

Ellsberg's Paradox and the Value of Chances

Richard Bradley

London School of Economics and Political Science

August 11, 2014

Abstract

What value should we put on chances? This paper examines the hypothesis that, contra the widely accepted theory of von Neumann and Morgenstern, chances can have diminishing marginal value. The hypothesis is defended by showing that it can be used to explain both the typical pattern of preferences observed in the Ellsberg paradox and the intuition that lotteries are the best way to distribute indivisible good.

1 Introduction

Chances matter. My chances of getting cancer influence my decisions about what to eat, how much exercise to do and whether to smoke. The chances of rainfall affect where I choose to holiday. My chances of 'winning big' sway my decision for or against entering the National Lottery. But why do chances matter? There is a commonsense answer to this question. Chances matter to us because the things that they are chances-of matter to us. I care about the chances of cancer because I want to avoid cancer, about the chances of rainfall because I prefer sunshine, and so on.

The relation of value dependence between a chance and what it is a chance of can take more than one form. Firstly, chances can matter because they are indicators of states of affairs that we care about. Secondly, they can matter instrumentally because by affecting our chances we make it more or less likely that some state of affairs that we care about will be realised. And finally they might matter symbolically, such as when being given a chance to compete in a tournament matters because it symbolises my recognition as a worthy competitor.¹

How much do chances matter? Or to put the question slightly differently, how much value should we attach to having a chance of a certain magnitude of obtaining some good? There is a well-established answer to this question too, in its contemporary form deriving from the work of von Neumann and Morgenstern [20] on decision making under risk. What von Neumann and Morgenstern's theory (vN-M for short) tells us, in essence, is that *the value of obtaining a chance of a good is a positive linear function of the chance*. Suppose, for instance, that

¹A detailed view on this value dependence would require an account of what chances are. But for our purposes all that must be ruled out is a hardcore subjectivist interpretation of them, because to make sense of the idea that they matter, they must be something that we can be right or wrong about. No so long ago this would have been enough for me to lose half of my readership, but hopefully no longer!

we are offered the option of a chance x of some good G and that we value G to degree g . Then the vN-M theory tells us to value this option to degree $x \times g$, i.e. that the value of a chance x of G is the value of G discounted by x . The discount reflects the fact that the option does not deliver G with certainty; its magnitude the degree to which the chance of G that it offers falls short of such certainty.

Von Neumann and Morgenstern's theory is not, of course, restricted in its application to options involving a single outcome. In general it says that the value of a probability distribution over a set of mutually exclusive outcomes (a 'lottery') is an additive linear function of the probability-utility products of each outcome. The additivity property is controversial, however, so I will stick to the simple cases where only linearity is at stake. But even for these simple cases, I will argue, the vN-M theory gets it wrong: the value of a chance x of G can permissibly deviate from the x -discounted value of G . My main focus will be on the hypothesis that it is frequently greater than this value, but in principle it can deviate in both directions. An example due to Edi Karni nicely illustrates this point.²

Consider a mountain climber who typically derives at least some of her enjoyment of climbing from confronting risk. For her the activity is of little worth if there is no associated chance of death or injury, even though it is to be avoided if the chances of death or injury are too high. Indeed there is an optimal region of risk, when the chances of death or injury are high enough to require courage of the climber, but not so high as to make the activity foolish. I think that, even if we don't share the climber's taste for risks of this kind, many of us will recognise the contribution that risks of losses or failures makes to the worth of activities we value.

Now the values that the mountain climber puts on the different chances of death and injury are clearly incompatible with the vN-M theory. When the values of the chances of an outcome are a positive linear function of the chances two things must be the case. Firstly if the outcome has a positive value (i.e. it is a good) then so too must a chance of this outcome and if it has a negative value (it's a bad) then so too must the chance. And secondly, the marginal values of the chances must be constant; for instance, the difference in value between a quarter chance and a half chance must be the same as the difference between a half chance and a three-quarter chance. Neither holds in Karni's example. For the value the mountain climber puts on the chances of death and injury (the bads) are not uniformly negative. And furthermore the marginal values of the chances vary with their magnitude.

It is my contention that the case of mountain climber is a dramatic example of a commonplace phenomenon: that we value chances non-linearly. In particular, we very often (permissibly) attach greater value to increases in our chances of a good when they are low than when they are high: a pattern of valuation that implies risk aversion with respect to chance. I will defend this claim indirectly by showing that it helps us understand a couple of phenomena that decision theorists have had considerable difficulty accommodating. The first of these is the Ellsberg Paradox: the conflict between the choices typically observed in Ellsberg's experiments and the rationality axioms of Savage's ver-

²The example is based on one that Edi Karni made in a seminar discussion and reported to me by Peter Wakker.

	red	black	yellow
B_1	\$100	\$0	\$0
B_2	\$0	\$100	\$0
B_3	\$100	\$0	\$100
B_4	\$0	\$100	\$100

Table 1: The Ellsberg Paradox

sion of Bayesian decision theory. The second is the conflict between Harsanyi's Utilitarianism and the common intuition that it is better (morally preferable) to distribute an indivisible good amongst claimants to it by means of a lottery than by simply giving it to one of them. That a single thesis about value can help resolve two such distinct questions is, I claim, strong evidence in its favour.

2 Ellsberg's Paradox

2.1 The Single-Urn Experiment

In a justly famous paper on decision making under uncertainty, Ellsberg [8] presents two experiments that he claims reveal a flaw in Savage's decision theory. Let us begin by recalling the second of these: the single-urn experiment that is depicted in Table 1. In the set-up Ellsberg describes, an urn is said to contain 90 balls, 30 of which are red, and the remaining 60 are black or yellow in an unknown proportion. Subjects are asked to choose between two bets. The first, B_1 , pays \$100 if, in a random draw from the urn, a red ball is drawn. The second, B_2 , pays \$100 if a black ball is drawn. Most subjects express a preference for B_1 over B_2 . In a second choice problem, subjects are asked to choose between B_3 and B_4 , which pay out \$100 in the events 'red or yellow' and 'black or yellow' respectively. Here, most subjects express a preference for B_4 over B_3 .

As can easily be verified, the preferences $B_1 \succ B_2$ and $B_4 \succ B_3$ are inconsistent with the Bayesian prescription to maximise subjective expected utility. For whatever probability is assigned to the possible states of the world and whatever utility is assigned to the monetary consequences, the expected utility of B_1 can exceed that of B_2 only if the expected utility of B_3 exceeds that of B_4 . Indeed this pattern of preferences - hereafter 'the Ellsberg preferences' - violates more than one of the postulates of rational preference that are the foundation of Savage's canonical formulation of Bayesian decision theory. That they violate the Sure-thing principle (Savage's P2) - the requirement that preferences be separable across states of the world - is perhaps obvious. But they also violate Savage's P4, his axiom of qualitative probability which, as Machina and Schmeidler [12] have shown, is not implied by the violation of the Sure-Thing principle. To see this, note that it follows from Savage's definition of the qualitative probability relation that $B_1 \succ B_2$ iff the event 'Red' is more probable than the event 'black' and that $B_4 \succ B_3$ iff the event 'black or yellow' is more probable than the event 'red or yellow'. But this are inconsistent, since the laws of probability require that 'red' is more probable than 'black' iff for any event X disjoint with both, 'red or X' is more probable than 'black or X'.

Ellsberg’s explanation for these observations was that agents are averse to what he calls *ambiguity*, this being the lack of information as to the precise probability distribution over the state space. In the first choice situation in which the subjects find themselves they are given information which makes it reasonable for them to put the probability of drawing a red ball at one-third, but with regard to the probability of a black ball they know only that it is no more than two-thirds. In view of this many people, Ellsberg conjectured, would ‘play it safe’ and opt for the lottery with a known probability of paying out over the one in which there is a good deal of uncertainty about the probability of a win. Similar reasoning would lead them, in the second choice problem, to pick lottery B_4 which has a ‘known’ probability of two-thirds of paying out over B_3 with its unknown probability of a win.

2.2 Reframing the Decision Problem

Since Ellsberg’s paper there have been numerous experiments reporting ambiguity aversion, at least in set-ups similar to his, and a variety of attempts made to model it.³ Although these models differ in various ways, they all take it as given that ambiguity aversion is not only inconsistent with Savage’s Sure-Thing Principle, but also with view that the individuals based their decisions on precise probabilities for the contingencies upon which the consequences of their choice depend. But if Ellsberg’s conjecture about how subjects perceive the decision problem they face is correct, then Table 1 does not provide the correct representation of their decision problem. A properly specified decision problem, from Savage’s point of view, is one in which the descriptions of states are maximally specific with regard to the presence or absence of all factors relevant to the determination of consequences that are causally independent of the actions available to the agent, and in which descriptions of consequences are maximally specific with regard to features that matter to the agent’s evaluation of their desirability. Now according to Ellsberg – correctly I believe – agents will regard the distribution of balls in the urn as relevant to the determination of the consequences: the monetary gains attendant on a draw. This means that strictly speaking the states of the world are not draws of red, black or yellow but combinations of distributions of balls in the urn plus the draw from it; such as ‘The urn contains 30 red balls, 25 black balls and 35 yellow balls. A yellow ball is drawn’ and ‘The urn contains 30 red balls, 60 black balls and no yellow balls. A red ball is drawn’.

Does the distribution from which a ball is drawn matter, given the specification of the ball actually drawn? To answer this we need to draw up the fully refined decision problem and see whether this framing can make a difference. Writing out all the states would be rather tedious however, so I will consider a simple variant of Ellsberg’s set-up in which the agent knows that the urn contains either 60 black or 60 yellow balls in addition to the 30 red balls, but no other combinations. This set-up has the same coarse grained representation as Ellsberg’s (i.e. that given by Table 1), but a much simpler refined one in which consequences are of the type described above. I shall assume that agents’ choices will exhibit the same pattern in the simplified set-up as in Ellsberg’s problem. The assumption is justified both by introspection and, perhaps more

³See Wakker [21] and Trautmann et al [18] for a review of the literature.

	30 red, 60 black	30 red, 60 yellow
B_1	$\frac{1}{3}$	$\frac{1}{3}$
B_2	$\frac{2}{3}$	0
B_3	$\frac{1}{3}$	1
B_4	$\frac{2}{3}$	$\frac{2}{3}$

Table 2: The Reframed Ellsberg Paradox

reliably, by the fact that every theory of decision making under ambiguity that I am aware of requires that it be so.

To make our main claim it will not be necessary to consider a fully refined representation of this problem; an alternative coarse-grained representation will suffice in which states are individuated by distributions of balls in the urn and consequences are chances of monetary gains. This is illustrated in Table 2 in which the cell entries are the chances of winning \$100. Thus an entry of x indicates that making that choice when the world is in that state gives the agent a chance of x of winning \$100 and a chance of $1 - x$ of winning nothing.

In this representation the betting acts B_1 to B_4 are functions from possible ball distributions to chances. Now let us suppose that our agent is a subjective expected utility maximiser à la Savage and moreover that she regards the two possible states of the world as equally likely (perhaps in virtue of symmetry considerations). In this case, a preference for B_1 over B_2 reveals that for the agent $U(\frac{1}{3}) > 0.5U(\frac{2}{3}) + 0.5U(0)$, while a preference for B_4 over B_3 reveals that for the agent $U(\frac{2}{3}) > 0.5U(\frac{1}{3}) + 0.5U(1)$. Together these imply that:

$$U(\frac{1}{3}) - U(0) > U(\frac{2}{3}) - U(\frac{1}{3}) > U(1) - U(\frac{2}{3}) \quad (1)$$

So we can conclude that a probabilistically sophisticated agent who maximises subjective expected utility *can* have Ellsberg preferences over betting acts, provided she values gains in chances of monetary payoffs less as the minimum/maximum chance rises, i.e. if the chances of money have diminishing marginal utility for her.

2.3 Chance Risk Aversion

When a good has diminishing marginal utility for an agent, she is said to have risk averse preferences for it. But what does it mean to be risk averse with respect to *chances*? Risk aversion with respect to any divisible good is canonically identified with a preference for acts which yield some fixed quantity of the good independently of the state of the world, over acts which have the same expected quantity of the good but which yield different quantities depending on the state of the world. For instance, someone who is risk averse with respect to money will prefer an act which always pays \$50 over one which pays either \$100 or nothing depending on the toss of a fair coin. Similarly someone who is risk averse with respect to the chances of receiving some good (divisible or

	30 red, 60 black	30 red, 60 yellow
B_1	$\frac{1}{3}$	$\frac{1}{3}$
B_2	$\frac{2}{3}$	0
B_5	0	$\frac{2}{3}$

Table 3: Hedging Acts

otherwise) will prefer acts which yield constant chances of getting the good over those with the same expected chances when the chances vary by state of the world. Consider Table 3, for instance. Someone who regards the two states of the worlds as equiprobable will be indifferent between betting acts B_1 and B_5 . If, furthermore, they are risk neutral with regard to the chances of monetary gain that are the outcomes of these acts they will regard B_1 as equally good as both B_2 and B_5 . But if they are risk averse with respect to these chances they will prefer B_1 over the other two.

Risk attitudes to goods and risk attitudes to the chances of these goods are logically independent. One could be risk neutral with regard to money, but risk averse with respect to the chances of obtaining it. Or just the other way around.⁴ But in one crucial respect they are similar: there is nothing particularly rational or irrational about having one risk attitude rather than another. We certainly do, as a matter of fact, care about the chances of outcomes as well as the outcomes themselves. There is a difference, we tend to think, between having no lottery ticket at all and having a lottery ticket which is not, in fact, a winner. And between succeeding at a task when the chance of doing so was low and succeeding at it when the chance of doing so was very high. This being so, why should we not value gains and losses in chances differently depending on what the reference point for these changes are (our initial chance endowment), just as we value gains and losses of money and other goods differently depending on our initial endowment?

Agents who are risk averse with respect to chances will display an inclination to hedge against chances in the sense of preferring mixtures of equally preferred acts to the latter. To explain what I mean by this, let f and g be any two acts whose outcomes are chances of some good (such as the lotteries in the Table 3). Then for any $\alpha \in [0, 1]$, the α -mixture of f and g , denoted $\alpha f + (1 - \alpha)g$, is an act whose consequence in each state of the world, s , is defined by:

$$(\alpha f + (1 - \alpha)g)(s) = \alpha f(s) + (1 - \alpha)g(s)$$

Now someone who prefers to hedge their chances is just someone who for any f and g , such that $f \approx g$, will prefer the mixture $\alpha f + (1 - \alpha)g$ to either of them. For instance, in Table 3, B_1 can be regarded as an equal weighted mixture of B_2 and B_5 . Suppose that an agent regards B_2 and B_5 as equally good. Then if she

⁴Though in private correspondence, Ittay Nissan suggested to me that when chances are tradeable at their expected monetary value, then risk aversion with respect to money will explain the risk aversion with respect to chances.

is risk neutral with respect to chances she will regard B1 as good as both, but if she is risk averse she will prefer B1.

This latter pattern of preference, for mixtures of acts over equally good elements of the mix, is what is termed *ambiguity aversion* in the decision theoretic literature. The experiments of Ellsberg and others provides strong evidence that at least some people are ambiguity averse in some contexts. As we noted earlier, such ambiguity aversion is typically taken to involve a violation of probabilistic sophistication generally and subjective expected utility maximisation in particular. And this in turn has given a great impetus to work on alternatives to Bayesian decision theory. But it is now evident that if agents are risk averse with respect to chances then in maximising subjective expected utility they will exhibit ambiguity aversion. So this phenomenon, and Ellsberg's paradox in particular, does not require us to abandon Bayesianism.

This claim is likely to arouse suspicion. Did we not begin by showing that the ambiguity averse preferences of Ellsberg's subjects were inconsistent with Savage's postulates? So is the conclusion that they are not, not just an artifice of the reframing of the decision problem, which succeeds in 'hiding' the violation of the Sure-Thing principle 'evident' in the framing given by Table 1. To see why the answer is 'no', an explanation needs to be provided as to why the two framings of Ellsberg's set-up lead to different conclusions. First, let me concede that for equation 1 not to imply a violation of subjective expected utility theory, it cannot be that, for any chance x of obtaining the \$100, $U(x) = x.U(\$100)$. For were this the case it would follow that $U(\frac{1}{3}) - U(0) = \frac{1}{3}U(\$100) = U(\$100) - \frac{2}{3}U(\$100) = U(1) - U(\frac{2}{3})$, contrary to equation 1. But this is precisely what is required by Savage's theory when the decision problem is given as in Table 1, i.e. when the consequences are taken to be the monetary prizes. So either one these framings is at fault or, contrary to my earlier claim, preferences that are risk averse with respect to chances are not consistent with subjective expected utility theory.

The former answer is the correct one. The problem with the first framing of the decision problem is that the winning of the \$100, or otherwise, is not all that matters if agents care about the chances of outcomes as well as the outcomes themselves. Winning nothing when you pick B_1 and a black ball is drawn may not be as bad as winning nothing when you pick B_2 and a yellow ball is drawn. For in the latter case, but not the former, you not only win nothing, but you also had no chance of winning anything. So in this representation, consequences are state-dependent. But, as observed earlier, to apply Savage's theory you must start with a description of consequences which are adequate in the sense that they are maximally specific with regard to all that matters to the agent. So consequences in the simplified Ellsberg set-up should be objects of the form 'Win \$100 when the chance of winning is x ' and not simply 'Win \$100'. Once we describe them fully, however, the apparent inconsistency between subjective expected utility theory and preferences that are risk averse with respect to chances disappears.

This is good news for Savage, but bad news for von Neumann and Morgenstern. Their theory rules out risk aversion with respect to chances because, as we saw earlier, if the value of a chance of good is a linear function of the chance, then the chances must have constant marginal values. So the price of making the Ellsberg preferences consistent with Bayesianism is that we must reject the vN-M theory as a descriptively accurate theory of choice under risk.

Normatively, we face a trilemma: it is not possible to regard the Ellsberg preferences as rational and to regard both Savage’s theory and the vN-M theory as normatively correct. Some authors resolve it by concluding that the Ellsberg preferences are irrational; many more that Savage’s theory must be abandoned. Here I have argued for a third response: rejecting the vN-M theory.

An important, if somewhat technical, qualification: Rejection of the vN-M theory does not entail rejection of the vN-M axioms of preference over lotteries. These axioms imply the existence of an expected utility representation of preference, unique up to affine transformation, but unique *as a type of representation* only up to positive monotonic transformation. In the simple case to which we restricted ourselves where there is only one good G , lotteries are just chances of G , and the vN-M theory says that rational preference for these lotteries can be represented by the identity function I on chances (with 1 being the utility of G). But then $\phi(I)$ also ordinally represents these preferences whenever ϕ is a positive monotone transformation of I ; hence in particular when it is a concave transformation. So the vN-M axioms do not settle the question of whether I or $\phi(I)$ is the correct measure of the agent’s degrees of preference. When only preferences amongst risky options are at stake there is perhaps no meaningful way of settling the issue (or, for that matter, point in doing so), but when we widen consideration to uncertain options the situation changes. Whereas the hypotheses that, respectively, I or $\phi(I)$ is the correct measure of the agent’s degrees of preference, are observationally equivalent with respect to her choices amongst lotteries, these hypotheses conjoined with Bayesian decision theory are not observationally equivalent with respect to her choices amongst acts with risky consequences.

2.4 Chances of Losses

Let us take stock and state our position more precisely using a framework akin to the Anscombe and Aumann [2] one in which much of the decision theoretic literature on ambiguity is couched. Let \Pr a measure of the agent’s degrees of belief on the set of states $\mathcal{S} = \{S_i\}$ and let acts be functions from states of the world to consequences, which in this context are just the chances of gaining some good. Then on the theory I am advocating the function V on acts such that, for any act f , $V(f) = \sum_i \phi(f(s_i)) \cdot \Pr(s_i)$ is a subjective expected utility representation of the preferences of a rational agent, with ϕ a transformation of the chances (of the identity function I on chances) that represents her attitude risk attitudes to chances (her ambiguity attitudes). In particular ϕ is convex for ambiguity averse agents.

The function V is an expectation based on (subjective) probabilities for (objective) probabilities, with the notable feature that the two tiers of probability are not reducible to a single one. This is a feature of a number of other models of ambiguity aversion such as those of Segal [14], Klibanoff et al [11] and Ergin and Gul [9]. In particular, formally speaking (and setting aside small differences in vocabulary) our theory is just the special case of the smooth ambiguity model of Klibanoff, Marinacci and Mukerji that is obtained when it is restricted to single-good lotteries. In their more general model, ϕ is a transformation, not of I , but of an expected utility representation of the agent’s preferences over lotteries. This is, of course, not the only direction in which our theory can be generalised, but it is certainly a natural way to do so. But where we do differ

	red	black	yellow
B_1	-\$100	\$0	\$0
B_2	\$0	-\$100	\$0
B_3	-\$100	\$0	-\$100
B_4	\$0	-\$100	-\$100

Table 4: The Negative Ellsberg Problem

is on the interpretation of the model. They seem to view I (more generally, vN-M expected utility) as an appropriate measure of the utility of the lotteries and ϕ as a transformation induced by psychological ambiguity aversion. I view $\phi(I)$ itself as the appropriate measure of the desirability of the lotteries, with ϕ encoding chance risk aversion.

What is at stake between these interpretations? On the account presented here, *behavioural* ambiguity aversion is explained by risk aversion with respect to chances. But it is only one possible explanation for this phenomenon. As we have seen, Ellsberg explained this pattern of preferences in terms of what might be called *psychological* ambiguity aversion: a dislike of options with outcomes contingent on unknown chances. In the main, Ellsberg’s explanation has been accepted in the literature (though economists do not, on the whole, attach much significance to psychological claims) and debate has centred on how to model it formally. So if we are to favour one account over the other, something more than its ability to explain the Ellsberg preferences must be offered in support of it. One reason to prefer the risk aversion account is that it does not require giving up Bayesian standards of rationality, but for someone who was equally attached to the vN-M theory this would not be decisive.

More evidence is required. In our discussion thus far we have only considered lotteries involving chances of obtaining some good. But what about the chances of losing some good or of gaining a bad? Consider, for example, the negative image of the Ellsberg set-up, displayed in Table 4, in which agents must choose between lotteries that involve only losses of \$100.

If individuals are psychologically ambiguity averse in the standard sense of having a preference for lotteries in which the chances of the outcomes are known over those in which they are not, then they will exhibit the same pattern of preferences in the negative Ellsberg problem as they do in the standard Ellsberg set-up, namely $B_1 \succeq B_2$ and $B_4 \succeq B_3$. The empirical data is fairly sparse, but on the whole does not support this prediction. Sujoy and Roy [6] and Trautmann et al [17], for instance, report ambiguity aversion in choice experiments involving monetary gains, but ambiguity seeking behaviour in those involving monetary losses. Recent experimental studies of Alex Voorhoeve, Lisa Stewart and Ken Binmore find little evidence of either ambiguity averse or ambiguity seeking choice. See Wakker [21] and Trautmann and van de Kuilen [18] for a review of the experimental evidence.

What does our theory have to say about preferences over the lotteries in the negative Ellsberg problem? Once again the problem requires reframing in order to allow for the modelling of agents’ attitudes to the chances of losses. This is done in Table 5 in which a cell entry of $-x$ indicates a chance of x of a ‘gain’ of -\$100 (i.e. of losing \$100), given the choice of -1 for the utility of a \$100 loss.

	30 red, 60 black	30 red, 60 yellow
B_1	$-\frac{1}{3}$	$-\frac{1}{3}$
B_2	$-\frac{2}{3}$	0
B_3	$-\frac{1}{3}$	-1
B_4	$-\frac{2}{3}$	$-\frac{2}{3}$

Table 5: Reframed Negative Ellsberg Problem

We see that:

1. $B_1 \succeq B_2 \Leftrightarrow U(-\frac{1}{3}) > 0.5 \cdot U(-\frac{2}{3}) + 0.5 \cdot U(0)$
2. $B_4 \succeq B_3 \Leftrightarrow U(-\frac{2}{3}) > 0.5 \cdot U(-\frac{1}{3}) + 0.5 \cdot U(-1)$

Now there are two salient possibilities here. The first is that it is the expected utilities that have diminishing marginal utility, in which case individuals will exhibit the pattern of preferences, $B_1 \succeq B_2$ and $B_4 \succeq B_3$, predicted by the hypothesis of psychological ambiguity aversion. This is what the smooth ambiguity model predicts, for instance, when concave φ over losses. The second possibility is that it is the chances themselves that have diminishing marginal utility, so that the utility of a chance x of a utility -1 will equal $-1 \cdot \varphi(x)$. In this case, the expected utilities will have increasing marginal utility in the domain of losses, because for concave φ :

$$-\varphi(1) - (-\varphi(\frac{2}{3})) > -\varphi(\frac{2}{3}) - (-\varphi(\frac{1}{3})) > -\varphi(\frac{1}{3}) - (-\varphi(0))$$

And this in turn implies that $B_2 \succeq B_1$ and $B_3 \succeq B_4$ – a reversal of the pattern of preferences in the standard Ellsberg problem and one which characterises ambiguity seeking preferences. So, in contrast to the psychological ambiguity aversion hypothesis, it is possible to explain ambiguity seeking preferences in the loss domain using the hypothesis of chance risk aversion.

3 What’s Good about Lotteries?

3.1 Utilitarianism and Equality

It is commonly thought that in distributing a good that is not perfectly divisible between two or more individual claimants on it, it is morally preferable to do so by means of a lottery rather than to simply give the good to one of them. Furthermore, if the individuals have equally strong claims on the good then they should have an equal chance of obtaining the good. In this case then, it seems that an equal weighted lottery is better than an unequal one. The challenge is to explain why this is so and under what conditions.

	Anne	Bob
<i>a</i>	G	0
<i>b</i>	0	G
<i>eq</i>	$\frac{1}{2}G$	$\frac{1}{2}G$

Table 6: Distribution of G

A common explanation is that it is morally better to use an equal chance lottery in these circumstances simply because this is the *way of distributing the good or that it provides the best combination of fairness and efficiency (see in particular [5] and [3]). A second explanation is that an equal weighted lottery is the best way of dealing with the moral uncertainty that we face in these cases (see [13]). In this section I want to explore a third explanation, namely that equal chance lotteries are morally preferable because they maximise the overall good whenever the individuals are risk averse with respect to the chances of receiving the good in question. These explanations are not mutually exclusive however and my interest in developing one of them should not be taken as an attempt to refute the other two.*

Suppose that some good G must be divided between two individuals, Anne and Bob. Classical Utilitarianism counsels dividing the good (to the extent that it can be) in such a way as to maximise total utility, where utility is an interpersonally comparable measure of the welfare that individuals obtain in virtue of receiving the good. It is commonly felt, however, that this method of distribution ignores an important consideration: Equality. Suppose for instance that G is divisible and equally valuable (in welfare terms) to Ann and Bob, and that the options for distribution are those displayed in Table 6. In such a case while it seems that Utilitarians should be indifferent between *a*, *b* and *eq*, their egalitarian critics argue that *eq* is to be preferred because it gives Ann and Bob equal shares of a good that benefits each equally.

Intuition clearly favours the moral preference for *eq*. But utilitarians need not deny this. For they can explain the superiority of *eq* by appeal to the diminishing marginal utility of shares of G. If the utility difference between an $x\%$ and an $(x - d)\%$ share of G is greater than that between an $(x + d)\%$ and an $x\%$ share, then the distribution that will maximise total utility is indeed the equal one. Such thinking was in fact the basis for the Utilitarian egalitarianism of economists such as Pigou:

“... it is evident that any transference of income from a relatively rich man to a relatively poor man of similar temperament, since it enables more intense wants, to be satisfied at the expense of less intense wants, must increase the aggregate sum of satisfaction..” – Pigou [15, Part I, chapter 8.3]

3.2 Dividing the Chances

What about when G is not divisible? Egalitarians argue that the next best thing to giving equal shares of G to Ann and Bob is to give each an equal chance of obtaining G. Thus when the options are as displayed in Table 7, where G can go

	Anne	Bob
<i>A</i>	1	0
<i>B</i>	0	1
<i>Eq</i>	$\frac{1}{2}$	$\frac{1}{2}$

Table 7: Distribution of Chances

to Ann for certain, or to Bob for certain, or to each with a chance of one-half, the last of these options should be preferred to the other two.

When an option take the form of a lottery over distributions of goods, then its value to Ann or Bob depends on how the uncertainty associated with its outcome is resolved. The von Neumann-Morgenstern theory dictates in such a case that its value for each individual should be measured by its expected value. In our simple case this implies that for Ann the value of *A* is just the utility of *G*, of *B* is zero, and that of *Eq* is exactly half of the utility of *G*, since Ann has a 50% of obtaining *G* from this lottery. Similarly for Bob, except that it is now *B* that has the value of *G*. So a Utilitarianism based on the vN-M theory, such as that of Harsanyi [10], must regard *A*, *B* and *Eq* as equally good distributions since the impartial sum of the Ann and Bob's utilities is same for each.

That Utilitarianism has this implication is widely regarded as presenting it with a serious difficulty. Peter Diamond [7], for instance, offered precisely such an example as a refutation of Harsanyi's theory. But once again a utilitarian can refuse to bite this particular bullet and instead attempt to explain the intuition in favour of *Eq* in some other way. One obvious strategy is to question whether we have the description of the outcomes right in Table 7. Broome [4], for instance, argues that whether someone is treated fairly or unfairly by a distributive process is an outcome of the process that should figure in our appraisal of it. What makes a lottery fair is that each person's chances of obtaining the good is proportional to the strength of the claims on it. Hence what makes an equal lottery better than an unequal one, in a situation in which individuals claims are equally strong, is that an unequal lottery treats at least one of the individuals unfairly. Lottery *A*, for instance, not only distributes *G* to Ann, but treats Bob unfairly in doing so. But being treated unfairly is bad for Bob. So the expected utility of lottery *A* is somewhat less than the utility of *G* and hence of lottery *Eq*. The same goes for lottery *B*.

Broome's strategy of redescription allows him to reconcile expected utility theory with the contention that its better to use lotteries to distribute goods in the circumstances under discussion and that more equal lotteries are better than less equal ones (because the greater the difference between the strength of someone's claims and the chances they are given, the more unfairly they are treated). The problem is that it requires him to treat fairness as a property of the *outcomes* of lotteries. Now whether an outcome of lottery in some state of the world has the property of being such that the affected individuals were treated fairly or not depends on what the outcome of the lottery would have been in other states of the world, i.e. on global features of the lottery. But, as Broome recognises, this creates a conflict with a basic requirement of the decision-theoretic framework that he adopts, namely that all combinations of

state-outcome pairs are possible.⁵ Still, I don't think this problem is crippling and, as Stefánsson [16] shows, can be solved by moving to a framework in which counterfactual properties are explicitly represented.

The harm of being treated unfairly resides, it would seem, in being given chances that are less than those implied by the strength of one's claim. But why is the bad done to the individual who gets a chance of a good less than the strength of her claim on it not offset by the good done to the individual who gets a higher chance than mandated by their claim? I want to explore the possibility that the answer lies precisely in the diminishing marginal utility of chances.

3.3 The Welfarist Case for Lotteries

To provide an alternative Utilitarian welfarist explanation of the moral value of distributing the indivisible good G to Ann and Bob by means of a lottery, three assumptions are required:

1. *Chance Risk Aversion*: Ann and Bob are risk averse with respect to chances of G
2. *Utilitarianism*: The goodness of a lottery is the sum of its utility to the affected individuals.
3. *Interpersonal Symmetry*: G has equal utility for Anne and Bob.

Let U_A and U_B be wellbeing measures on the space of outcomes (chances of G in this case) for Ann and Bob respectively. Let V be a utility defined on the space of lotteries that measures overall goodness. A lottery L can be expressed by a pair of positive numbers (l_A, l_B) such that $l_A + l_B = 1$; representing the chances of G afforded by L to Ann and Bob respectively. Then Utilitarianism can be expressed more precisely in this context as the requirement that, for any lottery $L = (l_A, l_B)$:

$$V(L) = U_A(l_A) + U_B(l_B)$$

Interpersonal Symmetry requires a bit more unpacking. We assumed that neither Ann nor Bob had a stronger claim on G. For Utilitarians this tends to mean something like G affords Ann and Bob an equal gain in welfare. But all that we really require for the argument is that Ann's and Bob's obtaining (or not) of G weighs equally in the judgement of overall good. That is, it is not better, or worse, overall that Ann obtains G than that Bob does. Understood this way, Interpersonal Symmetry allows us to co-scale U_A and U_B so that:

$$\begin{aligned} U_A(1) &= g = U_B(1) \\ U_A(0) &= 0 = U_B(0) \end{aligned}$$

where this co-scaling reflects the judgement that their claims on G are equal. (So strictly we should say that U_A and U_B are components of V that ordinally represent Ann's and Bob's welfare).

⁵This requirement of Savage's decision theory is dubbed the Rectangular Field assumption by Broome.

Now Chance Risk Aversion implies that:

$$\begin{aligned} U_A\left(\frac{1}{2}\right) &> \frac{1}{2}U_A(1) \\ U_B\left(\frac{1}{2}\right) &> \frac{1}{2}U_B(1) \end{aligned}$$

So it follows from Utilitarianism that:

$$\begin{aligned} V(A) &= U_A(1) = g \\ V(B) &= U_B(1) = g \\ V(Eq) &= U_A\left(\frac{1}{2}\right) + U_B\left(\frac{1}{2}\right) > \frac{1}{2}U_A(1) + \frac{1}{2}U_B(1) = g \end{aligned}$$

In other words, the equal-weighted lottery Eq is better overall than either giving G to Ann or giving it to Bob. It should be evident furthermore that nothing in the argument depends on the lottery giving Ann and Bob equal chances of G ; any lottery will be better under our assumptions than simply giving the good to either of them. So the hypothesis of Chance Risk Aversion supports a Utilitarian explanation of why it better to distribute a good by lottery in these circumstances.

What about the second part of the issue: Are more equal lotteries better than less equal ones? Consider any two lotteries $L1 = (\frac{1}{n}, \frac{1-n}{n})$ and $L2 = (\frac{1}{m}, \frac{1-m}{m})$, such that $m > n \geq 2$; hence such that $L1$ is more equal than $L2$. Then by Utilitarianism:

$$\begin{aligned} V(L1) &= U_A\left(\frac{1}{n}\right) + U_B\left(\frac{1-n}{n}\right) \\ V(L2) &= U_A\left(\frac{1}{m}\right) + U_B\left(\frac{1-m}{m}\right) \end{aligned}$$

Hence:

$$\begin{aligned} V(L1) &> V(L2) \\ \Leftrightarrow U_A\left(\frac{1}{n}\right) + U_B\left(\frac{1-n}{n}\right) &> U_A\left(\frac{1}{m}\right) + U_B\left(\frac{1-m}{m}\right) \\ \Leftrightarrow U_A\left(\frac{1}{n}\right) - U_A\left(\frac{1}{m}\right) &> U_B\left(\frac{1-m}{m}\right) - U_B\left(\frac{1-n}{n}\right) \end{aligned}$$

Now it is easy to see that if Ann and Bob are *equally* risk averse then it must be the case that $L1$ is better than $L2$. For in this case $U_B\left(\frac{1-m}{m}\right) - U_B\left(\frac{1-n}{n}\right) = U_A\left(\frac{1-m}{m}\right) - U_A\left(\frac{1-n}{n}\right)$ and then it simply follows from the fact that Ann is risk averse that when $m > n \geq 2$, $U_A\left(\frac{1}{n}\right) - U_A\left(\frac{1}{m}\right) > U_B\left(\frac{1-m}{m}\right) - U_B\left(\frac{1-n}{n}\right)$. But if they differ in their risk attitudes to chances then this will not always be the case. For example suppose that Ann is risk averse and Bob risk neutral; specifically that their utilities depend as follows on the distribution of chances $\{x, 1-x\}$ to Ann and Bob:

$$\begin{aligned} U_A(x) &= \sqrt{x} \\ U_B(x) &= 1-x \end{aligned}$$

Then the lottery $(\frac{1}{4}, \frac{3}{4})$ is better on a Utilitarian calculation than the perfectly equal lottery $(\frac{1}{2}, \frac{1}{2})$. Indeed as we can see from the graph below, $(\frac{1}{4}, \frac{3}{4})$ is the optimal distribution of chances to Ann and Bob.

3.4 Fairness and the Utility of Chances

Let's take stock. The hypothesis that individuals are risk averse with respect to chances of goods affords a broadly Utilitarian explanation of what is good about distributing a good by a lottery. Furthermore it explains why, when individuals are equally risk averse, a more equal lottery is better than a less equal one. For sure, the explanation it provides requires giving up some of Harsanyi's postulates – in particular his attachment to the von Neumann-Morgenstern theory – but this can be motivated quite independently of distributional concerns.

On the other hand, this line of thinking does not lead to the conclusion that equal lotteries are always better than unequal ones on a Utilitarian calculation. When individuals' risk attitudes differ they will not be; a lottery which is biased towards the less risk averse delivers greater total utility. Now the conclusion that equal lotteries are not always best is not in itself surprising. On Broome's account, for instance, an equal lottery will not be best if the strength of the individuals' claim differ. But what seems problematic is that the divergence from equality derives from an apparently morally irrelevant feature of the individual claimants – their risk attitudes – rather than a morally relevant one – the strength of their claims.

There are two responses one might make to this objection. The first is to argue that what is relevant to distributional issues, qua moral problems, is not what individuals prefer but what is good for them. So the risk attitudes that are relevant are not those based on the individuals' preference relations over chances, but their betterness relations. And although their preference-based risk attitudes can differ, their betterness-based ones should not, i.e. there is no reason why the marginal values of the chances of a good that is equally valuable for two individuals, should differ. I don't know whether this claim about betterness is true or not, but if it is, it successfully protects a goodness-based Utilitarianism from the objection. In any case, the question of goodness of chances deserves further investigation.

The second response is best brought out by a comparison between Broome's theory and mine. Broome does not commit to a view as to what sorts of reasons counts as claims, but let us assume that welfare benefits are *a* basis for a claim on a good and restrict ourselves to situations in which it is the only source of the individuals' claims. Now both theories imply that equal lotteries are morally preferable to unequal ones in cases where the chances of the good benefits each individual equally. For different reasons of course, but chance risk aversion and unfairness are not incompatible considerations and it could well be the case that equal lotteries are better for more than one reason.

What about the case where Ann and Bob's degree of chance risk aversion differs? What does fairness require in this case? That depends on how one assesses the strength of their claims in this situation. One might argue that the strength of their claims is independent of their risk attitudes and derive only from what the welfare benefit of G would be if they were to obtain it. But I do not think considerations of fairness need to apply in that way. If Ann and Bob have equal claims to welfare and Ann gets greater welfare from a chance of the good than Bob, then equalising welfare requires giving Bob a greater share of the chances. What is disturbing about this conclusion is the thought that Ann's greater risk aversion may derive from her being in a position of greater need (more desperate perhaps), in which case our intuition is that, if anything,

the lottery should be biased in her favour. But if Ann were really worse off (in a morally relevant way) then, contrary to assumption, Bob and Ann's claims to the good would not have been equal. So this thought is irrelevant here.

4 Conclusion

I have argued that, contrary to the vN-M theory, we value the chances of goods non-linearly and in particular that increases in our chances of a good are more valuable when they are low than when they are high. This risk aversion with respect to chances provides an explanation of the patterns of behaviour typically observed in the Ellsberg paradox. Moreover it is an explanation that is quite consistent with the orthodox theory of subjective expected utility maximisation due to Savage. This should not be taken to be blanket endorsement of Savage's theory. There are many situations in which we are unable to assign probabilities to all relevant contingencies and in which Savages's theory will not apply. But the Ellsberg set-up is not one of them. It's lesson is not that rationality does not require probabilistic sophistication, but rather that rationality does not require risk neutrality with respect to chances.

Chance risk aversion has consequences not only for rational individual behaviour. It also has implications for distributional issues and teleological ethics more generally. I illustrated this by considering whether chance risk aversion could serve as the basis for a utilitarian preference for distributing goods by lotteries and for an explanation of why more equal lotteries are better than less equal ones. The second of these questions is still open I think. But the central message remains: The moral assessment of distributions of chances requires an adequate theory of how chances matter. In particular, risk aversion with respect to chances, if normatively permissible, would undermine the claim of Harsanyi that expected utility representations of an agent's preferences are the correct measure of her wellbeing, construed in terms of preference satisfaction. Similarly, risk aversion in the betterness relation over chances, if permissible, would undermine Broome's claim that expected utility provides the correct measure of goodness. So there is much at stake here.⁶

References

- [1] Mohammed Abdellaoui, Frank Vossman and Martin Weber (2005) Choice-Based Elicitation and Decomposition of Decision Weights for Gains and Losses Under Uncertainty, *Management Science* Vol. 51.9, p. 1384–1399
- [2] Anscombe, F. J. and R. J. Aumann (1963) "A Definition of Subjective Probability", *Annals of Mathematical Statistics* 34: 199-205
- [3] Broome, John. 1990. Fairness. *Proceedings of the Aristotelian Society*, 91: 87–102.

⁶I am very grateful to Hilary Greaves, Alex Voorhoeve, Ittay Nissan and Orri Stefánsson for their comments on an earlier draft, as well as the audiences of the LSE Choice Group Seminars and the EIPE Research Seminars and the participants in the Conference on Rational Choice and Philosophy at Vanderbilt University. The research for this paper was supported by the Arts and Humanities Research Council (grant reference: AH/J006033/1).

- [4] Broome, John (1991) *Weighing Goods*, Cambridge, Mass., Basil Blackwell
- [5] Broome, John. 1994. Fairness versus doing the most good. *The Hasting Center Report* 24 (4): 36–39.
- [6] Sujoy Chakravarty & Jaideep Roy (2009). "Recursive expected utility and the separation of attitudes towards risk and ambiguity: an experimental study" *Theory and Decision* vol. 66(3):199–228
- [7] Diamond, P. "Cardinal Welfare, Individualistic Ethics and Interpersonal Comparisons of Utility: Comment", *Journal of Political Economy* 75: 765–6
- [8] Ellsberg, D. (1961) "Risk, ambiguity, and the Savage axioms", *Quarterly Journal of Economics*, 75: 643–669
- [9] Ergin, H. and F. Gul (2009) "A theory of subjective compound lotteries", *Journal of Economic Theory* 144: 899–929
- [10] Harsanyi, J. (1955) "Cardinal Welfare, Individualistic Ethics and Interpersonal Comparisons of Utility: Comment", *Journal of Political Economy* 63: 309–21
- [11] Klibanoff, P., Marinacci, M. and S. Mukerji (2005) "A smooth model of decision making under ambiguity", *Econometrica* 73: 1849–1892.
- [12] Machina, M.J. and Schmeidler, D. (1992) "A more robust definition of subjective probability". *Econometrica*, 60: 745–780.
- [13] Nissan–Rozen, I. (2012) "Doing the best one can: a new justification for the use of lotteries"; *Erasmus Journal for Philosophy and Economics* 5: 45–72.
- [14] Segal, U. (1990) "Two-Stage Lotteries without the Reduction Axiom", *Econometrica* 58: 349–377.
- [15] Pigou, A.C. (1952) *The Economics of Welfare*, 4th ed., London: Macmillan.
- [16] Stefansson, H. O. (forthcoming) "Fair Chance and Modal Consequentialism", *Economics and Philosophy*
- [17] Trautmann, Stefan T., Ferdinand M. Vieider, and Peter P. Wakker. 2011. "Preference Reversals for Ambiguity Aversion." *Management Science*, 57: 1320–1333.
- [18] Trautmann, S., & van de Kuilen, G. (forthcoming). "Ambiguity attitude". In *Blackwell Handbook of Behavioural Decision Making*.
- [19] Weymark, J. (1991) "A Reconsideration of the Harsanyi–Sen Debate on Utilitarianism," in Jon Elster & John E. Roemer (eds.) (1991). *Interpersonal Comparisons of Well-Being*. Cambridge:Cambridge University Press, pp 255 - 340
- [20] von Neumann, J. and Morgenstern, O. (1944) *Theory of Games and Economic Behaviour*, Princeton: Princeton University Press

- [21] Wakker, P. (2010) *Prospect Theory: For Risk and Uncertainty*, Cambridge University Press