Bounded Degree Subgraphs of Dense Graphs

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(joint work with Mathias Schacht & Anusch Taraz)

Subgraph containment problem

Question

- Given a graph H, which conditions on an n-vertex graph $G = (V, E)$ ensure $H \subseteq G$?

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H is a spanning subgraph of G.

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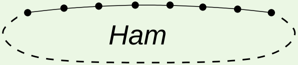
Classical example:

- $\delta(G) \geq \frac{1}{2}n \quad \Rightarrow$ 

$\subseteq G$ Dirac'52

This talk

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From Small Graphs to Spanning Graphs

- A graph H of constant size is forced in G, when

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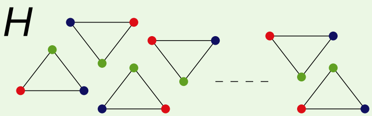
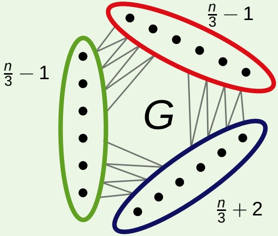


$\not\subseteq$



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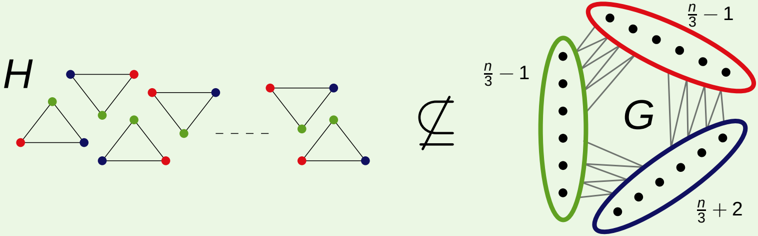

- Does $\delta(G) \geq \left(\frac{\chi(H)-1}{\chi(H)} + o(1)\right) n$ imply $H \subseteq G$?

Big Graphs

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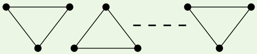
- $\delta(G) \geq \frac{r-1}{r}n \Rightarrow \lfloor \frac{n}{r} \rfloor$ disj. copies of $K_r \subseteq G$.

HAJNAL, SZEMERÉDI'69

Big Graphs

Does $\delta(G) \geq \left(\frac{\chi(H)-1}{\chi(H)} + o(1)\right)n$ imply $H \subseteq G$?

- $\delta(G) \geq \frac{2}{3}n \quad \Rightarrow$  $\subseteq G$

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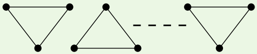
- $\delta(G) \geq \frac{2}{3}n \quad \Rightarrow$  $\subseteq G$

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- $\delta(G) \geq \frac{\chi(F)-1}{\chi(F)}n \quad \Rightarrow \quad \left\lfloor \frac{n}{|F|} \right\rfloor$ disj. copies of $F \subseteq G$

  KOMLÓS, SÁRKÖZY, AND SZEMERÉDI '01

  Alon-Yuster conjecture

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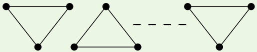
- $\delta(G) \geq \frac{2}{3}n \quad \Rightarrow$  $\subseteq G$

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- $\delta(G) \geq \frac{r-1}{r}n \Rightarrow (Ham)^r \subseteq G$.

  FAN, KIERSTEAD '95

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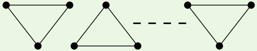
- $\delta(G) \geq \frac{2}{3}n \quad \Rightarrow$  $\subseteq G$

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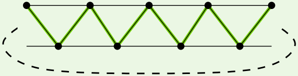
- $\delta(G) \geq \frac{2}{3}n \quad \Rightarrow$ Ham^2  $\subseteq G$

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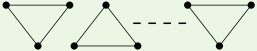
- $\delta(G) \geq \frac{2}{3}n \quad \Rightarrow$  $\subseteq G$

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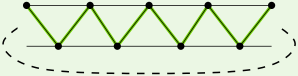
- $\delta(G) \geq \frac{2}{3}n \quad \Rightarrow \quad \mathrm{Ham}^2$  $\subseteq G$

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- $\delta(G) \geq (\frac{1}{2} + \gamma)n \Rightarrow$ every spanning tree with $\Delta(T) \leq \frac{cn}{\log n} \subseteq G$.

  Tree universality

  KOMLÓS, SÁRKÖZY, AND SZEMERÉDI'95

Generalizing Conjecture

Naı̈ve conjecture

H

G

Generalizing Conjecture

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H

- $\chi(H) = k$

G

- $\delta(G) \geq (\frac{k-1}{k} + \gamma)n$

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- $\chi(H) = k$
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Generalizing Conjecture

Naı̈ve conjecture

For all k, $\Delta \geq 1$, and $\gamma > 0$ exists n_0 s.t.

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- $\Delta(H) \leq \Delta$

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$\Longrightarrow$ G contains H.

Generalizing Conjecture

Naïve conjecture

For all $k, \Delta \geq 1$, and $\gamma > 0$ exists n_0 s.t.

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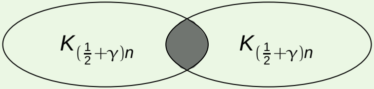
Counterexample:

- H : random bipartite graph on $\frac{n}{2} + \frac{n}{2}$ vertices with $\Delta(H) \leq \Delta$.
- G : two cliques of size $\left(\frac{1}{2} + \gamma\right) n$ sharing $2\gamma n$ vertices.



Conjecture of Bollobás and Komlós

Conjecture

For all $k, \Delta \geq 1$, and $\gamma > 0$ exists n_0 and $\beta > 0$ s.t.

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Conjecture of Bollobás and Komlós

Theorem

For all k, $\Delta \geq 1$, and $\gamma > 0$ exists n_0 and $\beta > 0$ s.t.

H

- $\chi(H) = k$
- $\Delta(H) \leq \Delta$
- $\mathrm{bw}(H) \leq \beta n$

G

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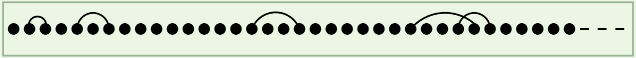
Bandwidth:

- $\mathrm{bw}(G) \leq b$ if there is a labelling of $V(G)$ by $1, \ldots, n$ s.t. for all $\{i, j\} \in E(G)$ we have $|i - j| \leq b$.



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Examples for H:

- Hamiltonian cycles (bandwidth 2)

- graphs of constant tree width (bandwidth $O(n/\log_\Delta n)$)
- bounded degree planar graphs (bandwidth $O(n/\log_\Delta n)$)

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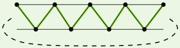
- Abbasi '98 annouced 2-chromatic case (see also Hiệp Hàn '06)
- additional γn is necessary
- Proof uses regularity lemma, blow-up lemma, and affirmative solution of Pósa's conjecture

$G \supseteq$ 

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Naïve conjecture

- $\chi(H) = 2$, $\Delta(H) \leq \Delta$
- $\delta(G) \geq (\frac{2-1}{2} + \gamma)n$ $\Longrightarrow$ G contains H

Counterexample:

- H : random bipartite graph on $\frac{n}{2} + \frac{n}{2}$ vertices with $\Delta(H) \leq \Delta$.
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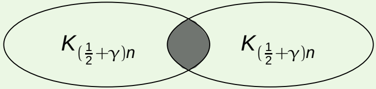

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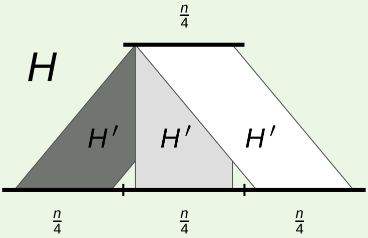

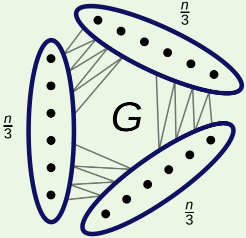

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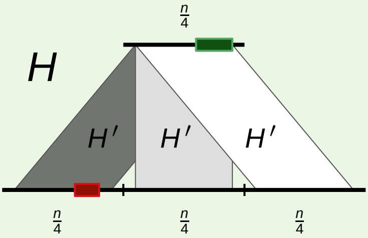

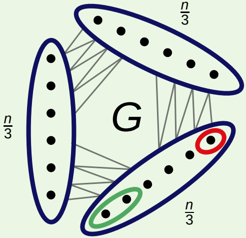

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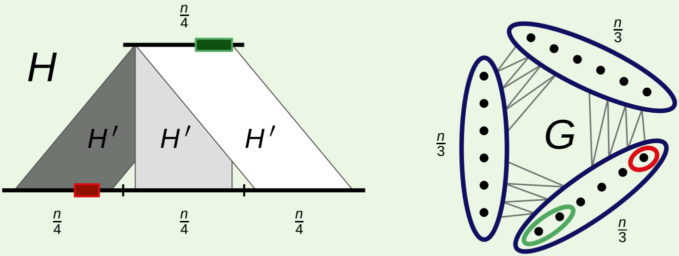

- Obstacle: G has big subsets A and B with $e(A, B)$ empty.

Quasi-random graphs

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- (n, d, λ)-graphs G (i.e. d-regular with $\lambda(G) \leq \lambda$) have

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Observation

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Because:

- H has an $\Delta + 1$-colouring with colour classes of equal size. (THM OF H.Sz.)

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- Every equi-partition of G into $\Delta+1$ parts is ε-regular.
- There is such a partition that is also super-regular.
- By blow-up lemma, G contains H. (KOMLÓS, SÁRKÖZY, SZEMERÉDI)

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What about sparser graphs?

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What about sparser graphs?

- If $\lambda(G) = o\left(\frac{d^3}{n^2 \log n}\right)$ (i.e. $d \gg n^{4/5} \log^{2/5} n$) then G has a triangle factor.

KRIVELEVICH, SUDAKOV, SZABÓ'04

Concluding remarks

- $\chi(H) = k$, $\Delta(H) \leq \Delta$, $\mathrm{bw}(H) \leq \beta n$
- $\delta(G) \geq (\frac{k-1}{k} + \gamma)n$ $\Longrightarrow$ G contains H

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- How many copies of H do we get?
  (for bipartite H see Person'07)

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Bă Łakashah!