PROPERLY COLOURED COPIES AND RAINBOW COPIES OF LARGE GRAPHS WITH SMALL MAXIMUM DEGREE

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Abstract. Let G be a graph on n vertices with maximum degree Δ. We use the Lovász local lemma to show the following two results about colourings χ of the edges of the complete graph K_n. If for each vertex v of K_n the colouring χ assigns each colour to at most $(n-2)/(22.4\Delta^2)$ edges emanating from v, then there is a copy of G in K_n which is properly edge-coloured by χ. This improves on a result of Alon, Jiang, Miller, and Pritikin [Random Struct. Algorithms 23(4), 409–433, 2003]. On the other hand, if χ assigns each colour to at most $n/(51\Delta^2)$ edges of K_n, then there is a copy of G in K_n such that each edge of G receives a different colour from χ. This proves a conjecture of Frieze and Krivelevich [Electron. J. Comb. 15(1), R59, 2008].

Our proofs rely on a framework developed by Lu and Székely [Electron. J. Comb. 14(1), R63, 2007] for applying the local lemma to random injections. In order to improve the constants in our results we use a version of the local lemma due to Bissacot, Fernández, Procacci, and Scoppola [preprint, arXiv:0910.1824].

Keywords. Ramsey theory; local lemma; rainbow colourings; proper edge colourings

1. Results

A form of the canonical Ramsey theorem [9] states that, for any graph G, if n is large enough, then every colouring of the edges of the complete graph K_n contains at least one of the following types of *G-copies*: a monochromatic copy of G; a copy of G such that each edge has a different colour; a copy of G such that, after we order the vertices of G appropriately, the colour of each edge is determined solely by the first vertex of this edge with respect to this ordering. The classical Ramsey theorem [16] is a special case of this result where the *number* of colours used in the colouring is restricted. In this paper we consider restrictions on the edge colourings of K_n which are of different nature and show that under such restrictions certain copies of *spanning* graphs G exist.

Let F be a graph. An edge colouring χ of F is called *k-bounded* if it does not use any colour more than k times and it is called *locally k-bounded* if for every vertex v of F it does not use any colour more than k times on the edges incident to v. We say that χ is *proper* if intersecting edges of F receive different colours, and that it is *rainbow* if all edges of F receive different colours. Clearly, proper colourings are exactly the locally 1-bounded colourings and the rainbow colourings are exactly the 1-bounded colourings.

Given a graph G on n vertices we will consider problems of the following type: For which numbers k does the complete graph K_n have one of the following properties?

(a) Every locally k-bounded colouring χ of K_n contains a copy of G which is properly coloured by χ.

(b) Every k-bounded colouring χ of K_n contains a copy of G which is rainbow coloured by χ.

We also say that a colouring of K_n is *G-proper* if it has the property asserted by (a), and *G-rainbow* if it has the property asserted by (b).

Our proofs apply a probabilistic method and rely on the Lovász local lemma [10].

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1.1. **Properly coloured subgraphs.** An old conjecture of Bollobás and Erdős [5] states that for $k = \lfloor n/2 \rfloor$ every locally k-bounded colouring of K_n is C_n-proper, where C_n is the cycle on n vertices (that is, a Hamilton cycle). Bollobás and Erdős showed that this is true for $k < n/69$. This was improved by by Chen and Daykin [7] to $k \leq n/17$, and by Shearer [18] to $k < n/7$. Currently, the best result is due to Alon and Gutin [2], who showed that for any $\varepsilon > 0$ there is an n_0 such that for all $n \geq n_0$ we can choose $k = \lfloor (1 - 2^{-1/2} - \varepsilon)n \rfloor$.

Generalising from C_n to a larger class of graphs, Shearer [18] proposed the following conjecture. A *cherry* in a graph G is a copy of a path of length 2 in G.

Conjecture 1 (Shearer [18])**.** *For fixed k and p, if n is sufficiently large and if G is an n-vertex graph containing at most pn cherries, then every locally k-bounded edge colouring of K_n is G-proper.*

This conjecture considers the case of constant k, but allows graphs G with maximum degree $\Omega(\sqrt{n})$. Replacing the global condition on the number of cherries in G by a local condition, Alon, Jiang, Miller and Pritikin [3] proved that, for fixed k and Δ, if n is sufficiently large and G is an n-vertex graph with maximum degree Δ, then every locally k-bounded edge colouring of K_n is G-proper. More precisely they showed the following result.

Theorem 2 (Alon, Jiang, Miller and Pritikin [3])**.** *Let G be an n-vertex graph with maximum degree Δ. If k satisfies $216(3k+2\Delta)^7(\Delta+1)^{20}k < n$ then any locally k-bounded edge colouring of K_n is G-proper.*

Notice that for graphs G whose maximum degree is bounded by a constant, the parameter k can be of order $\Omega(n^{1/8})$ in this theorem. The authors of [3] also suspected that it should be possible to replace this bound on k by a bound which is linear in n.

As a consequence of Theorem 3 below, we show that this is indeed possible (see Corollary 4). Theorem 3 uses the global condition on the total number of cherries from the conjecture of Shearer together with a local condition on the distribution of these cherries.

Theorem 3. *Let G be a graph on n vertices containing at most pn cherries such that each vertex is contained in at most q cherries. For $k \leq \frac{1}{3}\left(\frac{5}{6}\right)^5(n-2)/(q+3p)$ every locally k-bounded edge colouring of K_n is G-proper.*

Observe that a graph with maximum degree Δ contains at most $\binom{\Delta}{2}n \leq \frac{1}{2}\Delta^2 n$ cherries, and each of its vertices is contained in at most $\binom{\Delta}{2} + \Delta(\Delta-1) \leq \frac{3}{2}\Delta^2$ cherries. Therefore we immediately obtain the following corollary.

Corollary 4. *If G has n vertices and maximum degree $\Delta > 0$, then any locally $\big((n-2)/(22.4\Delta^2)\big)$-bounded edge colouring of K_n is G-proper.*

As mentioned earlier, this corollary asserts that for bounded degree graphs G we can even choose k linear in n, which clearly is best possible up to the value of the constant. Moreover, on the other extreme, this corollary implies that for each constant k there is a constant $c = c(k) > 0$ such that we can even take G with maximum degree $c\sqrt{n}$. Note that, therefore, Conjecture 1 does hold for graphs G with maximum degree $c\sqrt{n}$.

In comparison, all graphs with maximum degree Δ which contain at most pn cherries satisfy $\sqrt{2pn} \geq \big(2\binom{\Delta}{2}\big)^{1/2} \approx \Delta$. It follows that the graphs G covered by Conjecture 1 have maximum degree less than roughly $\sqrt{2pn}$, where p is a constant. Hence, in terms of the maximum degree, our result achieves the 'right' order of magnitude.

1.2. **Rainbow subgraphs.** Erdős and Stein posed the question of determining the largest k such that every k-bounded edge colouring of K_n is C_n-rainbow (see [11]). Hahn and Thomassen [14] conjectured that k can be linear. This was shown by Albert, Frieze, and Reed [1] (see also [17]), who improved on earlier sublinear bounds by Erdős, Nešetřil, and Rödl [11], by Hahn and Thomassen [14], and by Frieze and Reed [13].

Theorem 5 (Albert, Frieze and Reed [1])**.** *If $k \leq n/64$ then every k-bounded edge colouring of K_n is C_n-rainbow.*

Frieze and Krivelevich [12] extended this result and showed that there is a constant $c > 0$ such that for $k \leq cn$ every k-bounded edge colouring of K_n is C_ℓ-rainbow for all $3 \leq \ell \leq n$. Moreover, they considered almost spanning trees with bounded maximum degree and proved the following theorem.

Theorem 6 (Frieze and Krivelevich [12])**.** *For every real number $\varepsilon > 0$ and every integer $\Delta > 0$ there is a constant $c > 0$ such that the following holds for every tree T on at most $(1-\varepsilon)n$ vertices with maximum degree Δ. If $k \leq cn$ then every k-bounded edge colouring of K_n is T-rainbow.*

Frieze and Krivelevich also showed that for a special class of *spanning* trees T linearly bounded colourings of K_n are T-rainbow. They conjectured that this is true for all bounded degree trees.

In this paper we prove this conjecture. In fact, our result is more general and shows that the conclusion of the conjecture holds for all graphs with bounded maximum degree. This also improves on a result in [6], which considers almost spanning bipartite subgraphs G with sublinear bandwidth and constant maximum degree Δ. The result in [6] states that for $k \leq c(n/\log n)^{1/\Delta}$, where $c = c(\eta, \Delta) > 0$, every k-bounded colouring of K_n is G-rainbow, as long as G has at most $(1-\eta)n$ vertices.

Theorem 7. *Let G be a graph on n vertices with maximum degree $\Delta > 0$. For $k \leq n/(51\Delta^2)$ every k-bounded edge colouring of K_n is G-rainbow.*

Again, Theorem 7 can be applied to graphs with growing maximum degree; it asserts that constantly bounded edge colourings force rainbow copies of all graphs with maximum degree $c\sqrt{n}$ for some constant $c = c(k) > 0$.

2. The local lemma and random injections

Probabilistic existence proofs often try to estimate from above the probability that any of a set of bad events $\{X_i\}_{i\in[t]}$ occurs, with the goal of showing that $\mathbb{P}(\bigcup_{i\in[t]} X_i) < 1$ (here and in what follows, $[t] = \{1, \ldots, t\}$). In many classical applications the *union bound* $\mathbb{P}(\bigcup_{i\in[t]} X_i) \leq \sum_{i\in[t]} \mathbb{P}(X_i)$ is used for this purpose, which is obviously only a good estimate when the bad events are disjoint or almost disjoint. If the bad events, on the other hand, are mutually independent then $\mathbb{P}(\bigcap_{i\in[t]} \overline{X}_i) = \prod_{i\in[t]}(1-\mathbb{P}(X_i))$, which clearly implies that $\mathbb{P}(\bigcup_{i\in[t]} X_i) < 1$ iff no bad event occurs with probability 1. The local lemma is a compromise between these two extremes: it takes dependencies into account but gives a more optimistic upper bound for $\mathbb{P}(\bigcup_{i\in[t]} X_i)$ than the union bound, provided the dependencies are not too dense.

Before we can formulate this powerful tool we need some definitions. For a graph $G = (V, E)$ and a vertex $v \in V$ we denote the *neighbourhood* of v by $\Gamma_G(v) := \{u \in V : uv \in E\}$ and the *closed neighbourhood* by $\Gamma^*_G(v) := \Gamma_G(v) \cup \{v\}$. We may also omit the subscript G. We let $\Delta(G)$ denote the maximum degree of G.

Definition 8 (dependency graph)**.** *Let $\mathcal{X}$ be a set of events in some probability space. Then $D = (\mathcal{X}, E)$ is a* dependency graph *for $\mathcal{X}$ if every event $X \in \mathcal{X}$ is mutually independent of its non-adjacent events in D, i.e., for every set of events $\mathcal{Z} \subseteq \mathcal{X} \setminus \Gamma^*_D(X)$ we have that $\mathbb{P}(X | \bigcap_{Z\in\mathcal{Z}} \overline{Z}) = \mathbb{P}(X)$. We say that D is a* negative dependency graph *if $\mathbb{P}(X | \bigcap_{Z\in\mathcal{Z}} \overline{Z}) \leq \mathbb{P}(X)$ for all such X and $\mathcal{Z}$.*

The local lemma states that, if a given set $\mathcal{X}$ of bad events has a sufficiently sparse (negative) dependency graph compared to their probabilities, then with positive probability no bad event occurs.

Lemma 9 (Lovász local lemma [10, 8])**.** *Let $\mathcal{X} = \{X_i\}_{i\in[t]}$ be a set of events with negative dependency graph $D = (\mathcal{X}, E)$. If*

(i) we have

$$\mathbb{P}(X_i) < \frac{1}{e(\Delta(D)+1)} \qquad \textit{for all } i \in [t]\,,$$

or

(*ii*) *if there are numbers* $\{x_i\}_{i\in[t]}$ *in* $(0,1)$ *such that*

$$\mathbb{P}(X_i) \le x_i \prod_{X_iX_j\in E}(1-x_j) \qquad \textit{for all } i\in[t]\,, \tag{1}$$

then $\mathbb{P}(\bigcap_{i\in[t]} \overline{X}_i) > 0$.

Originally, the local lemma was proved in [10] with the stronger assumption that D is a dependency graph. Erdős and Spencer then observed in [8] that essentially the same proof applies when negative dependency graphs are used instead.

A quick calculation shows that Lemma 9(*ii*) implies Lemma 9(*i*) (it suffices to take $x_i := 1/(\Delta+1)$ for all i). Moreover, if in Lemma 9 we change from the variables $\{x_i\}_{i\in[t]}$ to the variables $\{\mu_i\}_{i\in[t]}$ defined by $x_i = \mu_i/(1+\mu_i)$, then Condition (1) becomes

$$\mathbb{P}(X_i) \le \frac{\mu_i}{\prod_{X_j\in\Gamma^*_D(X_i)}(1+\mu_j)} = \frac{\mu_i}{\sum_{R\subseteq\Gamma^*_D(X_i)}\prod_{X_j\in R}\mu_j}\,. \tag{2}$$

As usual, an empty product has value 1.

Using connections between the local lemma, independent set polynomials, and lattice gas partition functions, in [4] a version of the local lemma was recently established which often improves the constants in the upper bounds on $\mathbb{P}(X_i)$. This lemma states that the sum over all $R\subseteq\Gamma^*_D(X_i)$ on the right hand side of (2) can be replaced by a sum over all such R which form an independent set.

Lemma 10 (Bissacot, Fernández, Procacci and Scoppola [4])**.** *Let* $\mathcal{X}=\{X_i\}_{i\in[t]}$ *be a set of events with negative dependency graph* $D=(\mathcal{X},E)$*. For each* $i\in[t]$ *let* $\mathcal{R}_i$ *be the family of all subsets of* $\Gamma^*(X_i)$ *which are independent sets. If*

(*i*) *there is a positive number* μ *such that*

$$\mathbb{P}(X_i) \le \frac{\mu}{\sum_{R\in\mathcal{R}_i}\mu^{|R|}} \qquad \textit{for all } i\in[t]\,, \tag{3}$$

or

(*ii*) *if there are positive numbers* $\{\mu_i\}_{i\in[t]}$ *such that*

$$\mathbb{P}(X_i) \le \frac{\mu_i}{\sum_{R\in\mathcal{R}_i}\prod_{X_j\in R}\mu_j} \qquad \textit{for all } i\in[t]\,, \tag{4}$$

then $\mathbb{P}(\bigcap_{i\in[t]} \overline{X}_i) > 0$.

A *clique* in a graph G is the vertex set of a complete subgraph of G. Note that Lemma 10 provides better bounds than Lemma 9 when the neighbourhoods of events in the independence graph are 'dense', for instance, when they can be written as the union of few cliques. Hence, in an application of this lemma we shall aim at decomposing the closed neighbourhood of each vertex into cliques. We will rely on the following straightforward observation.

Remark. Assume we apply Lemma 10 to a negative dependency graph $D=(\mathcal{X},E)$ which satisfies the following condition for some integers ℓ and $\{q_j\}_{j\in[\ell]}$. For each vertex X_i we can write $\Gamma^*_D(X_i) = \bigcup_{j\in[\ell]} Q_{i,j}$ where $|Q_{i,j}|\le q_j$ and $Q_{i,j}$ is a clique in D. (Note that we do not require $Q_{i,j}\cap Q_{i,j'}=\emptyset$.) Then, if we replace Condition (3) by

$$\mathbb{P}(X_i) \le \frac{\mu}{\prod_{j\in[\ell]}(1+\mu q_j)} \qquad \text{for all } i\in[t]\,, \tag{3'}$$

then the conclusion of Lemma 10 remains valid.

Alternatively, assume that we are in the following (somewhat special) situation. There are two different types of vertices in D, that is, $\mathcal{X}=\mathcal{X}_1 \,\dot\cup\, \mathcal{X}_2$. With each type $s\in[2]$ of vertices we associate an integer ℓ_s. Moreover, suppose that for each $s\in[2]$ and each $i\in[t]$ with $X_i\in\mathcal{X}_s$ we can write $\Gamma^*_D(X_i) = \bigcup_{j\in[\ell_s]}\big(Q^{(1)}_{i,j} \,\dot\cup\, Q^{(2)}_{i,j}\big)$ such that for $s'\in[2]$ we have $Q^{(s')}_{i,j}\subseteq\mathcal{X}_{s'}$ and $|Q^{(s')}_{i,j}|\le q_{j,s'}$ for some $\{q_{j,s'}\}_{j\in[\ell_{s'}]}$. Assume in addition that $D\big[Q^{(1)}_{i,j}\cup Q^{(2)}_{i,j}\big]$ is a clique. Then we can replace Condition (4) by requiring that there are positive numbers μ_1 and μ_2 such that for each $s \in [2]$ and each $X_i \in \mathcal{X}_s$ we have

$$\mathbb{P}(X_i) \leq \frac{\mu_s}{\prod_{j\in[\ell_s]}\big(1+\mu_1 q_{j,1}+\mu_2 q_{j,2}\big)}\,. \tag{4'}$$

Next, we shall discuss how we construct negative dependency graphs in our applications of the local lemma. For this purpose we use a framework developed by Lu and Székely [15] suited for the case when the random experiment under consideration chooses an injection between two sets uniformly at random.

Let A and B be two finite sets. We denote by $\mathcal{I}(A,B)$ the set of injections from A to B. In the following we consider the probability space generated by drawing injections uniformly at random from $\mathcal{I}(A,B)$. We shall use the following distinguished type of events.

Definition 11 (canonical events, conflicts)**.** *A canonical event $X = X(A',B',\pi)$ for $\mathcal{I}(A,B)$ is determined by two sets $A' \subseteq A$, $B' \subseteq B$ and a bijection $\pi\colon A' \to B'$, and is defined as $X := \{\sigma \in \mathcal{I}(A,B)\colon \sigma|_{A'} = \pi|_{A'}\}$. We say that two canonical events $X(A_1',B_1',\pi_1)$ and $X(A_2',B_2',\pi_2)$* conflict *if there is no injection in $\mathcal{I}(A,B)$ that extends both π_1 and π_2.*

Lu and Székely showed that for canonical events there is a simple way of constructing a negative dependency graph: we only have to insert edges between conflicting events.

Lemma 12 (Lu and Székely [15])**.** *Let $\mathcal{X} = \{X_i\}_{i\in[t]}$ be a set of canonical events for $\mathcal{I}(A,B)$. Then the graph $D = (\mathcal{X},E)$ with*

$$E := \{X_iX_j\colon X_i \text{ and } X_j \text{ conflict}\}$$

is a negative dependency graph.

We also call the graph D constructed in this lemma the *canonical dependency graph* for $\mathcal{X}$. For a set $\mathcal{X}$ of canonical events in $\mathcal{I}(A,B)$, let the *intersection graph* for $\mathcal{X}$ be the graph $D' := (\mathcal{X},E)$ with precisely those edges X_1X_2 between events $X_1 = X_1(A_1,B_1,\pi_1)$ and $X_2 = X_2(A_2,B_2,\pi_2)$ with $(A_1\cap A_2)\cup(B_1\cap B_2) \neq \emptyset$. Observe that D' is a supergraph of the canonical dependency graph for $\mathcal{X}$ and hence D' is a negative dependency graph.

3. Proofs

For the proofs of Theorems 3 and 7 we combine the local lemma (Lemma 10) with Lemma 12. For this purpose, given a graph G on n vertices, we let $\mathcal{J} = \mathcal{I}\big(V(G),V(K_n)\big)$ be the set of embeddings of G into K_n (viewed as injections). Without loss of generality we assume $V(G) = V(K_n) = [n]$.

We will use certain canonical events $X(\vec{e},\vec{f};\vec{a},\vec{b})$ in $\mathcal{J}$, which will be defined from pairs of edges $\{e,f\}$ in G and pairs of edges $\{a,b\}$ in K_n. For the precise definition of $X(\vec{e},\vec{f};\vec{a},\vec{b})$, we introduce a way of orienting and labelling pairs of edges in G first.

For an edge $e = \{e_1,e_2\}$ of G we denote an ordered pair formed by its endvertices by $\vec{e}$ (so $\vec{e}$ is just an orientation of e). Given a pair of distinct edges of G, the canonical way we shall refer to it is as the ordered pair $(\vec{e},\vec{f})$, where $\vec{e} = (e_1,e_2)$, $\vec{f} = (f_1,f_2)$, $e_1 < e_2$, $f_1 < f_2$, and $\vec{e}$ is lexicographically smaller than $\vec{f}$, that is, either $e_1 < f_1$, or $e_1 = f_1$ and $e_2 < f_2$ (for simplicity, we say that $\vec{e}$ and $\vec{f}$ are *lexicographically ordered*). Note that, given a pair of distinct edges of G, the pair $(\vec{e},\vec{f})$ is uniquely defined. Let $\vec{a} = (a_1,a_2)$ and $\vec{b} = (b_1,b_2)$ be distinct pairs in $V(K_n)$.

We denote by $X(\vec{e},\vec{f};\vec{a},\vec{b})$ the event containing all embeddings of G into K_n which map $\vec{e}$ to $\vec{a}$, and $\vec{f}$ to $\vec{b}$, that is, $X(\vec{e},\vec{f};\vec{a},\vec{b})$ is the canonical event $X(e\cup f, a\cup b,\pi)$ with $\pi(e_1) = a_1$, $\pi(e_2) = a_2$, $\pi(f_1) = b_1$, and $\pi(f_2) = b_2$. It goes without saying that we only consider events $X(\vec{e},\vec{f};\vec{a},\vec{b})$ for which such an injection π exists, and $(\vec{e},\vec{f})$ is the canonical labelling of a pair of distinct edges of G. We say that $X(\vec{e},\vec{f};\vec{a},\vec{b})$ is of *disjoint type* if e and f (and hence also a and b) are disjoint, and otherwise we say that $X(\vec{e},\vec{f};\vec{a},\vec{b})$ is of *intersecting type*. We have

$$\mathbb{P}\big(X(\vec{e},\vec{f};\vec{a},\vec{b})\big) = \begin{cases} 1\,/\,(n)_3 & \text{if } X(\vec{e},\vec{f};\vec{a},\vec{b}) \text{ is of intersecting type,} \\ 1\,/\,(n)_4 & \text{otherwise,} \end{cases} \tag{5}$$

where $(n)_k = n!/(n-k)!$ is the falling factorial.

Observe that two canonical events $X(\vec{e},\vec{f};\vec{a},\vec{b})$ and $X(\vec{e}',\vec{f}';\vec{a}',\vec{b}')$ containing bijections π and π' conflict only if one of the following cases occurs. Either there is a vertex in $(e\cup f)\cap(e'\cup f')$ for which π and π' are inconsistent, or there is a vertex in $(a\cup b)\cap(a'\cup b')$ for which π and π' are inconsistent. In the first case $e\cup f$ and $e'\cup f'$ share some element and in the second case $a\cup b$ and $a'\cup b'$ share some element. This motivates the following definitions. We say that the events $X(\vec{e},\vec{f};\vec{a},\vec{b})$ and $X(\vec{e}',\vec{f}';\vec{a}',\vec{b}')$ are *G-intersecting* if $(e\cup f)\cap(e'\cup f')\neq\emptyset$ and that they are *K_n-intersecting* if $(a\cup b)\cap(a'\cup b')\neq\emptyset$.

3.1. **Properly coloured subgraphs.** In this section we prove Theorem 3. For this purpose we take a random embedding of G into K_n. A 'bad event' occurs if two adjacent edges e and f of G are mapped onto two (adjacent) edges of K_n with the same colour. We will use the local lemma to show that the probability that none of those bad events occurs is positive.

Proof of Theorem 3. Given G and a locally k-bounded edge colouring χ of K_n, we consider a random embedding $\sigma\colon V(G)\to V(K_n)$ of G into K_n (that is, a random injection from the set $\mathcal{J}$ defined above) and show that with positive probability σ has the desired property.

Let the canonical events $X(\vec{e},\vec{f};\vec{a},\vec{b})$ with $e,f\in E(G)$ and $a,b\in E(K_n)$ of $\mathcal{J}$ be as defined above. Let the set of bad events $\mathcal{X}$ be the set of events $X(\vec{e},\vec{f};\vec{a},\vec{b})$ of intersecting type such that $\chi(a)=\chi(b)$ and let D' be the intersection graph for $\mathcal{X}$. Observe that, if no bad event occurs, then σ provides a properly coloured copy of G. Our goal is to apply version (i) of Lemma 10 to show that $\mathbb{P}(\bigcap_{X\in\mathcal{X}}\overline{X})>0$. For this purpose it suffices to check Condition (3'). Therefore we will next analyse the closed neighbourhood $\Gamma^*_{D'}(X)$ of events X in D'.

Let $X=X(\vec{e},\vec{f};\vec{a},\vec{b})\in\mathcal{X}$ be fixed. Let $S_G(X)$ be the set of events in $\mathcal{X}$ that are G-intersecting with X and $S_{K_n}(X)$ be the set of events in $\mathcal{X}$ that are K_n-intersecting with X. Clearly we have $X\in S_G(X)\cap S_{K_n}(X)$. Thus $\Gamma^*_{D'}(X)=S_G(X)\cup S_{K_n}(X)$.

Observe that e and f form a cherry in G. Call its three vertices x, y and z. Assume that $X(\vec{e}',\vec{f}';\vec{a}',\vec{b}')\in\mathcal{X}$ is G-intersecting with X. Then the cherry (spanned by) e' and f' contains some vertex $x'\in\{x,y,z\}$. Recall that x' is contained in at most q cherries of G. Hence, given x' there are at most q choices for e',f'. Once $\{e',f'\}$ has been fixed, the vertices of $\vec{a}'$ can be chosen in $(n)_2$ ways, and for fixed $\vec{a}'$ there are at most k choices for the third vertex in $a'\cup b'$ since $\chi(a')=\chi(b')$ and χ is k-bounded. It follows that $S_G(X)$ is the union of three (overlapping) cliques in D' of order at most $q\cdot(n)_2k$ each.

Similarly, a and b form a cherry. Call its vertices u, v and w. If $X(\vec{e}',\vec{f}';\vec{a}',\vec{b}')\in\mathcal{X}$ is K_n-intersecting with $X(\vec{e},\vec{f};\vec{a},\vec{b})$, then the cherry a',b' contains some vertex $u'\in\{u,v,w\}$. The number of monochromatic cherries in K_n containing u' as an end point is at most $(n-1)k$ and the number of those containing u' as the middle point is at most $\frac{1}{2}(n-1)k$. Moreover, there are two injections from the vertices of a cherry in G to such a monochromatic cherry and there are at most pn cherries in G by hypothesis. We infer that $S_{K_n}(X)$ can be written as the union of three (overlapping) cliques in D' of order at most $2\big((n-1)k+\frac{1}{2}(n-1)k\big)\cdot pn = 3p\cdot(n)_2k$ each.

In conclusion $\Gamma^*_{D'}\big(X(\vec{e},\vec{f};\vec{a},\vec{b})\big)$ is the union of three cliques of order at most $q(n)_2k$ and three cliques of order at most $3p(n)_2k$. For checking Condition (3'), it is enough to show that there is a positive number μ such that

$$\mathbb{P}\big(X(\vec{e},\vec{f};\vec{a},\vec{b})\big) \overset{(5)}{=} \frac{1}{(n)_3} \le \frac{\mu}{\big(1+q(n)_2k\mu\big)^3\big(1+3p(n)_2k\mu\big)^3}\,.$$

We claim that for $\mu := \left(\frac{6}{5}\right)^6/(n)_3$ this inequality is satisfied. Indeed, since $k \le \frac{1}{3}\left(\frac{5}{6}\right)^5(n-2)/(q+3p)$ we have $k\mu \le \frac{2}{5}/\big((n)_2(q+3p)\big)$. This implies

$$\big(1+q(n)_2k\mu\big)^3\big(1+3p(n)_2k\mu\big)^3 \le \left(\left(1+\frac{2}{5}\cdot\frac{q}{q+3p}\right)\left(1+\frac{2}{5}\cdot\frac{3p}{q+3p}\right)\right)^3$$
$$= \left(1+\frac{2}{5}+\frac{4}{25}\cdot\frac{3qp}{(q+3p)^2}\right)^3 \le \left(1+\frac{2}{5}+\frac{4}{25}\cdot\frac{1}{4}\right)^3 = \left(\frac{6}{5}\right)^6,$$

where in the last inequality we used that $st/(s+t)^2 \le 1/4$ for all reals s and t with $s+t \ne 0$. Therefore we obtain

$$\frac{\mu}{\big(1+q(n)_2k\mu\big)^3\big(1+3p(n)_2k\mu\big)^3} \ge \frac{\left(\frac{6}{5}\right)^6}{\left(\frac{6}{5}\right)^6\cdot(n)_3} = \mathbb{P}\big(X(\vec{e},\vec{f};\vec{a},\vec{b})\big)\,,$$

as required. □

3.2. **Rainbow subgraphs.** In this section we prove Theorem 7. Its proof follows the strategy of the proof for Theorem 3. The major difference is that now we have to consider both, canonical events of intersecting and of disjoint type. This is why we will consider bad events with very different probabilities and therefore we will apply version (ii) of the local lemma in the form of Lemma 10.

Proof of Theorem 7. Observe that for $n < 77$ the condition of Theorem 7 implies $k \le 1$ and therefore in this case the theorem holds trivially. Hence we assume $n \ge 77$.

Given G and a k-bounded edge colouring χ of K_n, let $\mathcal{J}$ and its canonical events $X(\vec{e},\vec{f};\vec{a},\vec{b})$ be as defined at the beginning of Section 3, and let $\sigma\colon V(G) \to V(K_n)$ be a random injection from $\mathcal{J}$. Let m denote the number of edges in G.

Let the set of bad events $\mathcal{X}$ be the set of canonical events $X(\vec{e},\vec{f};\vec{a},\vec{b})$ (of intersecting and disjoint type) such that $\chi(a) = \chi(b)$ and let D' be the intersection graph for $\mathcal{X}$. Again, if no bad event occurs, then σ gives a rainbow copy of G. Hence, if we can apply version (ii) of Lemma 10 to show that $\mathbb{P}(\bigcap_{B\in\mathcal{X}}\overline{B}) > 0$, we are done.

For Lemma 10(ii) it suffices to define for every bad event $B \in \mathcal{X}$ a positive number μ_B such that (4') is satisfied . In order to find out how we should set these numbers we will investigate the neighbourhood of B in D', write it as a union of cliques, and derive bounds on their sizes.

Let $X = X(\vec{e},\vec{f};\vec{a},\vec{b})$ be an arbitrary bad event. If X is of intersecting type then we have $e \cup f = \{x_1,x_2,x_3\}$ and $a \cup b = \{u_1,u_2,u_3\}$; otherwise we have $e \cup f = \{x_1,x_2,x_3,x_4\}$ and $a \cup b = \{u_1,u_2,u_3,u_4\}$. In either case, for $\ell \in [4]$, let $Q_G^{(\ell)}$ denote the set of bad events which are G-intersecting with X and contain x_ℓ. Analogously, let $Q_{K_n}^{(\ell)}$ denote the set of bad events which are K_n-intersecting with X and contain u_ℓ. Hence we have

$$\Gamma^*_{D'}(X) = \begin{cases} \bigcup_{\ell\in[3]}\left(Q_G^{(\ell)} \cup Q_{K_n}^{(\ell)}\right) & \text{if } X \text{ is of intersecting type}\,,\\ \bigcup_{\ell\in[4]}\left(Q_G^{(\ell)} \cup Q_{K_n}^{(\ell)}\right) & \text{if } X \text{ is of disjoint type}\,. \end{cases}$$

Moreover, we write $Q^{(\ell)}_{\mathrm{int}(G)}$ for the set of those events in $Q_G^{(\ell)}$ that are of intersecting type and $Q^{(\ell)}_{\mathrm{dis}(G)}$ for the set of those of disjoint type; therefore $Q_G^{(\ell)} = Q^{(\ell)}_{\mathrm{int}(G)} \,\dot\cup\, Q^{(\ell)}_{\mathrm{dis}(G)}$. Analogously $Q_{K_n}^{(\ell)} = Q^{(\ell)}_{\mathrm{int}(K_n)} \,\dot\cup\, Q^{(\ell)}_{\mathrm{dis}(K_n)}$.

Claim 13. For each ℓ the sets $\{X\} \cup Q_G^{(\ell)}$ and $Q_{K_n}^{(\ell)}$ are cliques in D' and we have

$$\begin{aligned} \big|Q^{(\ell)}_{\mathrm{int}(G)}\big| &\le \gamma_{\mathrm{int}} := \tfrac{3}{2}\Delta^2 n^2 k\,, & \big|Q^{(\ell)}_{\mathrm{dis}(G)}\big| &\le \gamma_{\mathrm{dis}} := \Delta^2 n^3 k\,,\\ \big|Q^{(\ell)}_{\mathrm{int}(K_n)}\big| &\le \kappa_{\mathrm{int}} := \Delta^2 n^2 k\,, & \big|Q^{(\ell)}_{\mathrm{dis}(K_n)}\big| &\le \kappa_{\mathrm{dis}} := \Delta^2 n^3 k\,. \end{aligned} \tag{6}$$

Proof. Clearly $Q_G^{(\ell)}$ is a clique in D' since each of its events contains x_ℓ, and $Q_{K_n}^{(\ell)}$ is a clique since each of its events contains u_ℓ.

The bound on the number of events $X' = X(\vec{e}\,', \vec{f}'; \vec{a}', \vec{b}')$ in $Q^{(\ell)}_{\mathrm{int}(G)}$ follows from the following facts. We have $x_\ell \in e' \cup f'$ and there are at most $\frac{3}{2}\Delta^2$ cherries in G containing x_ℓ (recall that $\vec{e}\,'$ and $\vec{f}'$ are lexicographically ordered). The two edges forming the cherry a', b' can be chosen in $\binom{n}{2}k$ ways, and there are 2 isomorphisms from the cherry e', f' to this cherry.

We claim that there are at most $\Delta m \cdot (n)_2(2k)$ events $X' = X(\vec{e}\,', \vec{f}'; \vec{a}', \vec{b}')$ in $Q^{(\ell)}_{\mathrm{dis}(G)}$. Indeed, note first that there are at most Δ ways to form an edge in G containing x_ℓ and for the second edge of e', f' there are at most m possibilities. Moreover, for $\vec{a}'$ we have $(n)_2$ choices. Since $\chi(a') = \chi(b')$ this leaves at most k choices for b' and there are 2 ways to obtain $\vec{b}'$ from b'. The bound claimed in (6) then follows from $m \le \frac{1}{2}\Delta n$.

For the events $X' = X(\vec{e}\,', \vec{f}'; \vec{a}', \vec{b}')$ in $Q^{(\ell)}_{\mathrm{int}(K_n)}$ observe that there are at most $\binom{\Delta}{2}n \le \frac{1}{2}\Delta^2 n$ cherries in G. It follows that we can choose $\vec{e}\,', \vec{f}'$ in at most $\frac{1}{2}\Delta^2 n$ ways. Moreover, $u_\ell \in a' \cup b'$. For finding a monochromatic cherry in K_n which contains u_ℓ we have at most nk choices, and there are two isomorphisms from the cherry e', f' to this cherry. Hence there are at most $\Delta^2 n^2 k$ such events.

Finally, we check that the number of events $X' = X(\vec{e}\,', \vec{f}'; \vec{a}', \vec{b}')$ contained in $Q^{(\ell)}_{\mathrm{dis}(K_n)}$ is at most $\frac{1}{8}\Delta^2 n^2 \cdot 4 \cdot n(2k)$. Indeed, there are at most $\binom{m}{2} \le \frac{1}{8}\Delta^2 n^2$ possibilities to choose $\vec{e}\,', \vec{f}'$, and u_ℓ can be each of the 4 vertices in $a' \cup b'$. Moreover, for the second vertex in the edge of X' which contains x, say a', there are then at most n possibilities, and for $\vec{b}'$ we have again $2k$ choices. □

Now we define for each $B \in \mathcal{X}$ the number

$$\mu_B := \begin{cases} \mu_{\mathrm{int}} := \left(\frac{7}{5n}\right)^3 & \text{if } B \text{ is of intersecting type,} \\ \mu_{\mathrm{dis}} := \left(\frac{7}{5n}\right)^4 & \text{if } B \text{ is of disjoint type,} \end{cases} \tag{7}$$

and claim that with these numbers Condition (4') is satisfied. Indeed, let $B_{\mathrm{int}} \in \mathcal{X}$ be an arbitrary intersecting type event and $B_{\mathrm{dis}} \in \mathcal{X}$ an arbitrary disjoint type event. Claim 13 and Condition (4') imply together with (7) that it is sufficient to check the two conditions

$$\begin{aligned} \mathbb{P}(B_{\mathrm{int}}) &\le \frac{\mu_{\mathrm{int}}}{(1+\gamma_{\mathrm{int}}\mu_{\mathrm{int}}+\gamma_{\mathrm{dis}}\mu_{\mathrm{dis}})^3(1+\kappa_{\mathrm{int}}\mu_{\mathrm{int}}+\kappa_{\mathrm{dis}}\mu_{\mathrm{dis}})^3} =: p_{\mathrm{int}}\,, \\ \mathbb{P}(B_{\mathrm{dis}}) &\le \frac{\mu_{\mathrm{dis}}}{(1+\gamma_{\mathrm{int}}\mu_{\mathrm{int}}+\gamma_{\mathrm{dis}}\mu_{\mathrm{dis}})^4(1+\kappa_{\mathrm{int}}\mu_{\mathrm{int}}+\kappa_{\mathrm{dis}}\mu_{\mathrm{dis}})^4} =: p_{\mathrm{dis}}\,. \end{aligned} \tag{4''}$$

Since $k \le n/(51\Delta^2)$ we have

$$\begin{aligned} &(1+\gamma_{\mathrm{int}}\mu_{\mathrm{int}}+\gamma_{\mathrm{dis}}\mu_{\mathrm{dis}})(1+\kappa_{\mathrm{int}}\mu_{\mathrm{int}}+\kappa_{\mathrm{dis}}\mu_{\mathrm{dis}}) \\ &\quad\overset{(6)}{\le} (1+\tfrac{3}{2}\Delta^2n^2k\mu_{\mathrm{int}}+\Delta^2n^3k\mu_{\mathrm{dis}})(1+\Delta^2n^2k\mu_{\mathrm{int}}+\Delta^2n^3k\mu_{\mathrm{dis}}) \\ &\quad\overset{(7)}{\le} \Big(1+\tfrac{3}{2}\cdot\tfrac{1}{51}\big(\tfrac{7}{5}\big)^3+\tfrac{1}{51}\big(\tfrac{7}{5}\big)^4\Big)\Big(1+\tfrac{1}{51}\big(\tfrac{7}{5}\big)^3+\tfrac{1}{51}\big(\tfrac{7}{5}\big)^4\Big) \le \tfrac{50}{51}\cdot\tfrac{7}{5}\,, \end{aligned}$$

where the last inequality can be easily verified numerically. This implies the second part of (4'') because

$$p_{\mathrm{dis}} \ge \frac{\mu_{\mathrm{dis}}}{\left(\frac{50}{51}\cdot\frac{7}{5}\right)^4} \overset{(7)}{=} \left(\frac{51}{50n}\right)^4 \ge \frac{1}{(n)_4} \overset{(5)}{=} \mathbb{P}(B_{\mathrm{dis}})\,,$$

where we used $n \ge 77$ in the last inequality. The first part of (4'') follows analogously. Hence we can apply Lemma 10(ii) to conclude that $\mathbb{P}(\bigcap_{B\in\mathcal{X}} \overline{B}) > 0$, which finishes our proof. □

4. Concluding remarks

In this paper we showed how the framework developed by Lu and Székely [15] for applying the local lemma to random injections can be used for obtaining results about copies of spanning graphs in bounded or locally bounded edge colourings of K_n. In our proofs we used the version of the local lemma given in Lemma 10, which enabled us to obtain better constants than Lemma 9 would have yielded. We close with two remarks:

(*i*) If $n \geq 100$ then in Theorem 7 the constant 51 can be improved to 42, using calculations analogous to those in the proof of Theorem 7.

(*ii*) Albert, Frieze, and Reed [1] used the local lemma in the form of Lemma 9 to obtain Theorem 5. Using Lemma 10 instead, one can improve the constant 64 in Theorem 5 to 38, if n is sufficiently large.

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