# Bandwidth, treewidth, separators, expansion, and universality

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Technische Universität München

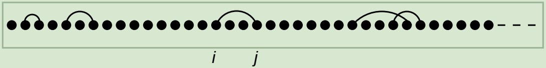
Topological & Geometric Graph Theory, 2008

*(joint work with Klaas P. Pruessmann, Anusch Taraz & Andreas Würfl)*

# The bandwidth of planar graphs of bounded degree

## Bandwidth:

Let $G$ be a graph on $n$ vertices. Then $\mathrm{bw}(G) \leq b$ if there is a labelling of $V(G)$ by $1, \ldots, n$ s.t. for all $\{i, j\} \in E(G)$ we have $|i - j| \leq b$.



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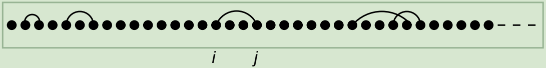


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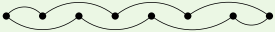
- Hamiltonian cycle: bandwidth 2



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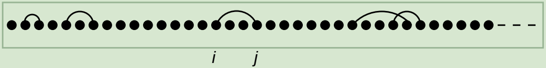


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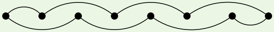

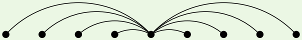

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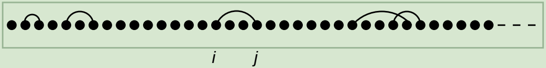


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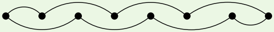

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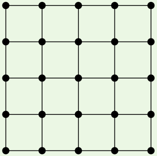

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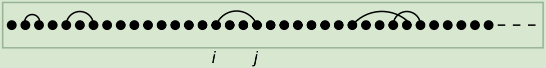


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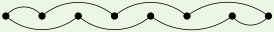

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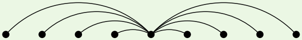

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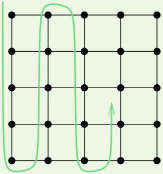

# The bandwidth of planar graphs of bounded degree

**Theorem** CHUNG'88

Let $T$ be a tree on $n$ vertices and maximal degree $\Delta(T) \leq \Delta$. Then $T$ has bandwidth at most $O(n/\log_\Delta n)$.

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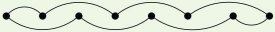

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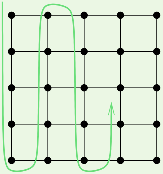

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**Question: What about planar graphs?**

Let $\Delta$ be constant. Do planar graphs $G$ on $n$ vertices with $\Delta(G) \leq \Delta$ have bandwidth $o(n)$?

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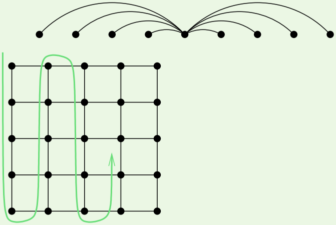

# Bandwidth, treewidth, separators, and expansion

**Theorem** (B, Pruessmann, Taraz, Wuerfl)

Let $\mathcal{G}$ be a hereditary class of graphs with max. degree $\Delta$. Then t.f.a.e.:

- All $n$-vertex $G \in \mathcal{G}$ have bandwidth $o(n)$,
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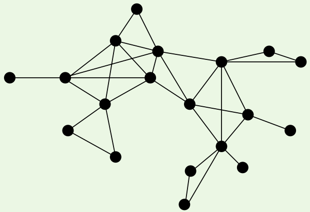

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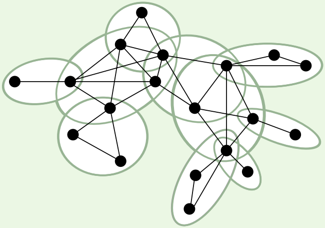

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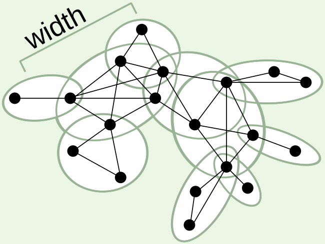


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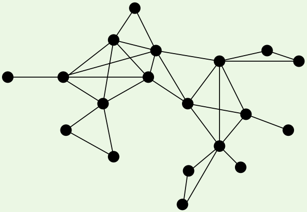

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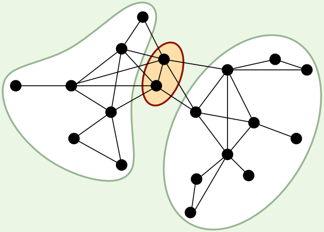

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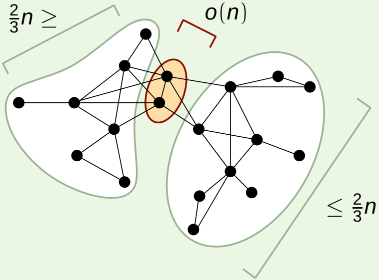

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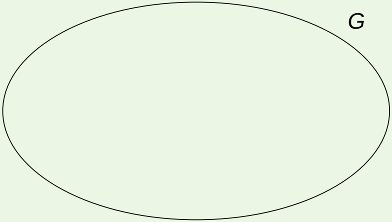
For all $G' \subseteq G$ on $n' \geq b$ vertices there is $U \subseteq V(G')$ with $|U| \leq \frac{n'}{2}$ such that $|N(U)| \leq \varepsilon |U|$.



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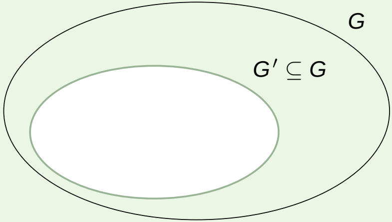

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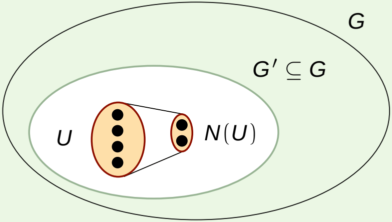

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**Theorem** (Lipton & Tarjan '79)

Planar graphs $G$ on $n$ vertices have $(o(n), 2/3)$-separators.

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**Theorem** Alon, Seymour, Thomas '90

$F$-minor free $G$ on $n$ vertices have $(|F|^{3/2}o(n), 2/3)$-separators.

# Bandwidth versus path partition width

## Path partition

A partition $V(G) = V_1 \dot\cup \ldots \dot\cup V_t$ of $G$ such that edges of $G$ only run within the $V_i$ and between $V_i$ and $V_{i+1}$.
The width of this partition is $\max_i |V_i|$, and $\mathrm{ppw}(G)$ is the minimal width of a path partition of $G$.

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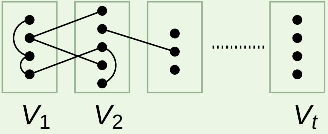

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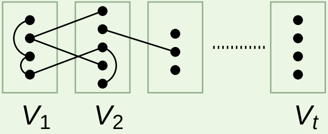


## Observation

For all $G$ we have $\text{ppw}(G) \le \text{bw}(G) \le 2\,\text{ppw}(G)$.

# The proof

- All $n$-vertex $G = (V, E) \in \mathcal{G}$ have $\big(o(n), 2/3\big)$-separators $\quad \Rightarrow$
- All $n$-vertex $G \in \mathcal{G}$ have bandwidth $o(n)$.

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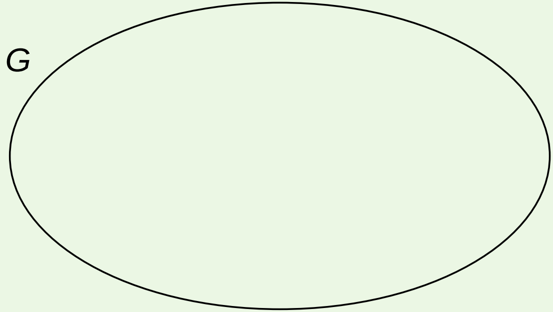

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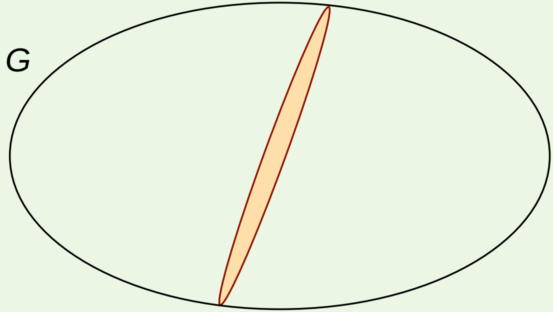

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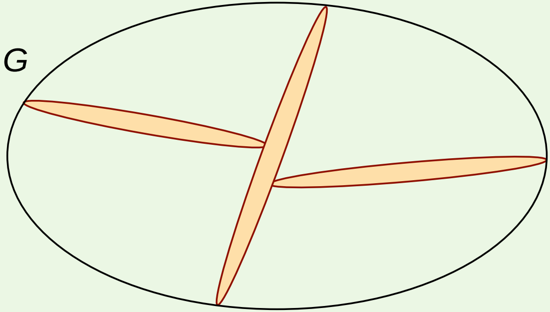

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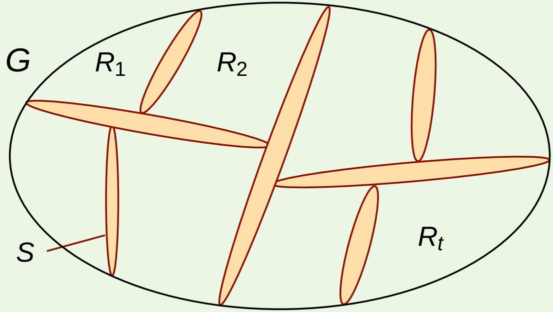

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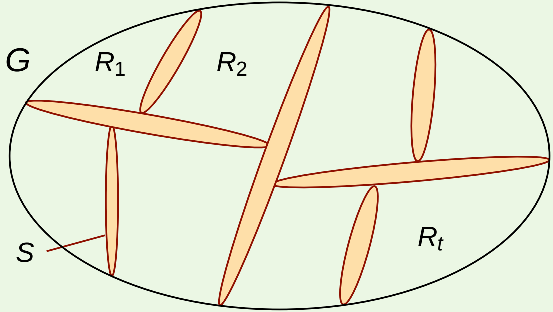


- $B_i := \{v : v \in R_j, i < j, \operatorname{dist}(v, S) = i\}$

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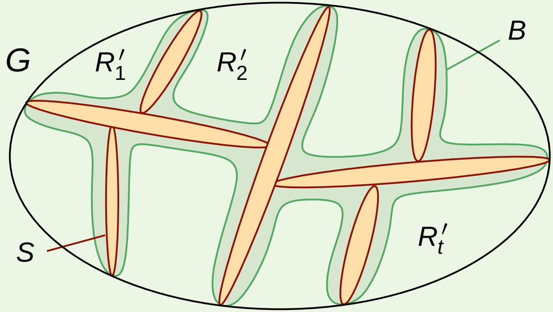


- $B_i := \{v : v \in R_j, i < j, \text{dist}(v, S) = i\}$
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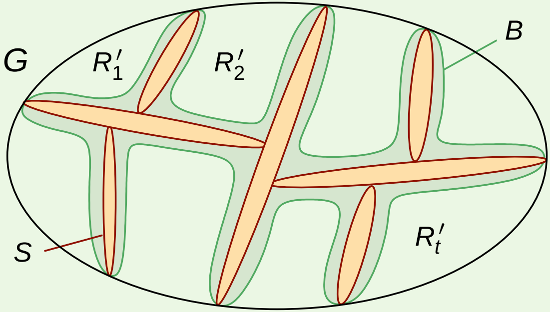


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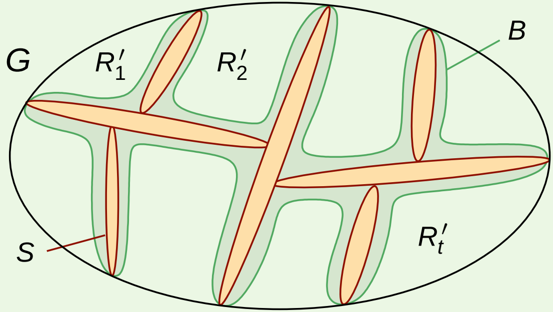


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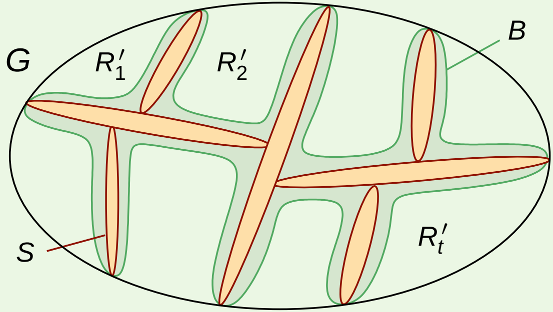


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# A consequence: universality for planar graphs

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Every $n$-vertex graph $H$ with min. degree $\delta(H) \geq (\frac{3}{4} + \gamma)n$ contains every $n$-vertex planar graph $G$ with max. degree $\Delta$.

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B, Schacht, Taraz'08

For all $k$, $\Delta \geq 1$, and $\gamma > 0$ exists $n_0$ and $\beta > 0$ s.t. for all $n \geq n_0$

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Kühn, Osthus, Taraz 2005

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# Concluding remarks

Let $\mathcal{G}$ be a hereditary class of graphs with max. degree $\Delta$. Then

- Bandwidth, treewidth, separators, expansion are sublinearly equivalent for $\mathcal{G}$.
- Each of them implies that all $H$ with $\delta(H) \geq (\frac{r-1}{r} + \gamma)n$ are universal for the class of $r$-chromatic graphs in $\mathcal{G}$.

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## Question

Let $G$ be an $r$-chromatic expander on $n$ vertices (i.e. $\forall U \subseteq V(G)$ with $|U| \leq \frac{n}{2}$ we have $|N(U)| \geq \varepsilon U$ for some constant $\varepsilon > 0$).
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Merci beaucoup.