An approximate blow-up lemma for sparse pseudorandom graphs [1]

Peter Allen [2]   Julia Böttcher [2]

Department of Mathematics, London School of Economics
Houghton Street, London WC2A 2AE, U.K.

Hiệp Hàn [3,4]   Yoshiharu Kohayakawa [5,4]

Instituto de Matemática e Estatística, Universidade de São Paulo
Rua do Matão 1010, 05508–090 São Paulo, Brazil

Yury Person [6]

Institut für Mathematik, Freie Universität Berlin
Arnimallee 3, D-14195 Berlin, Germany

Abstract

We state a sparse approximate version of the blow-up lemma, showing that regular partitions in sufficiently pseudorandom graphs behave almost like complete partite graphs for embedding graphs with maximum degree Δ. We show that (p, γ)-jumbled graphs, with $\gamma = o(p^{\max(2\Delta, \Delta+3/2)}n)$, are "sufficiently pseudorandom".

The approach extends to random graphs $G_{n,p}$ with $p \gg \left(\frac{\log n}{n}\right)^{1/\Delta}$.

Keywords: blow-up lemma, sparse regularity lemma, pseudorandom graphs, random graphs

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1 Introduction

The Regularity Method has proven to be a fundamental approach to extremal graph theory and Ramsey theory. Applications of Szemerédi's regularity lemma combined with the counting lemma and the blow-up lemma have led to many deep results: it is hard to overstate the importance of this method. One of its limitations, however, is that it is only useful for dense graphs. The development of a Regularity Method for sparse graphs has been an extensively studied subject in the last decades, but it has turned out to be difficult. While a sparse analogue of the regularity lemma was established by the senior author [7] and Rödl in the 1990s, it was only recently that major breakthroughs concerning the sparse counting lemmas have been achieved [5,4].

Here we continue this line of investigation by proving an almost spanning embedding lemma for subgraphs of random or pseudorandom graphs, hence, establishing an approximate sparse analogue of the blow-up lemma. In the dense case the blow-up lemma is a powerful tool in proving emergence of large or even spanning subgraphs in graphs with sufficiently high minimum degree. As applications of our result, we obtain (approximate) sparse analogues of many of these results and in this note we shall present two of them (see Theorem 1.1 and Theorem 1.2). We formulate our results in terms of pseudorandom graphs, as this is the somewhat harder case, and will comment on the random graph versions in due course.

In the dense case the first result is the resolution of the the Bollobás-Komlós conjecture proven by the second author and her co-authors [3] and the second result is a related result in Ramsey theory established by the first author and his co-authors [1]. If the philosophy of these two results (and of extremal combinatorics in general) is subsumed by saying that dense structures must exhibit certain patterns, then the sparse analogues state that (essentially) the same patterns also emerge in *relatively dense* substructures of *sparse ambient pseudorandom* structures.

In our case, the ambient setting is given by a sparse pseudorandom graph Γ which can informally be described as a graph with the "characteristics" of a truly random graph of the same edge density. There are many equivalent ways

to make this intuition formal, one being the following notion of jumbledness. Given $p = p(n)$ and $\gamma = \gamma(n)$, we say that an n-vertex graph Γ is (p, γ)-*jumbled* if for all disjoint $X, Y \subset V(\Gamma)$ we have

$$\left| e(X, Y) - p|X||Y| \right| \leq \gamma \sqrt{|X||Y|}.$$

It was Thomason [10] who coined the term jumbledness and initiated the systematic study of pseudorandom graphs. This topic has undoubtedly become a central subject in combinatorics. We refer to the survey [9] for further details.

The conjecture of Bollobás and Komlós addresses the emergence of spanning subgraphs in graphs with high minimum degree and states that these subgraphs can indeed be found if they have bounded maximum degree and limited expansion as captured by the following notion of bandwidth (see [2]). A graph H is said to have bandwidth at most b, denoted $bw(H) \leq b$, if there exists a labelling of the vertices by numbers $1, ..., n$, such that every edge ij of the graph satisfies $|i - j| \leq b$. Confirming the conjecture of Bollobás and Komlós it is proven in [3] that an n-vertex, k-colourable graph H with bounded maximum degree and sublinear bandwidth can be found in a graph G of the same order if minimum degree of G satisfies $\delta(G) > \left(\frac{k-1}{k} + o(1) \right) |V(G)|$. Our (approximate) sparse analogue reads as follows.

Theorem 1.1 *For all k, $\Delta \in \mathbb{N}$, δ, $\nu > 0$ there exist $c > 0$, $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and every density $p = p(n) > 0$ the following holds. Suppose H is an n-vertex, k-colourable graph with maximum degree at most Δ and bandwidth at most βn. Let Γ be a $(1 + \nu)n$-vertex, (p, γ)-jumbled graph with $\gamma = cp^{\max(2\Delta, \Delta + 3/2)} n$, and let $G \subset \Gamma$ be a subgraph with $\delta(G) > \left(\frac{k-1}{k} + \delta \right) p|V(\Gamma)|$. Then G contains a copy of H.*

The dense case corresponds to replacing Γ by the complete graph and, as mentioned, this case is resolved in [3] with the error term $\nu = 0$. In the sparse case, however, this error term cannot be removed without further assumptions, see e.g. [6, Proposition 6.3]. Further, the condition on the minimum degree of G can also be shown to be tight in the sense that $\delta > 0$ can not be completely omitted. We refer to [3] for further details.

The second application stems from Ramsey theory. For a given graph H, the Ramsey number $R(H)$ is defined to be the smallest n such that, any 2-colouring of the edges of K_n exhibits a monochromatic H.

It is easily shown that if H is connected then

$$R(H) \geq (\chi(H) - 1)(|H| - 1) + 1$$

and it is interesting to ask for which graphs this inequality is sharp up to a constant factor. In [1] it is shown that this holds for the same graphs H as considered in Theorem 1.1. Our sparse analogue of this result is as follows.

Theorem 1.2 *For any $\Delta \in \mathbb{N}$ there exist $\beta > 0$, $c > 0$ and n_0 such that for all $n > n_0$ and all densities $p = p(n) > 0$ the following holds. Let H be an n-vertex graph with maximum degree $\Delta(H) \leq \Delta$ and bandwidth bounded by βn. Further, let Γ be a (p, γ)-jumbled graph on $(2\chi(H) + 4)n$ vertices with $\gamma = cp^{\max(2\Delta, \Delta+3/2)}n$. Then however the edges of Γ are coloured with blue and red, there is a monochromatic copy of H.*

Theorems 1.1 and 1.2 follow from Theorem 1.3 stated below. In order to formally state the result we need further definitions.

Given $p = p(n)$ and a pair (A, B) of disjoint subsets of $V(G)$, we define the *p-relative density* of (A, B) by

$$d_{G,p}(A, B) = \frac{e_G(A, B)}{p|A||B|}.$$

We say that (A, B) is *(ε, p)-regular in G* if for all $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$ we have

$$\left| d_{G,p}(A', B') - d_{G,p}(A, B) \right| \leq \varepsilon.$$

If further we have $d_{G,p}(A', B') \geq \delta - \varepsilon$ for all $A' \subseteq A$ with $|A'| \geq \varepsilon|A|$ and $B' \subseteq B$ with $|B'| \geq \varepsilon|B|$, we say that (A, B) is an *(ε, δ, p)-regular pair in G*.

It follows from the sparse regularity lemma [7] that subgraphs of jumbled graphs can be partitioned into a constant number of vertex classes such that almost all pairs of these classes are regular.

Our goal is to embed a given graph H into a graph G which has almost the same order and which "agrees" with the rough structure of H in the following sense. Given a graph R on the vertex set $[r]$ and a graph H, a partition $V(H) = X_1 \dot{\cup} \cdots \dot{\cup} X_r$ of H is called an *R-partition* if for every edge xy of H with $x \in X_i$ and $y \in X_j$ the pair ij is an edge of R. Further, a partition $V(G) = V_1 \dot{\cup} \cdots \dot{\cup} V_r$ of G is called an *(ε, δ, p)-regular R-partition* if for each edge ij of R, the pair (V_i, V_j) is (ε, δ, p)-regular in G. Finally, these partitions are called *κ-balanced* if $|V_i| \leq \kappa|V_j|$ for all $i, j \in [r]$.

Our main result then reads as follows.

Theorem 1.3 *Given $\Delta \in \mathbb{N}$, $\mu, \delta > 0$ and $\kappa \geq 1$ there exists $\varepsilon > 0$ such that for all T there exist $c > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ and densities $p = p(n) > 0$.*

Let R be any graph on $r \leq T$ vertices and H a graph with $\Delta(H) \leq \Delta$ and an R-partition $V(H) = X_1 \dot\cup \cdots \dot\cup X_r$. Let Γ be a (p, γ)-jumbled graph with $\gamma = cp^{\max(2\Delta, \Delta+3/2)}n$, and let G be a subgraph of Γ with an (ε, δ, p)-regular κ-balanced R-partition $V(G) = V_1 \dot\cup \cdots \dot\cup V_r$ with $|V_i| \geq n$ and $|V_i| \geq |X_i| + \mu n$ for each $i \in [r]$.

Then there exists an embedding $\varphi : V(H) \to V(G)$ of H into G such that $\varphi(X_i) \subset V_i$ for all $i \in [r]$.

For the dense case $\Gamma = K_n$ the result was first proven in [8]. In fact the main result of that paper was to show that, with stronger conditions called "super-regularity" of the graph G, we can even take $\mu = 0$, i.e. embed H as a *spanning* subgraph of G. The sparse analogue of this stronger result will be the subject of future work. We note that our proof of Theorem 1.3 yields a randomised polynomial time algorithm for finding an embedding φ.

Theorems 1.1 and 1.2 follow from Theorem 1.3 by a standard application of the regularity method. We omit the details here in favour of a rough sketch of how Theorem 1.3 is proven. A very similar method shows that we can (with high probability) replace the (p, γ)-jumbled graph Γ with the random graph $\Gamma = G_{n,p}$ provided that $p \gg \left(\frac{\log n}{n}\right)^{1/\Delta}$, and consequently we can do the same in Theorems 1.1 and 1.2.

2 Sketch of the proof of Theorem 1.3

The proof consists of two steps. First, we embed most of H in G by using a random greedy algorithm similar to that of Komlós, Sárközy and Szemerédi [8]. This algorithm succeeds in embedding a large proportion of H, but as in the dense case it fails for a tiny *linear* portion (the "queue" in the original algorithm); we have no control over which vertices will fail. In order to be prepared for this tiny proportion, we set aside at the start a small fraction of each cluster in G which, however, will be much larger than the number of failed vertices for that cluster.

The random greedy algorithm will embed only into the main body of each cluster, but we demand that this algorithm maintains good properties for embedding both into the main body of each cluster and into the set-aside. After the random greedy algorithm finishes, we embed the failed vertices into the set-aside.

In this second step we have the additional difficulty that each vertex is restricted to a common neighbourhood of its previously embedded neighbours in H, but we gain in that the number of failed vertices in each cluster is much

smaller than the size of the set-aside into which we embed. We embed clusters one after another. In order to embed the failed vertices of a cluster in H to the corresponding set-aside in G, we construct a random matching according to the following process. We choose the vertex of the current cluster in H with fewest acceptable candidates in G for embedding, and embed it randomly into the acceptable candidates. We repeat this process until the cluster of H is embedded, and move on to the next cluster. The advantage of this process over (for example) verifying Hall's condition to obtain *some* matching is that the vertices of a pseudorandom graph *can* "conspire to misbehave", but such conspiracy can only involve a small number of vertices, which a random selection is very unlikely to pick.

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