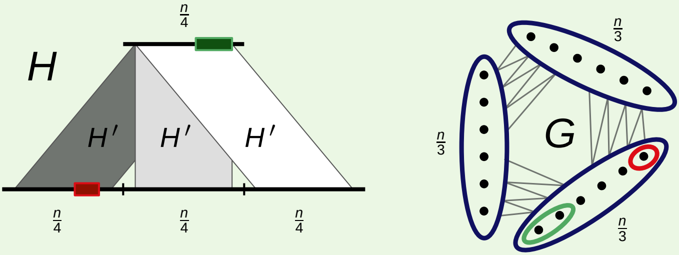

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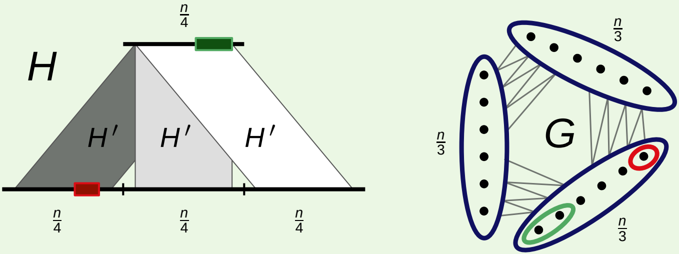

- Reason: G has big subsets A and B with $e(A, B)$ empty.

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Chritical chromatic number:

- $\sigma := \min_{\text{col. of } H} |\text{smallest colour class}|$
- $\chi_{cr}(H) := (\chi(H) - 1)\frac{|H|}{|H|-\sigma}$
- it follows that $\chi(H) - 1 \leq \chi_{cr}(H) \leq \chi(H)$

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Theorem

For all k, $\Delta \geq 1$, and $\gamma > 0$ exists n_0 and $\beta > 0$ s.t.

- H has a $(k+1)$-colouring that colours $\leq \beta n$ vertices with 0 and is $(8k\beta n, 4k\beta n)$-zero free, $\Delta(H) \leq \Delta$, $\mathrm{bw}(H) \leq \beta n$
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Assume $V(G)$ is labelled with $[n]$.

A $(k+1)$-colouring is (x, y)-zero free if for all $t \in [n]$ there is $t' \in [t, t+x]$ s.t. no vertex in $[t', t'+y]$ is coloured 0. (colours $\{0, \ldots, k\}$)

Concluding remarks

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- When can the chromatic number be replaced by the chritical chromatic number? SEE KÜHN, OSTHUS 2007+
- What about spanning graphs H of non-constant max. degree?
- Which further restrictions on G allow us to ommit the bandwidth restriction on H?