Pay Caps in Knowledge Economies

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Abstract

We study the effects of pay caps and other wage policies in a knowledge economy featuring heterogeneously skilled individuals, occupational choice, many-to-one matching and firm size constraints. We define and characterize stable matchings for the knowledge economy with pay caps. Quantitatively, pay caps decrease the income share of the top 1% at a small cost to GDP. However, the bottom 50% are also substantially harmed by the policy. Compared to minimum wage/maximum rent policies, the pay cap is more effective at reducing the top 1% income share for a fixed level of inefficiency.

1 Introduction

There is an ongoing policy debate on whether the pay ratio of top corporate executives relative to the median worker should be capped. Several proposals have been made for the value of the cap, typically ranging from 10 to 50, which are justified by an implicit trade-off between improving the equality of the income distribution and avoiding inefficiencies. Estimates of the magnitude of this trade-off and the overall effect of the policy are, however, largely unavailable

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since, with the exception of Caselli (2025), there is little economic analysis of the topic. An important obstacle to progress, limiting our ability to inform the policy debate, is that the existence of a pay cap undermines the standard undercutting logic that justifies competitive equilibrium wages; hence there is a need to define a more general solution concept.

Pay caps and other wage regulation policies (such as minimum wage and maximum rent policies) affect the structure of wages directly and also indirectly through changes in the organization of firms that optimally combine the knowledge of their members. Thus, it is natural to study the effects of these policies in the context of knowledge economies, as introduced by Garicano and Rossi-Hansberg (2004), which feature the occupational choices of individuals to become managers, workers or self-employed and the many-to-one matching of workers to managers into firms whose production depends on members' knowledge.[1] However, the conceptual and technical challenges involved in modelling pay caps and other policy constraints are especially severe in knowledge economies since the wage plays a crucial role in the sorting of agents in the assortative matching problem. In this paper, we address these challenges by defining and characterizing the appropriate solution concept for knowledge economies with policy constraints, and we use our results to quantify the effects of imposing pay caps and other wage regulations.

At a conceptual level, the standard practice in the study of knowledge economies is to define an intuitive and tailor-made notion of competitive equilibrium. A policy constraint on wages makes it unclear what this definition should be, especially when one wants its existence to be generally guaranteed. We address this issue by showing, in Section 2, that a policy-constrained economy fits the general framework of large many-to-one matching markets with occupational choice introduced in Carmona and Laohakunakorn (2024) and extended in Carmona, Laohakunakorn, and Tsujiyama (2025) (CLT henceforth); this then allows us to use their notion of stable matching as a solution concept and their existence result to establish the existence of stable matchings in the policy-constrained economy.

In the absence of the policy, stable matchings are characterized by the following method (see e.g. CLT). First, stable matchings are equivalently described by an assignment, describing

[1] Garicano and Rossi-Hansberg (2004) build on earlier ideas from Lucas (1978) and Rosen (1982) but one important difference is that in Garicano and Rossi-Hansberg's (2004) model, as in the models we consider, there are upper bounds on the number of workers that can be hired (motivated by organizational constraints).

who matches with whom, and an earnings function, describing the payment received by each individual type. Then, each of these two elements is obtained independently of the other by solving a (possibly infinite dimensional) linear programming problem. The presence of the policy constraint implies that this method is not applicable: one problem is that it is possible for individuals of the same type to have different earnings depending on whether the policy constraint is binding or not in their matches. Despite this, we show in Section 3 that stable matchings can still be characterized by an assignment and the analog of an earnings function which now describes the lowest payment that individuals of each type receive. We also show that, in the case of a finite set of individual types, the assignments and earnings functions are jointly (part of) the solution of a mixed-integer linear programming problem; this allows us to compute stable matchings numerically and is the core of our computational method.

In Section 4, we solve a number of simple examples to illustrate our model and its solutions. These examples demonstrate that pay caps often benefit those in the middle of the income distribution at the expense of the top and the bottom. In particular, as the pay cap becomes tighter, the worst individuals are not worth hiring at any wage satisfying the pay cap and become self-employed, i.e. drop out of the economy. For low enough pay cap, all individuals become self-employed (which is maximally inefficient). However, following a not too low pay cap, the worst worker (who is in the middle of the distribution since he is more skilled than the self-employed who drop out) is better off since his wage is increased mechanically by the binding pay cap and this effect also extends to nearby workers. The top managers are worse off since they pay higher wages. These effects are clearly and analytically demonstrated in our examples.

The generality of our approach allows us to go beyond Caselli's (2025) model by considering a production function that depends on both on the manager's and the workers' knowledge. This then allows us, in Section 5, to quantitatively analyse a knowledge economy, introduced in CLT, which accounts well for U.S. wage distribution (in particular, for the wage polarization over 1988-2008), the U.S. establishment-size distribution, the top 1% income share and its evolution. We evaluate the impact of a pay-cap policy in this economy and find that a pay cap of 10 reduces the top 1% income share from 12.86% to 9.95% while having a less than a 1% impact on GDP. The reason is that under the parameters of this economy, all individuals

remain matched in firms following the pay cap. Thus, the pay cap has a first order effect on inequality by mechanically increasing the wages paid by those for which it binds, but a higher order effect on GDP through the rematching of individuals. In addition, in comparison to minimum wage/maximum rent policies, the pay cap is more effective at reducing the top 1% income share for a fixed level of inefficiency, although at the cost of a larger decrease in the bottom 50% income share.

2 Model

In this section we describe the economy, represent it as a large many-to-one matching market with occupational choice and apply the definition of stable matching to it.

2.1 Economy

There is a large number of individuals characterized by their knowledge, with Z denoting the set of knowledge levels; we assume that Z is a compact metric space. The knowledge distribution is denoted by ν and is a non-zero, finite, Borel measure on Z such that its support, $\text{supp}(\nu)$, is equal to Z. Individuals can be workers, managers or self-employed.

A firm consists of one manager and several workers of the same type. Production equals $F(z, z', n)$ when the manager's type is z, the worker's type is z' and there are n workers, where $F : Z^2 \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. A manager of type z can hire exactly $n(z, z')$ workers of type z', where $n : Z^2 \to \mathbb{R}_{++}$ is continuous. Define $s : Z^2 \to \mathbb{R}_+$ by setting, for each $(z, z') \in Z^2$, $s(z, z') = F(z, z', n(z, z'))$, i.e. $s(z, z')$ is the total surplus when z matches with z'. Workers in a given match are paid a common wage $c \in \mathbb{R}_+$ and the manager's income equals total surplus minus the wage bill, $s(z, z') - cn(z, z')$. There is a pay-cap $\xi \geq 1$ on the manager's rent, requiring that it is at most ξ times the wage paid to workers: $s(z, z') - cn(z, z') \leq \xi c$. A self-employed individual of type z produces $U_s(z)$, where $U_s : Z \to \mathbb{R}_+$ is continuous.

The pay-cap policy imposes a lower bound on wages: $c \geq \frac{s(z,z')}{\xi + n(z,z')}$ for each $(z, z') \in Z^2$. In general, we consider policies described by a continuous function $p : Z^2 \to \mathbb{R}_+$ and which require, for each $(z, z') \in Z^2$,

$$c \geq p(z, z'). \tag{1}$$

In addition to pay-cap policies, examples include a minimum wage $\underline{c}$ policy and a maximum rent $\bar{r}$ policy described, respectively, by $p(z,z') = \underline{c}$ and $p(z,z') = \frac{s(z,z')-\bar{r}}{n(z,z')}$ for each $(z,z') \in Z^2$.

Concrete examples of knowledge economies include the setting of Garicano and Rossi-Hansberg (2004) in which $Z = [0,\bar{z}]$ with $\bar{z} \in (0,1)$, ν is atomless, $F(z,z',n) = zn$, $n(z,z') = n(z') = \frac{1}{h(1-z')}$ and $U_s(z) = z$. Another example is the knowledge economy considered in CLT and obtained by changing F and n to $F(z,z',n) = z^{\alpha}n$ and $n(z') = \frac{1}{h(1-z')^{\sigma}}$ with $\alpha, \sigma > 0$.

2.2 Matching with occupational choice

We represent the economy in a simplified version of the general framework of markets with occupational choice of CLT. A matching market with occupational choice (a market, henceforth) is specified by a list $E = (Z, \nu, C, P_m, P_s, (\succ_z)_{z\in Z})$, where Z is the set of individual types, ν is the type distribution, C is the set of contracts, P_m is a correspondence describing the matches that are feasible for managers of type z, P_s is the same for self-employed individuals and $\succ_z$ describes the preferences of an individual of type z, for each $z \in Z$. A dummy type $\emptyset \notin Z$ is used to represent self-employed individuals, and we let $Z_\emptyset = Z \cup \{\emptyset\}$, with the assumption that $\emptyset$ is an isolated point in $Z_\emptyset$.

The representation of the setting in Section 2.1 above as a market is as follows. First, let Z and ν be as in Section 2.1. Second, we let contracts be wages and set $C = [0,\bar{c}]$, with $\bar{c} > \max_{(z,z')\in Z^2} \max\left\{\frac{s(z,z')}{n(z,z')}, p(z,z')\right\}$. It is feasible for a manager of type $z \in Z$ to be matched with workers of type z' at wage c provided that the policy inequality (1) holds. Thus, let

$$P_m(z) = \{(z',c) \in Z \times C : c \geq p(z,z')\}.$$

Self-employed individuals do not hire workers and do not pay wages, hence let

$$P_s(z) = \{(\emptyset, 0)\}.$$

Finally, to complete the description of a market, we must specify preferences for each type of individual. An individual of type z has preferences over his occupation $a \in \{w,s,m\}$ and his match $(\tilde{z},c)$, which belongs to $Z \times C$ when $a \in \{w,m\}$ and $(Z_\emptyset \setminus Z) \times C$ when $a \in \{s\}$.

For each $z \in Z$, define:

$$U_z(a, \tilde{z}, c) = \begin{cases} c & \text{if } a = w \text{ and } (\tilde{z}, c) \in Z \times C, \\ U_s(z) & \text{if } a = s \text{ and } (\tilde{z}, c) \in (Z_\emptyset \setminus Z) \times C, \\ s(z, \tilde{z}) - cn(z, \tilde{z}) & \text{if } a = m \text{ and } (\tilde{z}, c) \in Z \times C. \end{cases}$$

Then the utility function U_z represents the preferences of an individual of type z.

2.3 Stable matchings

A matching in a market E is a measure on $Z \times Z_\emptyset \times C$ that describes the occupational choices of individuals and the way they are matched. Thus, matches are of the form (z, z', c) and the occupational choices are described by the place in the match each individual occupies: if $z' \in Z$, then the first coordinate refers to managers and the second to workers and, when $z' = \emptyset$, the first coordinate refers to a self-employed individual. Then μ roughly specifies how many matches described by each possible (z, z', c) there are.

For convenience, let $X = Z \times C$ and $X_\emptyset = X \cup \{(\emptyset, 0)\}$. Formally, a *matching* is a Borel measure μ on $Z \times X_\emptyset$ such that

1. $\text{supp}(\mu) \cap (Z \times X) \subseteq \text{graph}(P_m)$, and

2. $\nu_M + \nu_S + \nu_W = \nu$

where, for each Borel subset B of Z, $\nu_M(B) = \mu(B \times X)$, $\nu_S(B) = \mu(B \times (X_\emptyset \setminus X))$ and $\nu_W(B) = \int_{Z \times X} 1_B(z') n(z, z') \mathrm{d}\mu(z, z', c)$. Condition 1 requires that the matches that form (i.e. that belong to the support of μ) satisfy the feasibility constraints in P_m (the one in P_s has been imposed in the definition of $X_\emptyset$) and condition 2 requires that everyone in the market is accounted for.

Towards the definition of stability: Roughly, a matching is stable if no one can increase his utility by choosing an occupation and matching with someone from his targets as a member of that occupation. We start by defining an individual's *targets*, those who are better off with him than in their current match.

Given a matching μ, a type $z \in Z$ and an occupation $a \in \{w, m, s\}$, individuals of type z can target certain types and wages (z^*, c) when choosing occupation a provided that someone of type z^* is better off with someone of type z with occupation a and with wage c than in his current match. The set of such (z^*, c) is denoted by $T_z^a(\mu)$.

The targets for the prospective self-employed are the type-wage pairs that are feasible for an unmatched individual: For each $z \in Z$, let $T_z^s(\mu) = \{(\emptyset, 0)\}$.

The targets of prospective managers are as follows. Someone of type z^* can be targeted at wage c by anyone of type z who chooses to be a manager when c is bigger than the payment that such type z^* individual obtains in his current match. If such z^* individual is a worker, then

(a) there exists $(\hat{z}, \hat{c}) \in Z \times C$ such that $(\hat{z}, z^*, \hat{c}) \in \text{supp}(\mu)$ and $c > \hat{c}$,

i.e. $\hat{z}$ is the type of the manager that hires him and $\hat{c}$ is his wage in μ; since such z^* individual is a worker, his utility in the current match is $\hat{c}$ and, hence, $c > \hat{c}$ for him to prefer to be matched with a type z at wage c. If such z^* individual is self-employed, then

(b) $(z^*, \emptyset, 0) \in \text{supp}(\mu)$ and $c > U_s(z^*)$

since $(z^*, \emptyset, 0) \in \text{supp}(\mu)$ indicates that he his self-employed and, hence, $c > U_s(z^*)$ for him to prefer to be matched with a type z at wage c. Finally, when such z^* individual is a manager,

(c) there exists $(z', c') \in X$ such that $(z^*, z', c') \in \text{supp}(\mu)$ and $c > s(z^*, z') - c'n(z^*, z')$.

Thus, for each $z \in Z$, let $T_z^m(\mu)$ be the set of $(z^*, c) \in Z \times C$ such that (a), (b) or (c) hold.[2]

Matches are triples (z, z', c) and consist, in particular, of a manager of type z and workforce described by (z', c) if $(z', c) \in X$ or a self-employed of type z if $(z', c) = (\emptyset, 0) \in X_\emptyset \setminus X$. Stability requires that none of these individuals can increase utility by changing occupation and matching with occupation-specific targets. We decompose this requirement by final occupation and note that it is enough to consider the case where such final occupation is m or s.[3]

[2]It follows by Theorem 2.1 in the supplementary material of CLT that the targets $T_z^w(\mu)$ of prospective workers are not needed for the definition of a stable matching.

[3]This also follows by Theorem 2.1 in the supplementary material of CLT. Intuitively, if an individual can become strictly better off as a worker, the individual who hires him must also become strictly better off as a manager.

The set $S_M(\mu)$ below describes matches (z, z', c) such that no one in it gains by changing his occupation-match pair by matching with his targets while becoming a manager (from being a worker as in condition (ii) below or self-employed as in condition (iii)) or while remaining a manager (as in condition (i) below). Formally, let $S_M(\mu)$ be the set of $(z, z', c) \in Z \times X_\emptyset$ such that, if $(z', c) \in X$, then

(i) there does not exist $(\tilde{z}, \tilde{c}) \in P_m(z)$ such that $(\tilde{z}, \tilde{c}) \in T_z^m(\mu)$ and $s(z, \tilde{z}) - \tilde{c}n(z, \tilde{z}) > s(z, z') - cn(z, z')$,

(ii) there does not exist $(\tilde{z}, \tilde{c}) \in P_m(z')$ such that $(\tilde{z}, \tilde{c}) \in T_{z'}^m(\mu)$ and $s(z', \tilde{z}) - \tilde{c}n(z', \tilde{z}) > c$,

and, if $(z', c) = (\emptyset, 0)$, then

(iii) there does not exist $(\tilde{z}, \tilde{c}) \in P_m(z)$ such that $(\tilde{z}, \tilde{c}) \in T_z^m(\mu)$ and $s(z, \tilde{z}) - \tilde{c}n(z, \tilde{z}) > U_s(z)$.

The set $IR(\mu)$ describes matches (z, z', c) such that no one in it gains by changing his occupation to become self-employed. Formally, let $IR(\mu)$ be the set of $(z, z', c) \in Z \times X_\emptyset$ such that, if $(z', c) \in X$, then $s(z, z') - cn(z, z') \geq U_s(z)$ and $c \geq U_s(z')$. A matching μ is *stable* if $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$.

3 Existence, characterization and computation

In this section we establish general properties of stable matchings. The first one is existence, which is obtained by applying the general existence result in CLT.

Theorem 1 *A stable matching exists.*

The second general result shows that stable matches are characterized by a pair formed by an assignment, describing who matches whom, and an earnings function, describing the payment that each individual receives.

An assignment is a measure over pairs of types $(z, z') \in Z \times Z_\emptyset$ indicating roughly how many matches feature a manager (respectively, self-employed) of type z and $n(z, z')$ workers

of type z' (respectively, the dummy type $\emptyset$), with the convention that $n(z,\emptyset) = 1$. An *assignment* is $\gamma \in M(Z \times Z_\emptyset)$ such that, for each Borel subset B of Z,

$$\gamma(B \times Z_\emptyset) + \int_{Z\times B} n(z,z')\mathrm{d}\gamma(z,z') = \nu(B).^4 \tag{2}$$

Note that $\gamma(B \times Z_\emptyset)$ is the measure of those who are managers or self-employed and have a type in B and that $\int_{Z\times B} n(z,z')\mathrm{d}\gamma(z,z')$ is the measure of those who are workers and have a type in B; thus (2) is simply an accounting identity. Given an assignment γ, we define the set of managers and self-employed individuals as:

$$M = \{z \in Z : (z,z') \in \mathrm{supp}(\gamma) \text{ for some } z' \in Z\} \text{ and}$$
$$S = \{z \in Z : (z,\emptyset) \in \mathrm{supp}(\gamma)\}.$$

An *earnings function* is a lower semicontinuous function $u : Z_\emptyset \to \mathbb{R}_+$ such that $u(\emptyset) = 0$. Thus, u specifies earnings for each type. A difficulty, which is addressed in the definition of stability for pairs (γ, u) is that different workers of the same type might have different earnings depending on the knowledge of the manager for whom they work due to the pay-cap constraint; the correct interpretation of $u(z)$ in this case is the lowest earnings of an individual with knowledge z.

An *outcome* is (γ, u) such that γ is an assignment, u is an earnings function,

1. $u|_M$ is continuous and $u|_S = U_s$, and

2. $u(z) + n(z,z')\max\{u(z'), p(z,z')\} = s(z,z')$ for each $(z,z') \in \mathrm{supp}(\gamma) \cap (Z \times Z)$.

An outcome (γ, u) is *stable* if

3. $u(z) + n(z,z')\max\{u(z'), p(z,z')\} \geq s(z,z')$ and $u(z) \geq U_s(z)$ for each $(z,z') \in Z \times Z$.

Condition 1 requires that the restriction of u to M is continuous and the restriction of u to S is U_s; the latter ensures that the self-employed payment is feasible. Condition 2 says that,

[4] In general, given a metric space T, $M(T)$ denotes the set of finite, Borel measures on T endowed with the weak (narrow) topology (see Varadarajan (1958) for details). Given two metric spaces T and Y and $\mu \in M(T \times Y)$, μ_T (respectively, μ_Y) denotes the marginal of μ on T (respectively, Y).

for matches with a manager of type z and workers of type z' that actually form (i.e. belong to the support of γ), the workers receive the maximum between the earnings specified by u, $u(z')$, and the one specified by the policy constraint, $p(z, z')$, and the manager receives the rest $s(z, z') - n(z, z') \max\{u(z'), p(z, z')\}$; this ensures that their payments are feasible. Condition 3 requires that no type z can hire type z' at a feasible wage, thus either $u(z')$ or $p(z, z')$, whichever is highest, and obtain a profit $s(z, z') - n(z, z') \max\{u(z'), p(z, z')\}$ higher than his earnings, i.e. this is the no-blocking condition. In addition, each individual of type z must earn at least his self-employed payment $U_s(z)$.

Let $f : Z \times Z_\emptyset \to Z \times Z_\emptyset \times \mathbb{R}_+$ be defined by setting, for each $(z, z') \in Z \times Z_\emptyset$,

$$f(z, z') = \begin{cases} (z, z', \max\{u(z'), p(z, z')\}) & \text{if } z' \in Z \\ (z, z', 0) & \text{if } z' = \emptyset. \end{cases}$$

The first two coordinates of f are simply the identity on $Z \times Z_\emptyset$ and the third coordinate specifies the wage that the workers receive in a match (z, z') (in the case $z' = \emptyset$, recall that the "wage" of type $\emptyset$ is zero). This function then allows us to obtain a stable matching from a stable outcome.

Theorem 2 *If μ is a stable matching, then there exists an earning function u such that $(\mu_{Z \times Z_\emptyset}, u)$ is a stable outcome. If (γ, u) is a stable outcome, then $\gamma \circ f^{-1}$ is a stable matching.*

Stable outcomes can be computed using the following method, which requires discretizing the set of types Z and the knowledge distribution ν, and works as follows. An assignment in the discrete case is $\gamma = (\gamma(z, z'))_{z \in Z, z' \in Z_\emptyset}$ and the earnings function is $u = (u(z))_{z \in Z_\emptyset}$. Stable

outcomes are then obtained by solving the following optimization problem (OP henceforth):

$$\min_{\gamma,u} \sum_{z\in Z}\sum_{z'\in Z} \gamma(z,z')\left(u(z)+n(z,z')\max\left\{u(z'),p(z,z')\right\}-s(z,z')\right) \tag{3}$$
$$+\sum_{z\in Z}\gamma(z,\emptyset)\left(u(z)-U_s(z)\right)$$

$$\text{subject to } \gamma(z,z')\geq 0 \text{ for each } z\in Z, z'\in Z_\emptyset, \tag{4}$$

$$\sum_{z'\in Z_\emptyset}\gamma(z,z')+\sum_{\hat{z}\in Z} n(\hat{z},z)\gamma(\hat{z},z)=\nu(z) \text{ for each } z\in Z, \tag{5}$$

$$u(z)+n(z,z')\max\left\{u(z'),p(z,z')\right\}\geq s(z,z') \text{ for each } z\in Z, z'\in Z, \tag{6}$$

$$u(z)\geq U_s(z) \text{ for each } z\in Z. \tag{7}$$

It follows from the definition of a stable outcome that (γ,u) is a stable outcome if and only if (γ,u) solves OP. Thus, Theorems 1 and 2 imply that OP has a solution.

The optimization problem OP can be transformed into a mixed-integer linear programming problem with big-M constraints (MILP henceforth). There are two sources of non-linearity in the OP. The first is the presence of the max operator in the objective function (3) and in constraint (6). To deal with this, in addition to the original variables γ and u, we introduce $w=(w(z,z'))_{z\in Z,z'\in Z}$ and $y=(y(z,z'))_{z\in Z,z'\in Z}$. The vector w is such that, in any solution to the MILP, $w(z,z')=\max\{u(z'),p(z,z')\}$ for each $(z,z')\in Z^2$, which is guaranteed by the choice of $y(z,z')\in\{0,1\}$ and the big-M constraints. Since M denotes the set of manager types, we use N instead in the big-M constraints and set

$$N=\max\left\{\max_{(z,z')\in Z^2} s(z,z'),\bar{c},\max_{z\in Z}U_s(z),\max_{(z,z')\in Z^2} n(z,z')\right\}.$$

The second source of non-linearities comes from the product terms $\gamma(z,z')u(z)$ in the objective function (3). We deal with this issue by exploiting the fact that solutions of OP satisfy the complementary conditions

$$\gamma(z,z')(u(z)+n(z,z')w(z,z')-s(z,z'))=0 \text{ for each } (z,z')\in Z^2, \text{ and}$$
$$\gamma(z,\emptyset)(u(z)-U_s(z))=0 \text{ for each } z\in Z,$$

which we formalize as linear constraints using binary variables $\lambda = (\lambda(z, z'))_{(z,z') \in Z \times Z_\emptyset}$. Once we impose the complementary conditions, the objective function becomes identically equal to zero and this will be the objective function of the MILP; let $L \equiv 0$.

The MILP is:

$$\min_{\gamma,u,w,y,\lambda} L(\gamma, u, w, y, \lambda) \tag{8}$$

$$\text{subject to } \gamma(z, z') \geq 0 \text{ for each } z \in Z, z' \in Z_\emptyset, \tag{9}$$

$$\text{for each } z \in Z : 0 \leq u(z) \leq N, \tag{10}$$

$$\sum_{z' \in Z_\emptyset} \gamma(z, z') + \sum_{\hat{z} \in Z} n(\hat{z}, z)\gamma(\hat{z}, z) = \nu(z) \text{ and} \tag{11}$$

$$u(z) \geq U_s(z), \tag{12}$$

$$\lambda(z, \emptyset) \in \{0, 1\}, \tag{13}$$

$$\gamma(z, \emptyset) \leq \nu(z)\lambda(z, \emptyset), \text{ and} \tag{14}$$

$$u(z) - U_s(z) \leq N(N + 1)(1 - \lambda(z, \emptyset)), \tag{15}$$

$$\text{for each } (z, z') \in Z^2 : 0 \leq w(z, z') \leq N, \tag{16}$$

$$y(z, z') \in \{0, 1\}, \tag{17}$$

$$w(z, z') \geq u(z'), \tag{18}$$

$$w(z, z') \leq u(z') + Ny(z, z'), \tag{19}$$

$$w(z, z') \geq p(z, z'), \tag{20}$$

$$w(z, z') \leq p(z, z') + N(1 - y(z, z')), \tag{21}$$

$$u(z) + n(z, z')\, w(z, z') \geq s(z, z'), \tag{22}$$

$$\lambda(z, z') \in \{0, 1\}, \tag{23}$$

$$\gamma(z, z') \leq \nu(z)\lambda(z, z'), \text{ and} \tag{24}$$

$$u(z) + n(z, z')w(z, z') - s(z, z') \leq N(N + 1)(1 - \lambda(z, z')). \tag{25}$$

Theorem 3 *If (γ, u) solves the OP, then there is (w, y, λ) such that $(\gamma, u, w, y, \lambda)$ solves the MILP. If $(\gamma, u, w, y, \lambda)$ solves the MILP, then (γ, u) solves the OP.*

Note that if $(\gamma, u, w, y, \lambda)$ solves the MILP for a given value of N and (10), (15), (16),

(19), (21) and (25) are nonbinding, then $(\gamma, u, w, y, \lambda)$ also solves the MILP for all larger values of N. Thus, when $\max\left\{\max_{(z,z')\in Z^2} s(z,z'), \bar{c}, \max_{z\in Z} U_s(z), \max_{(z,z')\in Z^2} n(z,z')\right\}$ is large, one may try a smaller value of N (for which the MILP can be solved quickly) and check ex post that the relevant constraints are nonbinding.

4 Examples

We provide several simple examples in this section to illustrate the model and its solutions. They demonstrate the following effects of pay caps:

1. As ξ decreases, the pay cap becomes binding for certain matches and increases the wages of workers.

2. Moreover, certain types eventually become self-employed causing the allocation to adjust.

3. Often, the bottom and the top of the income distribution become worse off and the middle becomes better off.

These points are clearly illustrated in a 3-type example which separates the effects acting through changes in wages and changes in allocation.

All of the examples consider the specification of Garicano and Rossi-Hansberg (2004), i.e. assume then $F(z,z',n) = zn$, $n(z,z') = n(z') = \frac{1}{h(1-z')}$ and $U_s(z) = z$ for each $z, z' \in Z$ and $n \in \mathbb{R}_+$.

Example 1: Let $Z = \{0, \bar{z}\}$ and $\nu(0) > n(0)\nu(\bar{z})$. In the absence of pay-caps, the unique stable outcome (γ, u) is such that type $\bar{z}$ is matched with type 0 and some type 0 are self-employed.[5] Type 0 individuals are indifferent to being workers or self-employed, hence their wage is 0. Thus, $\text{supp}(\gamma) = \{(\bar{z}, 0), (0, \emptyset)\}$, $\gamma(\bar{z}, 0) = \nu(\bar{z})$, $\gamma(0, \emptyset) = \nu(0) - n(0)\nu(\bar{z})$, $u(0) = 0$ and $u(\bar{z}) = \bar{z}n(0)$.

With a pay-cap constraint, managers need to be more knowledgeable than workers in any stable outcome, hence the only possibilities in terms of allocations are either for the

[5]The self-employed of type 0 can also match between themselves and produce zero; so uniqueness is up to this equivalent treatment of type zero individuals.

same one or everyone being self-employed. In the former case, the wage c is the lowest to satisfy the constraint: $c = \frac{\bar{z}n(0)}{\xi+n(0)}$. Thus, $\text{supp}(\gamma) = \{(\bar{z}, 0), (0, \emptyset)\}$, $\gamma(\bar{z}, 0) = \nu(\bar{z})$ and $\mu(0, \emptyset) = \nu(0) - n(0)\nu(\bar{z})$ as before, but now $u(0) = \frac{\bar{z}n(0)}{\xi+n(0)}$ and $u(\bar{z}) = \frac{\xi\bar{z}n(0)}{\xi+n(0)}$. This is possible if and only if managers' payment is at least their self-employed payment $\bar{z}$. Thus, if and only if $\frac{\xi n(0)}{\xi+n(0)} \geq 1 \Leftrightarrow \xi \geq \frac{1}{1-h}$. If $\xi < \frac{1}{1-h}$, then there is a unique stable outcome (γ, u) such that everyone is self-employed: $\text{supp}(\gamma) = \{(\bar{z}, \emptyset), (0, \emptyset)\}$, $\gamma(\bar{z}, \emptyset) = \nu(\bar{z})$, $\gamma(0, \emptyset) = \nu(0)$, $u(0) = 0$ and $u(\bar{z}) = \bar{z}$. Note that as ξ decreases, type 0 workers initially benefit from the pay cap, but eventually become self-employed with zero payoff.

Example 2: Let $Z = \{0, \tilde{z}, \bar{z}\}$. This is the smallest type space in which the wage distribution has a top, middle and bottom part. For concreteness, let $\bar{z} = 0.9$ and $\tilde{z} = 0.1$. Set $h = 0.5$, $\nu(0) = 1 = \nu(\bar{z})$ and $\nu(0.1) = 2.5$, which then implies that $n(0) = 2$, $n(0)\nu(\bar{z}) > \nu(0)$ and $n(\tilde{z})\nu(\bar{z}) < \nu(\tilde{z})$.

The characterization of stable outcomes in Example 2 is as follows: If $\xi \geq 45.45$, then the stable outcome (γ^*, u^*) is the same as in the absence of pay caps, with $\text{supp}(\gamma^*) = \{(0.9, 0), (0.9, 0.1), (0.1, 0)\}$,

$$\gamma^*(0.9, 0) = 0.07,$$
$$\gamma^*(0.9, 0.1) = 0.93,$$
$$\gamma^*(0.1, 0) = 0.43,$$

$u^*(0) = 0.0379$, $u^*(0.1) = 0.1241$ and $u^*(0.9) = 1.7241$. The most knowledgeable individuals are managers and hire both low- and middle-knowledge individuals. The middle ones are both workers and managers, and hire the least knowledgeable in the latter case. There is no self-employment. There is a lot of inequality in the wage distribution, as can be seen from the earnings function u^*.

If $16.36 < \xi < 45.45$, then there is a unique stable outcome (γ, u) defined by $\gamma = \gamma^*$ and

$$u(0) = \tilde{z} - \frac{u(\tilde{z})}{n(0)} < p(\bar{z}, 0),$$
$$u(\tilde{z}) = \bar{z} - \frac{u(\bar{z})}{n(\tilde{z})},$$
$$u(\bar{z}) = \frac{\xi \bar{z} n(0)}{\xi + n(0)}.$$

The assignment is the same as in the no pay-cap economy but now the pay cap binds in the match between $\bar{z}$ and 0. Type 0 workers hired by $\tilde{z}$ are paid $\tilde{c}(0) = u(0)$, but those hired by $\bar{z}$ are paid the pay cap, $c(0) = p(\bar{z}, 0)$. Panel A of Figure 1 describes the change in earnings as ξ varies in the interval $16.36 < \xi \leq 45.45$. As ξ decreases, the earnings of the bottom workers hired by the top managers increase mechanically by the pay cap. For the same reason, the earnings of the top managers decrease. Then the earnings of the middle workers increase since $u(\tilde{z}) = \bar{z} - \frac{u(\bar{z})}{n(\tilde{z})}$. Finally, the earnings of the bottom workers who are hired by the middle type $\tilde{z}$ decrease since $u(0) = \tilde{z} - \frac{u(\tilde{z})}{n(0)}$. In short, the policy creates another income group by splitting those with the least knowledge; the bottom level is now occupied by those with 0 knowledge who are hired by $\tilde{z}$ and their income decreases with ξ. Hence, the policy is bad for the extremes and good for middle.

If $9.82 < \xi < 16.36$, there is a unique stable outcome (γ, u) defined by

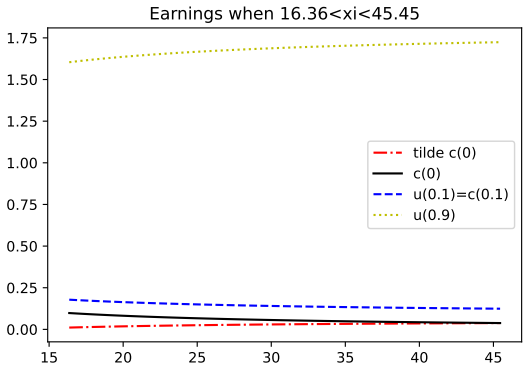
$$\gamma(\bar{z}, \tilde{z}) = 1, \gamma(\tilde{z}, 0) = 0.27, \gamma(0, \emptyset) = 0.44,$$

and

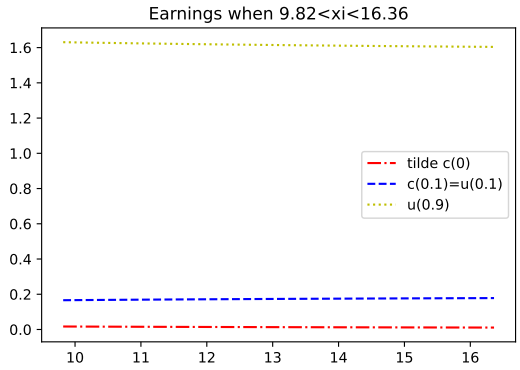
$$u(0) = 0 < p(\tilde{z}, 0),$$
$$u(\tilde{z}) = \frac{\xi \tilde{z} n(0)}{\xi + n(0)},$$
$$u(\bar{z}) = (\bar{z} - u(\tilde{z}))n(\tilde{z}).$$

The assignment changes since now type $\bar{z}$ only hires type $\tilde{z}$ and not type 0.[6] This is the

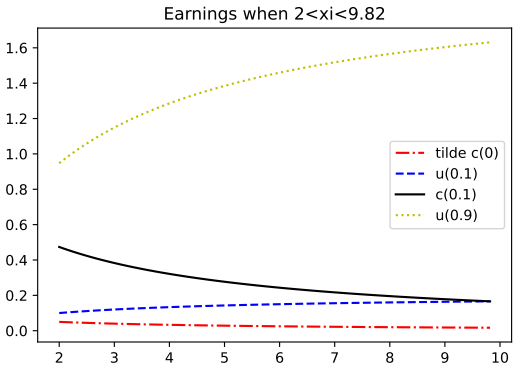
[6]When $\xi = 16.36$, there is a continuum of stable outcomes indexed by the measure of matches between $\bar{z}$ and 0.



(A) Earnings when $16.36 < \xi \leq 45.45$



(B) Earnings when $9.82 < \xi < 16.36$



(C) Earnings when $2 < \xi < 9.82$

Figure 1: Earnings in Example 2

result of the pay cap binding in the match between $\tilde{z}$ and 0, which caps type $\tilde{z}$'s outside option, making them more profitable to hire. Hence the remaining type 0 individuals become self-employed, creating a further group in the earnings distribution which is the worst off in the economy. Type 0 individuals who are hired by $\tilde{z}$ are paid the pay cap $\tilde{c}(0) = p(\tilde{z}, 0)$. Panel B of Figure 1 shows the change in earnings. As ξ decreases, $\tilde{c}(0)$ increases; hence the type 0 workers become better off and type $\tilde{z}$ becomes worse off; this then implies that the top managers become better off since $u(\bar{z}) = (\bar{z} - u(\tilde{z}))n(\tilde{z})$. The earnings of the self-employed of type 0 is zero independently of ξ (not shown in the figure).

If $2 < \xi \leq 9.82$, there is a unique stable outcome (γ, u) defined by

$$\gamma(\bar{z}, \tilde{z}) = 1, \gamma(\tilde{z}, 0) = 0.27, \gamma(0, \emptyset) = 0.44,$$

and

$$u(0) = 0 < p(\tilde{z}, 0),$$
$$u(\tilde{z}) = \frac{\xi \tilde{z} n(0)}{\xi + n(0)} \leq p(\bar{z}, \tilde{z}),$$
$$u(\bar{z}) = \frac{\xi \bar{z} n(\tilde{z})}{\xi + n(\tilde{z})}.$$

There is no change to the assignment but now pay caps in both firms are active. Hence, type $\tilde{z}$ individuals (who are managers in one firm and workers in the other) are split in terms of their earnings, with workers earning the pay cap $c(\tilde{z}) = p(\bar{z}, \tilde{z})$ and managers earning $u(\tilde{z}) = \frac{\xi \tilde{z} n(0)}{\xi + n(0)} \leq p(\bar{z}, \tilde{z})$. As before, type 0 is also split, with workers earning the pay cap $\tilde{c}(0) = p(\tilde{z}, 0)$ and the self-employed earning $u(0) = 0$. Panel C of Figure 1 shows the evolution of earnings: as ξ declines, workers, of type 0 and type $\tilde{z}$ earn more and, thus, managers, of type $\tilde{z}$ and $\bar{z}$, earn less. The worst off, which are the self-employed of type 0, are unchanged (not shown the the figure).

If $1.81 < \xi < 2$, there is a unique stable matching outcome (γ, u) defined by

$$\gamma(\bar{z}, \tilde{z}) = 1, \gamma(\tilde{z}, \emptyset) = 0.27, \gamma(0, \emptyset) = 1,$$

and

$$u(0) = 0,$$
$$u(\tilde{z}) = \tilde{z} < p(\bar{z}, \tilde{z}),$$
$$u(\bar{z}) = \frac{\xi \bar{z} n(\tilde{z})}{\xi + n(\tilde{z})}.$$

Now the type $\tilde{z}$ individuals who were managers become self-employed, earning $\tilde{z}$. The type $\tilde{z}$ workers continue to earn the pay cap $c(0.1) = p(\bar{z}, \tilde{z})$. All type 0 individuals are now self-employed. If $\xi < 1.81$, then everyone is self-employed.[7]

Example 3: Let $Z = [0, 1]$ with ν uniform, $h = 0.8$, and $\xi = 2$. In this example, there

[7]When $\xi = 2$, there is a continuum of stable outcomes indexed by the measure of type 0.1 who are managers and hire type 0; when $\xi = 1.81$ there is also a continuum of stable outcomes indexed by the measure of type 0.9 who are managers and hire type 0.1.

exists a stable outcome where individuals are partitioned into four sets:

$$\{[0, \hat{z}], [\hat{z}, z_1], [z_1, z_2], [z_2, 1]\}.$$

Those in $[0, \hat{z}]$ are self-employed, those in $[\hat{z}, z_1]$ are workers, those in $[z_1, z_2]$ are self-employed, and those in $[z_2, 1]$ are managers. In fact, for individuals in $(\hat{z}, 1]$, their outcome is exactly the same as in a (no pay cap) Garicano and Rossi-Hansberg (2004) economy where the lowest type is $\hat{z}$: the wage of the workers is given by a strictly increasing function $c : [\hat{z}, z_1] \to [0, 1]$, the worker of each manager is given by a strictly increasing function $\phi : [z_2, 1] \to [\hat{z}, z_1]$ such that ϕ is feasible and $\phi(z)$ is the optimal choice of worker for manager z given wages. The cutoff type $\hat{z}$ is defined as the solution to:

$$\frac{\xi n(\hat{z})}{\xi + n(\hat{z})} = 1,$$

which implies that any manager is indifferent between self-employment and hiring $\hat{z}$ at the pay cap and strictly prefers self-employment to hiring any $z' < \hat{z}$. Thus, $\hat{z}$ is hired at exactly the pay cap and all individuals $z' < \hat{z}$ must be self-employed.

Specifically, $\hat{z} = 3/8 = 0.375$, $z_1 = \frac{1}{8}(-2+\sqrt{65}) \approx 0.7578$, and $z_2 = \frac{1}{8}(15-\sqrt{65}) \approx 0.8672$,

$$\phi(z) = 1 - \sqrt{\kappa(z_1) - 2.5z} \text{ for each } z \in [z_2, 1], \tag{26}$$

$$c(z) = \frac{\kappa(z_1)}{2.5} + (1 - z)\left(\frac{2.5z_1 - \kappa(z_1) + (1 - z_1)(z_1 - z)}{2.5(1 - z_1)}\right) \text{ for each } z \in [\hat{z}, z_1], \tag{27}$$

where, for convenience, $\kappa(z_1) = \frac{1}{2}(2z_1^2 - 4z_1 + 7)$. Formally, the stable outcome (γ, u) satisfies:

1. $\gamma = \nu \circ \sigma^{-1}$, where $\sigma : [0, \hat{z}] \cup [z_1, 1] \to Z \times Z_\emptyset$ is such that

$$\sigma(z) = \begin{cases} (z, \emptyset) & \text{if } z \in [0, \hat{z}] \cup [z_1, z_2) \\ (z, \phi(z)) & \text{if } z \in [z_2, 1], \end{cases}$$

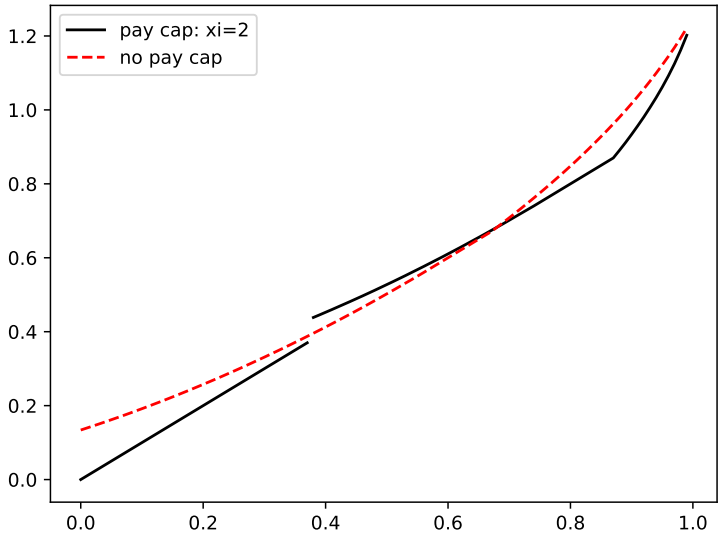


Figure 2: Earnings when $Z = [0, 1]$, ν uniform, $h = 0.8$

$$2.\ u(z) = \begin{cases} z & \text{if } z \in [0, \hat{z}] \cup (z_1, z_2] \\ c(z) & \text{if } z \in (\hat{z}, z_1] \\ (z - c(\phi(z))n(\phi(z)) & \text{if } z \in (z_2, 1]. \end{cases}$$

The meaning of condition 1 is that each $z \in [0, \hat{z}] \cup [z_1, z_2)$ is self-employed and each $z \in [z_2, 1]$ is a manager matched with $\phi(z)$. See Figure 2 for a plot of the earnings function defined by condition 2. Note that the earnings jump at $\hat{z}$, who is hired at the pay cap. Individuals in $(\hat{z}, 1]$ earn exactly the same as they would in a (no pay cap) Garicano and Rossi-Hansberg (2004) economy where the lowest type is $\hat{z}$ instead of 0, and the pay cap is non-binding for all such individuals. This implies that the outcome is stable if we only consider individuals in $(\hat{z}, 1]$. But since individuals in $[0, \hat{z}]$ are not profitable to hire at any wage satisfying the pay cap and get negative utility from hiring individuals better than them, the outcome is also stable for the whole economy – see the Appendix for details.

Relative to the no pay cap benchmark, those in $[0, \hat{z}]$ are worse off since they become self-employed. The worst worker $\hat{z}$ is hired at the pay cap and is better off (at least the lucky ones who are hired – some individuals of type $\hat{z}$ may remain self-employed) and the types just above $\hat{z}$ also benefit from higher wages. The top individuals are worse off under the pay

cap since they pay higher wages.

This example can be generalised. As long as $\xi < 5$, we have $\hat{z} > 0$. Conjecturing that $[0, \hat{z}]$ are self-employed and the outcome for $(\hat{z}, 1]$ is the same as in the Garicano and Rossi-Hansberg (2004) economy with lowest type $\hat{z}$, we can solve for z_1 and z_2. When $\xi > \frac{5}{2}(3 - \sqrt{6}) \approx 1.38$, $z_1 < z_2$ which implies that $u(z_2) = z_2$ and hence $c(\hat{z}) = \frac{z_2 n(\hat{z})}{\xi + n(\hat{z})}$. Thus, the pay cap is binding for $\hat{z}$ and it can be checked by direct computation that the pay cap is indeed satisfied for all $z' > \hat{z}$. Thus, for each $\xi \in (1.38, 5)$, the stable outcome takes the same form as above. When $\xi \geq 5$, the outcome is the same as in the no pay cap economy.

Example 4: Let $Z = [0, \bar{z}]$ and ν be atomless and such that $\text{supp}(\nu) = Z$. The pay-cap constraint affects only firms and can be avoided when their members become self-employed. This is what happens when ξ becomes sufficiently close to zero. In an assignment γ, $\gamma(Z \times Z)$ is the measure of managers and also of firms; there is a positive measure of firms if and only if ξ is not too low.

Theorem 4 *Let γ be a stable assignment. Then $\gamma(Z \times Z) > 0$ if and only if $\xi > \frac{1-\bar{z}h}{1-h}$. If γ is a stable assignment and $\xi \leq \frac{1-\bar{z}h}{1-h}$, then $\gamma = \nu \circ \sigma^{-1}$, where $\sigma(z) = (z, \emptyset)$ for each $z \in Z$.*

5 Quantitative application

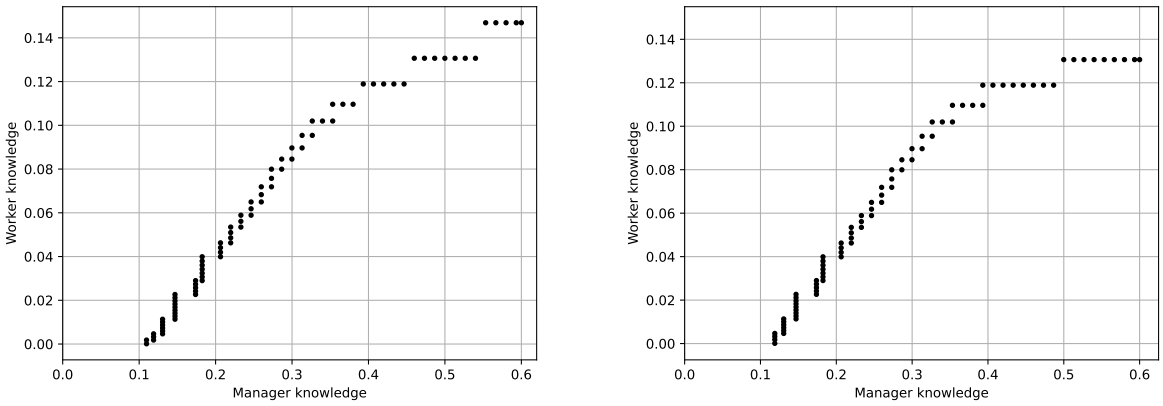
In this section, we quantitatively evaluate the effects of a pay cap using an extension of the Garicano and Rossi-Hansberg (2004) model. This economy is specified by setting $Z = [0.0001, 0.6]$ with types distributed according to a beta distribution $\text{Beta}(\beta_1, \beta_2)$, $n(z, z') = \frac{1}{h(1-z')^\sigma}$, $U_s(z) = z^\alpha$ and $F(z, z', n) = z^\alpha n$. In CLT, this model was calibrated to match the 2008 US wage (5th to 95th percentiles) and establishment size distributions. Here, we use the calibrated parameters $(\beta_1, \beta_2, h, \alpha, \sigma, h) = (1.18, 22.38, 0.92, 33.76, 0.24)$ and simulate the effect of different levels of pay caps.[8]

The results are shown in Figure 3. Lowering the pay cap reduces GDP, the top 1% income share and the bottom 50% income share. The share of the top 5% is slightly higher under a pay cap $\xi = 50$ relative to the unconstrained economy, but this share also decreases as the pay

[8]The codes used to generate the table and figure in this section can be found at https://klaohakunakorn.com/pccodes.zip.

Pay Cap	1	10	20	40	50	∞
GDP	0.0308	0.1032	0.1034	0.1036	0.1037	0.1038
Top 1% income share	0.0466	0.0995	0.1201	0.1270	0.1280	0.1286
Top 5% income share	0.1590	0.2296	0.2475	0.2495	0.2496	0.2477
Bottom 50 % income share	0.2070	0.2354	0.2393	0.2473	0.2484	0.2527

Figure 3: Effect of pay caps on GDP and income shares



(A) Pay cap $\xi = 20$ **(B)** No pay cap

Figure 4: Assignment with and without pay cap

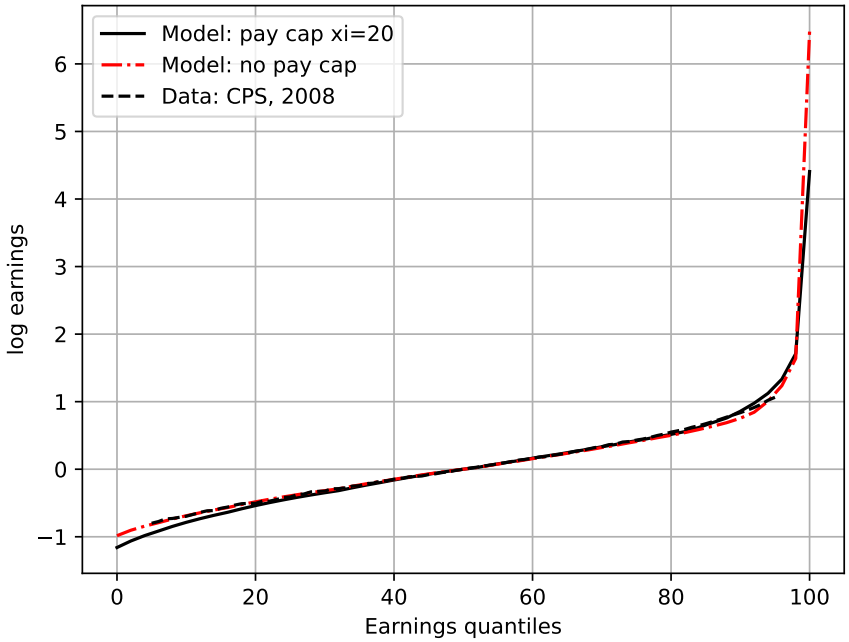


Figure 5: Income distribution with and without pay cap

	$\xi = 10$	$\bar{r} = 11.5$	$\underline{c} = 0.07$	No policy
GDP	0.1032	0.1032	0.1032	0.1038
Top 1% income share	0.0995	0.1243	0.1297	0.1286
Top 5% income share	0.2296	0.2399	0.2483	0.2477
Bottom 50 % income share	0.2354	0.2619	0.2504	0.2527

Figure 6: Comparison of wage policies

cap falls below $\xi = 50$. As in Caselli (2025), the implication is that the middle individuals belonging to the 50th to 95th percentiles are those who benefit from the introduction of a pay cap.

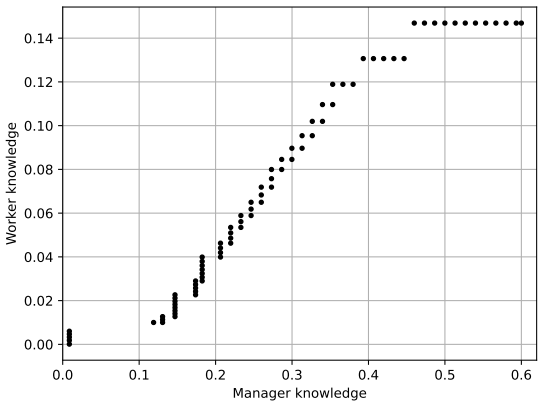
When $\xi = 1$, everyone becomes self-employed and GDP falls significantly. On the other hand, the effect on GDP is small for pay caps $\xi \in \{10, 20, 40\}$ since such pay caps do not generate any self-employment and the assignment of workers to managers remains similar. This can be seen in Figure 4 which plots the assignment of workers to managers for the economy with and without pay cap ($\xi = 20$). The pay cap binds for the top managers who respond by hiring better workers. Since the best worker becomes more knowledgeable, feasibility implies that the worst manager becomes less knowledgeable, i.e. some individuals become both managers and workers. Note that the median individual has knowledge level 0.0379 – given the parameters of the beta distribution, most of its mass is concentrated on individuals with very little knowledge.

Figure 5 shows the wage distribution generated by the model with and without pay cap ($\xi = 20$) compared to the data on log hourly wages in the US, with the log median wage normalized to zero.[9] From the figure, it can be seen that the pay cap decreases the top and bottom income shares, while raising those of the middle.[10]

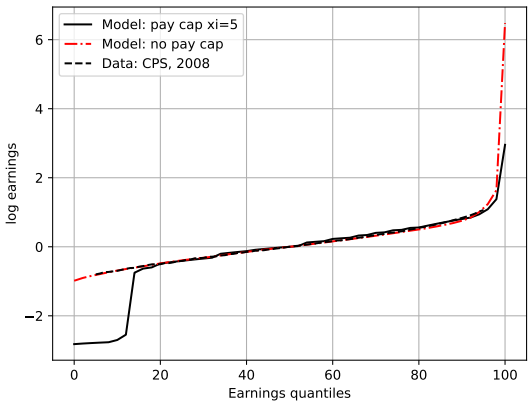
Finally, we compare the pay cap $\xi = 10$ to the minimum wage and maximum rent policy. We choose a minimum wage of 0.065 and a maximum rent of 11.5 to achieve the same level of inefficiency as the pay cap. Figure 6 shows that for the same level of inefficiency, the pay cap is most effective at reducing the top 1% income share, but at the cost of also reducing the bottom 50% income share.

[9]The data come from Acemoglu and Autor (2011) which relies on the Current Population Survey.

[10]Although the model without pay cap was calibrated to match the 5th–95th percentiles of the wage distribution, we note that it generates a top 1% income share that is close to recent estimates reported by Auten and Splinter (2024).



(A) Assignment



(B) Income distribution

	$\xi = 5$	$\bar{r} = 2.45$	$\underline{c} = 0.095$
GDP	0.0990	0.0994	0.0982
Top 1% income share	0.0610	0.1045	0.1364
Top 5% income share	0.1689	0.2322	0.2479
Bottom 50 % income share	0.2431	0.2578	0.2288

(C) Policy comparison

Figure 7: Assignment, income distribution and policy comparison, $\xi = 5$

One concern with the above analysis is that for all pay caps except $\xi = 1$, the introduction of the pay cap has very little effect on the matching and acts primarily through changes in the wage schedule. The effect of more aggressive pay caps may be of interest, especially because the pay cap in our model should be interpreted as the maximum ratio of *average* managerial wage to *average* worker wage (given that our firms consist of only two layers). Since policy proposals typically limit the CEO-median worker pay ratio, a given value of ξ in our model may correspond to a larger value of ξ in such proposals. Thus, Figure 7 shows the assignment, income distribution and policy comparison for $\xi = 5$, a pay cap which is aggressive enough to cause significant changes in the matching: in particular, the worst individuals no longer get hired by the top individuals in the economy and end up having to hire each other at significant cost to themselves. The comparison with the maximum wage/minimum rent policies is qualitatively similar to before, except the minimum wage is now the worst policy at both reducing the top 1% income share and maintaining the bottom 50% income share.

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A Proofs

A.1 Proof of Theorem 1

The proof of Theorem 1 is an application of Theorem 3.1 in the supplementary material of CLT. In the latter, a match is represented by a pair $(z, n(z,z')1_{(z',c))})$ and, in particular, P_m is alternately defined as:

$$P_m(z) = \left\{ n(z,z')1_{(z',c)} : (z',c) \in Z \times C \text{ and } c \geq p(z,z') \right\}$$

and $P_s(z) = \{1_{(\emptyset,0)}\}$. The following lemma is then useful since it shows that a match $(z, n(z,z')1_{(z',c)})$ can be represented in an equivalent way simply by the triple (z,z',c). Recall that $n(z,\emptyset) = 1$; thus, $1_{(\emptyset,0)} = n(z,\emptyset)1_{(\emptyset,0)}$ and, for each $(z,\delta) \in \text{graph}(P_m) \cup \text{graph}(P_s)$, there is $(z',c) \in Z_\emptyset \times C$ such that $\delta = n(z,z')1_{(z',c)}$. Let $g : \text{graph}(P_m) \cup \text{graph}(P_s) \to Z \times Z_\emptyset \times C$ be defined by setting, for each $(z,\delta) \in \text{graph}(P_m) \cup \text{graph}(P_s)$,

$$g(z,\delta) = (z,z',c)$$

where $(z',c) \in Z_\emptyset \times C$ is such that $\delta = n(z,z')1_{(z',c)}$. Let $X = \cup_{z \in Z} P_m(z)$ and $X_\emptyset = X \cup \{1_{(\emptyset,0)}\}$.

Lemma 1 *If $\mu \in M(Z \times X_\emptyset)$, then g is a homeomorphism between* $\text{supp}(\mu)$ *and* $g(\text{supp}(\mu))$.

Proof. It is clear that $g^{-1} : (z,z',c) \mapsto (z, n(z,z')1_{(z',c)})$ is 1-1 and continuous. If $(z, 1_{(\emptyset,0)}) \in \text{supp}(\mu)$, then g is continuous at $(z, 1_{(\emptyset,0)})$ since this is an isolated point of $\text{supp}(\mu)$. Thus, consider $(z,\delta) = n(z,z')1_{(z',c)}$ for some $(z,z',c) \in Z \times Z \times C$ and $\{z_k, \delta_k\}_{k=1}^\infty$ such that, for each $k \in \mathbb{N}$, $\delta_k = n(z_k, z_k')1_{(z_k', c_k)}$ for some $(z_k, z_k', c_k) \in Z \times Z \times C$ and $(z_k, \delta_k) \to (z,\delta)$. Let $\kappa : Z \times Z \times C \to \mathbb{R}$ be defined by setting, for each $(\tilde{z}, \hat{z}, \hat{c}) \in Z \times Z \times C$, $\kappa(\tilde{z}, \hat{z}, \hat{c}) = \max\{|\hat{z} - z|, |\hat{c} - c|\}/n(\tilde{z}, \hat{z})$. Then κ is bounded and continuous, and hence

$$\begin{aligned} |g(z_k,\delta_k) - g(z,\delta)| &= \max\{|z_k - z|, |z_k' - z'|, |c_k - c|\} = \max\left\{|z_k - z|, \int \kappa \mathrm{d}\delta_k\right\} \\ &\to \max\left\{|z - z|, \int \kappa \mathrm{d}\delta\right\} = \max\{|z-z|, |z'-z'|, |c-c|\} = 0. \end{aligned}$$

Thus, g is continuous. ■

Lemma 1 allows us to make the suitable adaptations to the conditions in Theorem 3.1 in CLT for the case where matches are described by elements of $Z \times Z_{\emptyset} \times C$, which we will do freely.

Let E denote the market. Lemma 2 below establishes a property needed to apply Theorem 3.1 in CLT.

Lemma 2 *E is rich.*

Proof. Let $(z, z', c, \mu) \in Z \times Z \times C \times M(Z \times Z_{\emptyset} \times C)$ be such that $(z, z', c) \in \text{supp}(\mu)$, V be open and such that $\Lambda(z, z', c, \mu) \cap V \neq \emptyset$. Let $(z^*, c^*) \in \Lambda(z, z', c, \mu) \cap V$, i.e. $(z^*, c^*) \in P_m(z) \cap (\{(z', c)\} \cup T_z^m(\mu)) \cap V$. Let V_{z^*} and V_{c^*} be neighborhoods of z^* and c^* respectively such that $V_{z^*} \times V_{c^*} \subseteq V$.

We claim that there is an open neighborhood O of (z, z', c, μ) such that, for each $(\hat{z}, \tilde{z}, \tilde{c}, \tilde{\mu}) \in O$, $\Lambda(\hat{z}, \tilde{z}, \tilde{c}, \tilde{\mu}) \cap V \neq \emptyset$. Suppose not; then there is a sequence $\{(z_j, z_j', c_j, \mu_j)\}_{j=1}^{\infty}$ such that $(z_j, z_j', c_j, \mu_j) \to (z, z', c, \mu)$ and

$$\Lambda(z_j, z_j', c_j, \mu_j) \cap V = \emptyset \text{ for each } j \in \mathbb{N}. \tag{28}$$

Suppose first that $(z^*, c^*) \in T_z^m(\mu)$. If $c^* = p(z, z^*)$, then recalling that $p(z, z^*) < \bar{c}$ and letting $\varepsilon > 0$ be such that $c^* + \varepsilon < \bar{c}$ and $(z^*, c^*) \in V$, it follows that $(z^*, c^* + \varepsilon) \in P_m(z) \cap T_z^m(\mu) \cap V$. Hence, we may assume that $c^* > p(z, z^*)$. Let O_z and O_{z^*} be open neighborhoods of z and z^*, respectively, such that $c^* > p(x, y)$ for each $(x, y) \in O_z \times O_{z^*}$. Then, by Lemma 3.9 in CLT, let, for each j sufficiently large, $z_j^* \in V_{z^*} \cap O_{z^*}$ be such that $(z_j^*, c^*) \in T_{z_j}^m(\mu_j)$. Hence, for each j sufficiently large, $c^* > p(z_j, z_j^*)$. Thus, for each j sufficiently large, $(z_j^*, c^*) \in \Lambda(z_j, z_j', c_j, \mu_j) \cap V_{z^*} \times V_{c^*} \subseteq \Lambda(z_j, z_j', c_j, \mu_j) \cap V$. But this contradicts (28).

Suppose next that $(z^*, c^*) = (z', c)$ and $c < \bar{c}$. In this case, letting $\varepsilon > 0$ be such that $c + \varepsilon < \bar{c}$ and $c + \varepsilon \in V_{c^*}$, it follows that $(z^*, c^* + \varepsilon) \in T_z^m(\mu)$. Hence, the case in the previous paragraph applies to $(z^*, c^* + \varepsilon)$ to yield a contradiction to (28).

Finally, consider the case $(z^*, c^*) = (z', \bar{c})$. In this case, let $z_j^* = z_j'$ and $c_j^* = c_j$ for each j. It follows that, for each j sufficiently large, $(z_j^*, c_j^*) = (z_j', c_j)$ and $c_j^* > p(z_j, z_j^*)$ (the

latter since $c_j^* \to \bar{c}$, $p(z_j, z_j^*) \to p(z, z')$ and $p(z, z') < \bar{c}$). Thus, for each j sufficiently large, $(z_j^*, c_j^*) \in \Lambda(z_j, z_j', c_j, \mu_j) \cap V_{z^*} \times V_{c^*} \subseteq \Lambda(z_j, z_j', c_j, \mu_j) \cap V$. But this contradicts (28).

It then follows that there is an open neighborhood O of (z, z', c, μ) such that, for each $(\hat{z}, \tilde{z}, \tilde{c}, \tilde{\mu}) \in O$, $\Lambda(\hat{z}, \tilde{z}, \tilde{c}, \tilde{\mu}) \cap V \neq \emptyset$. Since V is arbitrary, this shows that Λ is lower hemicontinuous at (z, z', c, μ).

An analogous argument shows that Λ_0 is lower hemicontinuous at any (z, μ) (in fact, it suffices to consider the first case of the argument regarding Λ). ■

We have that E is rational since, for each $z \in Z$, preferences of individuals of type z over (a, z', c) are represented by the utility function $U_z(a, z', c)$. The market E is also continuous since (i) $(z, a, z', c) \mapsto U_z(a, z', c)$ is continuous, (ii) P_m is closed and its range, $Z \times C$, is compact and (iii) $P_s \equiv \{(\emptyset, 0)\}$, hence P_s continuous with nonempty compact values. The market E is bounded since $\max_{(z,z') \in Z^2} n(z, z') < \infty$ and is rich by Lemma 2. It then follows by Theorem 3.1 in CLT that E has a stable matching.

A.2 Proof of Theorem 2

Part 1: If μ is a stable matching, then there exists an earning functions u such that $(\mu_{Z \times Z_\emptyset}, u)$ is a stable outcome.

We start by showing the existence of u. This is done by first considering the sets of manager, worker and self-employed types separately. Let

$$M = \{z \in Z : (z, z', c) \in \text{supp}(\mu) \text{ for some } (z', c) \in Z \times C\}, \tag{29}$$

$$S = \{z \in Z : (z, \emptyset, 0) \in \text{supp}(\mu)\} \text{ and} \tag{30}$$

$$W = \{z \in Z : (\hat{z}, z, c) \in \text{supp}(\mu) \text{ for some } (\hat{z}, c) \in Z \times C\} \tag{31}$$

denote the set of manager, self-employed and workers types, respectively.

The following is the equal treatment property for managers.

Lemma 3 *If $z, z', \hat{z} \in Z$ and $w', \hat{w} \in [0, 1]$ are such that $(z, z', w'), (z, \hat{z}, \hat{w}) \in \text{supp}(\mu)$, then $U_z(m, z', w') = U_z(m, \hat{z}, \hat{w})$.*

Proof. If $U_z(m, z', w') > U_z(m, \hat{z}, \hat{w})$, then, letting $\varepsilon > 0$ be such that $w' + \varepsilon < \bar{c}$

(note that $w' < \bar{c}$ since $U_z(m, z', w') > U_z(m, \hat{z}, \hat{w}) \geq U_s(z) \geq 0$) and $s(z, z') - (w' + \varepsilon)n(z, z') > U_z(m, \hat{z}, \hat{w})$, it follows that $(z', w'+\varepsilon) \in T_z^m(\mu)$, $w'+\varepsilon > p(z, z')$ and $U_z(m, z', w'+\varepsilon) > U_z(m, \hat{z}, \hat{w})$, a contradiction to the stability of μ. Thus, $U_z(m, z', w') \leq U_z(m, \hat{z}, \hat{w})$ and an analogous argument shows that $U_z(m, z', w') \geq U_z(m, \hat{z}, \hat{w})$; hence, $U_z(m, z', w') = U_z(m, \hat{z}, \hat{w})$. ■

Define $u_M : M \to \mathbb{R}_+$ by setting, for each $z \in M$, $u_M(z) = U_z(m, z', w')$, where $(z', w') \in Z \times \mathbb{R}_+$ is such that $(z, z', w') \in \text{supp}(\mu)$. Lemma 3 implies that the function u_M is well-defined.

Lemma 4 *u_M is continuous.*

Proof. Let $z \in M$ and $\{z_k\}_{k=1}^{\infty}$ be such that $z_k \to z$ and $z_k \in M$ for each k. We will show that $\lim_k u_M(z_k) = u_M(z)$ by showing that every subsequence of $\{u_M(z_k)\}_{k=1}^{\infty}$ has a further subsequence converging to $u_M(z)$.

Let $\{z_{k_j}\}_{j=1}^{\infty}$ and $\{u_M(z_{k_j})\}_{j=1}^{\infty}$ be subsequences of $\{z_k\}_{k=1}^{\infty}$ and $\{u_M(z_k)\}_{k=1}^{\infty}$ respectively. Then let $\{(z'_j, w_j)\}_{j=1}^{\infty}$ be such that $(z_{k_j}, z'_j, w_j) \in \text{supp}(\mu)$. Since $Z \times C$ is compact, there is a convergent subsequence $\{(z'_{j_l}, w_{j_l})\}_{l=1}^{\infty}$ of $\{(z'_j, w_j)\}_{k=1}^{\infty}$; let $(z', w) = \lim_l (z'_{j_l}, w_{j_l})$. Then $u_M(z_{k_{j_l}}) = s(z_{k_{j_l}}, z'_{j_l}) - w_{j_l} n(z_{k_{j_l}}, z'_{j_l}) \to s(z, z') - wn(z, z')$, $(z, z', w) \in \text{supp}(\mu)$ and, by Lemma 3, $u_M(z) = s(z, z') - wn(z, z') = \lim_l u_M(z_{k_{j_l}})$. ■

Define $u_S : S \to \mathbb{R}_+$ by setting, for each $z \in S$, $u_S(z) = U_s(z)$. Then u_S is continuous.

Let

$$U = \{(z, z', w) \in \text{supp}(\mu) : w > p(z, z')\},$$
$$R = \{(z, z', w) \in \text{supp}(\mu) : w = p(z, z')\}.$$

In U, equal treatment holds for workers.

Lemma 5 *If $z, \hat{z}, z' \in Z$ and $w, \hat{w} \in C$ are such that $(z, z', w) \in \text{supp}(\mu)$ and $(\hat{z}, z', \hat{w}) \in U$, then $\hat{w} \leq w$. If $(z, z', w) \in U$, then $w = \hat{w}$; if $(z, z', w) \in R$, then $w = p(z, z')$.*

Proof. Suppose that $(z, z', w) \in \text{supp}(\mu)$, $(\hat{z}, z', \hat{w}) \in U$ and $w < \hat{w}$. Since $\hat{w} > p(\hat{z}, z')$, there is $\varepsilon > 0$ such that $w < \hat{w} - \varepsilon$ and $\hat{w} - \varepsilon > p(\hat{z}, z')$. Then $(z', \hat{w} - \varepsilon) \in T_{\hat{z}}^m(\mu)$ and $s(\hat{z}, z') - (\hat{w} - \varepsilon)n(\hat{z}, z') > s(\hat{z}, z') - \hat{w}n(\hat{z}, z') = U_{\hat{z}}(m, z', \hat{w})$, a contradiction to the stability of μ. Thus, $w \geq \hat{w}$.

If $(z, z', w) \in U$, then the above shows that $w \leq \hat{w}$; hence, $w = \hat{w}$.

If $(z, z', w) \in R$, then $w = p(z, z')$ by definition. ■

Define $u_W : W \times M \to C$ by setting, for each $(z, \hat{z}) \in W \times M$, $u_W(z, \hat{z}) = w$, where $w \in C$ is such that $(\hat{z}, z, w) \in \text{supp}(\mu)$. Lemma 5 implies that the function u_W is well-defined. Indeed, if $(\hat{z}, z, w), (\hat{z}, z, \hat{w}) \in \text{supp}(\mu)$, then we may assume that $\hat{w} \geq w$ and, hence, $w = \hat{w}$ if $(\hat{z}, z, \hat{w}) \in U$; if $(\hat{z}, z, \hat{w}) \in R$, then $\hat{w} = p(\hat{z}, z)$ and $p(\hat{z}, z) = \hat{w} \geq w \geq p(\hat{z}, z)$ implies that $w = \hat{w}$.

Lemma 6 *u_W is continuous.*

Proof. Let $z \in M$, $\hat{z} \in M$, $\{z_k\}_{k=1}^{\infty}$ and $\{\hat{z}_k\}_{k=1}^{\infty}$ be such that $z_k \to z$, $\hat{z}_k \to \hat{z}$, $z_k \in W$ and $\hat{z}_k \in M$ for each k. We will show that $\lim_k u_W(z_k, \hat{z}_k) = u_W(z, \hat{z})$ by showing that every subsequence of $\{u_W(z_k, \hat{z}_k)\}_{k=1}^{\infty}$ has a further subsequence converging to $u_W(z, \hat{z})$.

Let $\{(z_{k_j}, \hat{z}_{k_j})\}_{j=1}^{\infty}$ and $\{u_W(z_{k_j}, \hat{z}_{k_j})\}_{j=1}^{\infty}$ be subsequences of $\{(z_k, \hat{z}_k)\}_{k=1}^{\infty}$ and $\{u_W(z_k)\}_{k=1}^{\infty}$ respectively. Then $(\hat{z}_{k_j}, z_{k_j}, u_W(z_{k_j}, \hat{z}_{k_j})) \in \text{supp}(\mu)$. Since C is compact, there is a convergent subsequence $\{u_W(z_{k_{j_l}}, \hat{z}_{k_{j_l}})\}_{l=1}^{\infty}$ of $\{u_W(z_{k_j}, \hat{z}_{k_j})\}_{k=1}^{\infty}$; let $w = \lim_l u_W(z_{k_{j_l}}, \hat{z}_{k_{j_l}})$. Then $(\hat{z}, z, w) = \lim_l (z_{k_{j_l}}, \hat{z}_{k_{j_l}}, u_W(z_{k_{j_l}}, \hat{z}_{k_{j_l}}))$ and, hence, $(\hat{z}, z, w) \in \text{supp}(\mu)$. Thus, $\lim_l u_W(z_{k_{j_l}}, \hat{z}_{k_{j_l}}) = u_W(z, \hat{z})$. ■

For each $z \in W$, let $M(z) = \{\hat{z} \in M : (\hat{z}, z, w) \in \text{supp}(\mu) \text{ for some } w \in C\}$. Note that $M(z)$ is closed since $\text{supp}(\mu)$ is closed and C is compact. Let $\underline{u}_W : W \to C$ be defined by setting, for each $z \in W$,

$$\underline{u}_W(z) = \min_{\hat{z} \in M(z)} u_W(z, \hat{z}).$$

Lemma 7 *$\underline{u}_W$ is lower semi-continuous.*

Proof. Let $\{z_k\}_{k=1}^{\infty}$ be such that $z_k \to z$ and let $u = \liminf_k \underline{u}_W(z_k)$. Then there is a subsequence $\{z_{k_j}\}_{j=1}^{\infty}$ of $\{z_k\}_{k=1}^{\infty}$ such that $\underline{u}_W(z_{k_j}) \to u$. Let $\{\hat{z}_j\}_{j=1}^{\infty}$ be such that $(\hat{z}_j, z_{k_j}, \underline{u}_W(z_{k_j})) \in \text{supp}(\mu)$. Since Z is compact, we may assume that $\{\hat{z}_j\}_{j=1}^{\infty}$ converges; let $\hat{z} = \lim_j \hat{z}_j$. Then $(\hat{z}, z, u) \in \text{supp}(\mu)$, hence $\hat{z} \in M(z)$, $u = u_W(z, \hat{z})$ and $\underline{u}_W(z) \leq u_W(z, \hat{z}) = u = \liminf_k \underline{u}_W(z_k)$. ■

Lemma 8 *M, W and S are closed.*

Proof. The closedness of M follows because $X = Z \times C$ is compact and $\text{supp}(\mu)$ is closed. The set W is closed since $Z \times X$ is compact and $\text{supp}(\mu)$ is closed. The closedness of S follows because $\text{supp}(\mu)$ is closed and $\text{supp}(\mu) \cap (Z \times (X_\emptyset \setminus X)) \subseteq Z \times \{(\emptyset, 0)\}$. ■

Lemma 9 *$Z = M \cup S \cup W$.*

Proof. Let $K = M \cup S \cup W$ and note that we have that $K \subseteq Z$ by definition.

Conversely, note first that K is closed by Lemmas 8. Furthermore, letting $\pi(\text{supp}(\mu))$ be the projection of $\text{supp}(\mu)$ in Z, we have that $\text{supp}(\mu) \subseteq \pi(\text{supp}(\mu)) \times X_\emptyset = (M \cup S) \times X_\emptyset \subseteq K \times X_\emptyset$ and, hence,

$$\mu(K \times X_\emptyset) \geq \mu(\text{supp}(\mu)) = \mu(\text{supp}(\mu) \cap (Z \times X_\emptyset)) = \mu(Z \times X_\emptyset).$$

Furthermore, $\text{supp}(\mu) \cap (Z \times X) \subseteq Z \times W \times C \subseteq Z \times K \times C$ by the definitions of W and K and, hence,

$$\begin{aligned}
&\nu(K) = \mu(K \times X) + \mu(K \times (X_\emptyset \setminus X)) + \int_{Z \times K \times C} n(z, z') \mathrm{d}\mu(z, z', w) \\
&= \mu(K \times X_\emptyset) + \int_{Z \times K \times C} n(z, z') \mathrm{d}\mu(z, z', w) \\
&\geq \mu(Z \times X_\emptyset) + \int_{(Z \times X) \cap \text{supp}(\mu)} n(z, z') \mathrm{d}\mu(z, z', w) = \nu(Z).
\end{aligned}$$

It then follows by the definition of $\text{supp}(\nu)$ that $Z = \text{supp}(\nu) \subseteq K$. ■

Let $\underline{u}_W(z) = \infty$ for each $z \notin W$ and, for each $A \in \{M, S\}$, define $\underline{u}_A : Z \to \overline{\mathbb{R}}_+$ by setting, for each $z \in Z$,

$$\underline{u}_A(z) = \begin{cases} u_A(z) & \text{if } z \in A, \\ \infty & \text{otherwise.} \end{cases}$$

Then define $u : Z \to \mathbb{R}_+$ by setting, for each $z \in Z$,

$$u(z) = \min\{\underline{u}_W(z), \underline{u}_S(z), \underline{u}_M(z)\}.$$

It follows that u is real-valued by Lemma 9 and that u is lower semi-continuous by Lemmas 4 and 7.

Lemma 10 *The following holds: (i)* $u_M(z) \leq \underline{u}_W(z)$ *if* $z \in M \cap W$*, (ii)* $u_M(z) = u_S(z)$ *if* $z \in M \cap S$*, (iii)* $u_S(z) \leq \underline{u}_W(z)$ *if* $z \in W \cap S$*, (iv)* $u_M(z) = \underline{u}_W(z)$ *if* $z \in M \cap W$ *and* $(\hat{z}, z, c) \in U$ *for some* $(\hat{z}, c) \in Z \times C$*, and (v)* $u_S(z) = \underline{u}_W(z)$ *if* $z \in W \cap S$ *and* $(\hat{z}, z, c) \in U$ *for some* $(\hat{z}, c) \in Z \times C$*.*

Proof. Let $z \in M \cap W$. Suppose that $u_M(z) > \underline{u}_W(z)$ and let $(z', w) \in Z \times C$ be such that $(z, z', w) \in \text{supp}(\mu)$. Let $\varepsilon > 0$ be such that $w + \varepsilon < \bar{c}$ (note that $w < \bar{c}$ since $u_M(z) > \underline{u}_W(z) \geq 0$) and $s(z, z') - (w + \varepsilon)n(z, z') > \underline{u}_W(z)$. Then $(z', w + \varepsilon) \in T_z^m(\mu)$ and $U_z(m, z', w + \varepsilon) > \underline{u}_W(z)$, contradicting the stability of μ. This establishes (i).

Let $z \in M \cap S$. The stability of μ implies that $u_M(z) \geq u_S(z)$. If $u_M(z) > u_S(z)$, then let $(z', w) \in Z \times C$ be such that $(z, z', w) \in \text{supp}(\mu)$ and $\varepsilon > 0$ be such that $w + \varepsilon < \bar{c}$ and $s(z, z') - (w+\varepsilon)n(z, z') > u_S(z)$. It then follows that $(z', w+\varepsilon) \in T_z^m(\mu)$ and $U_z(m, z', w+\varepsilon) > u_S(z)$, contradicting the stability of μ. Thus, (ii) follows.

Let $z \in W \cap S$. The stability of μ implies that $\underline{u}_W(z) \geq u_S(z)$. This establishes (iii).

To show (iv) and (v), let $z \in (M \cup S) \cap W$ and $(\hat{z}, c) \in Z \times C$ be such that $(\hat{z}, z, c) \in U$ and $u < \underline{u}_W(z)$ where $u = u_M(z)$ if $z \in M$ and $u = u_S(z)$ otherwise. Then $c = \underline{u}_W(z)$ by Lemma 5 and $c > p(\hat{z}, z)$. Let $\varepsilon > 0$ be such that $c - \varepsilon > u \geq 0$ (note that $c > u \geq U_s(z) \geq 0$) and $c - \varepsilon > p(\hat{z}, z)$. Then $(z, c - \varepsilon) \in P_m(\hat{z}) \cap T_{\hat{z}}^m(\mu)$ and $s(\hat{z}, z) - (c - \varepsilon)n(\hat{z}, z) > s(\hat{z}, z) - cn(\hat{z}, z)$, contradicting the stability of μ. ■

Let $g : \text{supp}(\mu) \to Z \times Z_\emptyset$ be defined by setting, for each $(z, z', c) \in \text{supp}(\mu)$, $g(z, z, c) = (z, z')$. Then g is a homeomorphism between $\text{supp}(\mu)$ and $g(\text{supp}(\mu)) \subseteq Z \times Z_\emptyset$ since g is the restriction of the projection of $Z \times Z_\emptyset \times C$ onto $Z \times Z_\emptyset$, C is compact and $c = u_W(z', z)$ for each $(z, z', c) \in \text{supp}(\mu)$. Furthermore, $\mu \circ g^{-1} = \mu_{Z \times Z_\emptyset}$ since, for each Borel subset B of $Z \times Z_\emptyset$,

$$\mu(g^{-1}(B)) = \mu(\text{supp}(\mu) \cap (B \times C)) = \mu(B \times C) = \mu_{Z \times Z_\emptyset}(B).$$

Let $\gamma = \mu_{Z \times Z_\emptyset}$ and extend the domain of u to $Z_\emptyset$ by defining $u(\emptyset) = 0$. It follows that u is an earnings function. We first show that γ is an assignment. Let B be a Borel subset of Z;

then

$$\begin{aligned}
&\gamma(B \times Z_\emptyset) + \int_{Z\times B} n(z,z')\mathrm{d}\gamma(z,z') = \\
&\gamma(B \times Z_\emptyset) + \int_{Z\times Z} 1_B(z')n(z,z')\mathrm{d}\gamma(z,z') = \\
&\mu(g^{-1}(B \times Z_\emptyset)) + \int_{Z\times X} 1_{Z\times B}(g(z,z',c))n(g(z,z',c))\mathrm{d}\mu(z,z',c) = \\
&\mu(B \times X_\emptyset) + \int_{Z\times X} 1_B(z')n(z,z')\mathrm{d}\mu(z,z',c) = \nu(B),
\end{aligned}$$

where the last equality follows because μ is a matching.

We next claim that (γ, u) is an outcome. Condition 1 follows immediately from Lemma 4, Lemma 10 and the definition of u_S.

We next establish condition 2. Note first that $\mathrm{supp}(\gamma) = g(\mathrm{supp}(\mu))$ by Carmona and Laohakunakorn (2024, Lemma 1) since g is an homeomorphism between two compact spaces. Let $(z,z') \in \mathrm{supp}(\gamma)$. Since $\mathrm{supp}(\gamma) \subseteq g(\mathrm{supp}(\mu))$, $(z,z',u_W(z',z)) \in \mathrm{supp}(\mu)$. If $z' \in Z$, then $z \in M$ and $z' \in W$. Thus, $u_W(z',z) \geq p(z,z')$ and

$$u_M(z) + n(z,z')u_W(z',z) = s(z,z') - u_W(z',z)n(z,z') + u_W(z',z)n(z,z') = s(z,z').$$

We have that $u_M(z) = u(z)$. If $u_W(z',z) > p(z,z')$, then $(z,z',u_W(z',z)) \in U$ and, by Lemmas 5 and 10, $u(z') = u_W(z',z) > p(z,z')$. Thus, $\max\{u(z'),p(z,z')\} = u(z')$ and, hence,

$$u(z) + n(z,z')\max\{u(z'),p(z,z')\} = u_M(z) + n(z,z')u_W(z',z) = s(z,z').$$

If $u_W(z',z) = p(z,z')$, then $u(z') \leq u_W(z',z) = p(z,z')$. Thus, $\max\{u(z'),p(z,z')\} = p(z,z') = u_W(z',z)$ and, hence,

$$u(z) + n(z,z')\max\{u(z'),p(z,z')\} = u_M(z) + n(z,z')u_W(z',z) = s(z,z').$$

We finally establish condition 3, i.e. show that (γ, u) is stable. Let $(z,z') \in Z \times Z$. We have that $u(z) \geq U_s(z)$ since μ is stable. We next show that $u(z) + n(z,z')\max\{u(z'),p(z,z')\} \geq$

$s(z,z')$ holds. Suppose not; then $u(z)+n(z,z')\max\{u(z'),p(z,z')\}<s(z,z')$. If $u(z')\geq p(z,z')$, then let $\varepsilon>0$ be such that $u(z')+\varepsilon<\bar{c}$ (note that $u(z')<\bar{c}$ since $0\leq U_s(z)\leq u(z)<s(z,z')-u(z')n(z,z')$) and $u(z)+n(z,z')(u(z')+\varepsilon)<s(z,z')$; then $(z',u(z')+\varepsilon)\in T_z^m(\mu)$ and $s(z,z')-(u(z')+\varepsilon)n(z,z')>u(z)$, a contradiction to the stability of μ. If $u(z')<p(z,z')$, then $(z',p(z,z'))\in T_z^m(\mu)$ and $s(z,z')-p(z,z')n(z,z')>u(z)$, a contradiction to the stability of μ.

Part 2: If (γ,u) is a stable outcome, then $\gamma\circ f^{-1}$ is a stable matching.

We start by showing that f is a homeomorphism.

Lemma 11 *If (γ,u) is stable, then f is a homeomorphism between* $\text{supp}(\gamma)$ *and* $f(\text{supp}(\gamma))$.

Proof. It is clear that f is injective. Since $\text{supp}(\gamma)$ is compact, it suffices to show that $f:\text{supp}(\gamma)\to f(\text{supp}(\gamma))$ is continuous because then $f^{-1}:f(\text{supp}(\gamma))\to\text{supp}(\gamma)$ is continuous too. Writing $f=(f_1,f_2,f_3)$, note that (f_1,f_2) is the identity on $Z\times Z_\emptyset$, which is continuous; hence, it suffices to show that f_3 is continuous. To this end, let $(z,z')\in\text{supp}(\gamma)$ and $\{(z_k,z_k')\}_{k=1}^\infty$ be such that $(z_k,z_k')\in\text{supp}(\gamma)$ for each $k\in\mathbb{N}$ and $(z_k,z_k')\to(z,z')$.

If $z'=\emptyset$, then $z_k'=\emptyset$ for each k sufficiently large and, thus, $f_3(z_k,z_k')=0=f(z,z')$. Thus, consider $z'\in Z$ and $z_k'\in Z$ for each k sufficiently large. Then

$$f_3(z,z')=\frac{s(z,z')-u_M(z)}{n(z,z')}\text{ and }f_3(z_k,z_k')=\frac{s(z_k,z_k')-u_M(z_k)}{n(z_k,z_k')}$$

for each k sufficiently large. Since u_M, s and n are continuous, it follows that $f_3(z,z')=\lim_k f_3(z_k,z_k')$. ■

Let $\mu=\gamma\circ f^{-1}$. We first show that μ is a matching. Let B be a Borel subset of Z; then $f^{-1}(B\times X_\emptyset)=B\times Z_\emptyset$ and

$$\begin{aligned}
&\mu(B\times X_\emptyset)+\int_{Z\times X}1_B(z')n(z,z')\mathrm{d}\mu(z,z',c)=\\
&\gamma\circ f^{-1}(B\times X_\emptyset)+\int_{Z\times X}1_B(z')n(z,z')\mathrm{d}\gamma\circ f^{-1}(z,z',c)=\\
&\gamma(B\times Z_\emptyset)+\int_{Z\times Z_\emptyset}1_B(z')n(z,z')\mathrm{d}\gamma(z,z')=\\
&\gamma(B\times Z_\emptyset)+\int_{Z\times B}n(z,z')\mathrm{d}\gamma(z,z')=\nu(B).
\end{aligned}$$

Furthermore, if $(z, z', c) \in \text{supp}(\mu) \cap (Z \times X)$ and $f^{-1}(z, z', c) = (z, z')$, then $(z, z') \in \text{supp}(\gamma)$ since $f^{-1}(\text{supp}(\mu)) = \text{supp}(\gamma)$ by Carmona and Laohakunakorn (2024, Lemma 1). Then $\text{supp}(\mu) = f(\text{supp}(\gamma))$ and, hence, $c = \max\{u(z'), p(z, z')\}$. Thus, $\text{supp}(\mu) \cap (Z \times X) \subseteq \text{graph}(P_m)$.

We conclude this proof by showing that μ is stable. Let $(z, z', c) \in \text{supp}(\mu)$; then $(z, z') = f^{-1}(z, z', c) \in \text{supp}(\gamma)$ and $c = \max\{u(z'), p(z, z')\}$ if $z' \in Z$ and $c = 0$ if $z' = \emptyset$.

Consider first the case where $z' \in Z$. We have that $U_{z'}(w, z, c) = c \geq u(z')$ and $U_z(m, z', c) = s(z, z') - cn(z, z') = u(z)$ by condition 2 in the definition of a stable outcome. Hence, $U_{z'}(w, z, c) \geq U_s(z')$ and $U_z(m, z', c) \geq U_s(z)$ by condition 3 in the definition of a stable outcome. Thus, $(z, z', c) \in IR(\mu)$ and, hence, $\text{supp}(\mu) \subseteq IR(\mu)$.

We next show that $\text{supp}(\mu) \subseteq S_M(\mu)$. Note first that if $(\tilde{z}, \tilde{c}) \in P_m(x) \cap T_x^m(\mu)$ for some $x \in \{z, z'\}$, then $\tilde{c} \geq \max\{u(\tilde{z}), p(x, \tilde{z})\}$. Since $(\tilde{z}, \tilde{c}) \in T_x^m(\mu)$, it follows that $(\tilde{z}, z^*, c^*) \in \text{supp}(\mu)$, or $(\tilde{z}, \emptyset, 0) \in \text{supp}(\mu)$ or $(z^*, \tilde{z}, c^*) \in \text{supp}(\mu)$ for some $(z^*, c^*) \in Z \times C$. Corresponding to each of these three cases, we have that $\tilde{c} > U_{\tilde{z}}(m, z^*, c^*) = u(\tilde{z})$ (by condition 2), or $\tilde{c} > U_{\tilde{z}}(s, \emptyset, 0) = u(\tilde{z})$ (by condition 1) or $\tilde{c} > c^* = \max\left\{u(\tilde{c}), \frac{s(z^*, \tilde{z})}{\xi + n(z^*, \tilde{z})}\right\}$ (by $(z^*, \tilde{z}, c^*) = f(z^*, \tilde{z})$); in either case, $\tilde{c} > u(\tilde{z})$. It follows from $(\tilde{z}, \tilde{c}) \in P_m(x)$ that $\tilde{c} \geq p(x, \tilde{z})$; hence, $\tilde{c} \geq \max\{u(\tilde{z}), p(x, \tilde{z})\}$.

Suppose that there is $(\tilde{z}, \tilde{c}) \in P_m(z) \cap T_z^m(\mu)$ such that $U_z(m, \tilde{z}, \tilde{c}) > U_z(m, z', c)$. Then $\tilde{c} \geq \max\{u(\tilde{z}), p(z, \tilde{z})\}$ by the above claim. Thus,

$$\begin{aligned} s(z, \tilde{z}) - n(z, \tilde{z}) \max\{u(\tilde{z}), p(z, \tilde{z})\} &\geq s(z, \tilde{z}) - \tilde{c} n(z, \tilde{z}) \\ &= U_z(m, \tilde{z}, \tilde{c}) > U_z(m, z', c) = u(z), \end{aligned}$$

a contradiction to condition 3 in the definition of a stable outcome.

Suppose next that there is $(\tilde{z}, \tilde{c}) \in P_m(z') \cap T_{z'}^m(\mu)$ such that $U_{z'}(m, \tilde{z}, \tilde{c}) > U_{z'}(w, z, c)$. Then, by the above claim, $\tilde{c} \geq \max\{u(\tilde{z}), p(z', \tilde{z})\}$. Thus,

$$\begin{aligned} s(z', \tilde{z}) - n(z', \tilde{z}) \max\{u(\tilde{z}), p(z', \tilde{z})\} &\geq s(z', \tilde{z}) - \tilde{c} n(z', \tilde{z}) \\ &= U_{z'}(m, \tilde{z}, \tilde{c}) > U_{z'}(w, z, c) = c \geq u(z'), \end{aligned}$$

a contradiction to condition 3 in the definition of a stable outcome.

Finally, consider the case where $z' = \emptyset$. Then $U_z(s, \emptyset, 0) = u(z)$ by condition 1 of the definition of a stable outcome. Suppose that there is $(\tilde{z}, \tilde{c}) \in P_m(z) \cap T_z^m(\mu)$ such that $U_z(m, \tilde{z}, \tilde{c}) > U_z(s, \emptyset, 0) = u(z)$. Then, by the above claim, $\tilde{c} \geq \max\{u(\tilde{z}), p(z, \tilde{z})\}$. Thus,

$$\begin{aligned} s(z, \tilde{z}) - n(z, \tilde{z}) \max\{u(\tilde{z}), p(z, \tilde{z})\} &\geq s(z, \tilde{z}) - \tilde{c} n(z, \tilde{z}) \\ &= U_z(m, \tilde{z}, \tilde{c}) > U_z(s, \emptyset, 0) = u(z), \end{aligned}$$

a contradiction to condition 3 in the definition of a stable outcome.

It then follows that $(z, z', c) \in S_M(\mu)$ and, hence, $\text{supp}(\mu) \subseteq S_M(\mu)$. Hence, $\text{supp}(\mu) \subseteq S_M(\mu) \cap IR(\mu)$ and it follows that μ is stable.

A.3 Proof of Theorem 3

Part 1: If (γ, u) solves OP, then there is (w, y, λ) such that $(\gamma, u, w, y, \lambda)$ solves MILP.

Let (γ, u) solve OP. For each $(z, z') \in Z^2$, define $w(z, z') = \max\{u(z'), p(z, z')\}$,

$$y(z, z') = \begin{cases} 0 & \text{if } w(z, z') = u(z'), \\ 1 & \text{if } w(z, z') > u(z'), \end{cases}$$

and, for each $(z, z') \in Z \times Z_\emptyset$, define

$$\lambda(z, z') = \begin{cases} 0 & \text{if } \gamma(z, z') = 0, \\ 1 & \text{if } \gamma(z, z') > 0. \end{cases}$$

Since (γ, u) solves OP if and only if (γ, u) is stable, it follows that the objective function (3) of OP is equal to zero. Hence, it suffices to show that $(\gamma, u, w, y, \lambda)$ satisfies all the constraints in MILP.

We have that (9) follows from (4). We have that, for each $z \in Z$, $0 \leq U_s(z) \leq u(z)$ by (7) and that there is $\hat{z} \in Z_\emptyset$ such that $(z, \hat{z}) \in \text{supp}(\gamma)$ or $(\hat{z}, z) \in \text{supp}(\gamma)$ by (5); thus, by the fact that the objective function of OP is equal to zero, either $u(z) = s(z, \hat{z}) - n(z, \hat{z}) w(z, \hat{z})$, $u(z) \leq \frac{s(\hat{z}, z) - u(\hat{z})}{n(\hat{z}, z)}$ or $u(z) = U_s(z)$ and (10) follows.

We have that (11) and (12) follow, respectively, by (5) and (7). The definition of w implies that (16) holds (since $p(z,z') = s(z,z')/(\xi + n(z,z')) \leq \bar{c} \leq N$ for each $(z,z') \in Z^2$) and that (18) and (20) hold. The definition of y implies that (17), (19) and (21) hold.

The definition of w and (6) imply that (22) holds. The definition of λ implies that (13) and (23) hold.

It follows from the above that the objective function (3) of OP is equal to zero at (γ, u). Since $w(z,z') = \max\{u(z'), p(z,z')\}$ for each $(z,z') \in Z^2$, this then implies that $\gamma(z,z')(u(z)+n(z,z')w(z,z')-s(z,z')) = 0$ for each $(z,z') \in Z^2$ and $\gamma(z,\emptyset)(u(z)-U_s(z)) = 0$ for each $z \in Z$. This then implies that (14), (15), (24) and (25) hold as follows.

Let $(z,z') \in Z^2$. If $\gamma(z,z') = 0$, then $\lambda(z,z') = 0$ and (24) holds; (25) also holds since $u(z)$, $n(z,z')$ and $w(z,z')$ are all at most N by (10), the definition of N and (16). If $\gamma(z,z') > 0$, then $u(z) + n(z,z')w(z,z') - s(z,z') = 0$ and $\lambda(z,z') = 1$, hence (25) holds; (24) also holds since $\gamma(z,z') \leq \nu(z)$ by (5).

Finally, let $z \in Z$. If $\gamma(z,\emptyset) = 0$, then $\lambda(z,\emptyset) = 0$ and (14) holds; (15) also holds since $u(z) \leq N$ by (10). If $\gamma(z,\emptyset) > 0$, then $u(z) - U_s(z) = 0$ and $\lambda(z,\emptyset) = 1$, hence (15) holds; (14) also holds since $\gamma(z,\emptyset) \leq \nu(z)$ by (5).

Part 2: If $(\gamma, u, w, y, \lambda)$ solves MILP then (γ, u) solves OP.

Let $(\gamma, u, w, y, \lambda)$ solve MILP. It then follows that (γ, u) satisfy all the constraints of OP. Indeed, (4) follows from (9), (5) follows from (11), and (7) follows from (12). Finally, (6) follows from (17)–(22) since (17)–(21) imply that $w(z,z') = \max\{u(z'), p(z,z')\}$ for each $(z,z') \in Z^2$ and then (22) implies (6).

The objective function (3) of OP is at least zero by (6). Thus, to show that (γ, u) is a solution of OP, it suffices to show that the value of the objective function (3) of OP at (γ, u) is zero.

We have that $w(z,z') = \max\{u(z'), p(z,z')\}$ for each $(z,z') \in Z^2$ as shown above. Furthermore, we have that $\gamma(z,z')(u(z)+n(z,z')w(z,z')-s(z,z')) = 0$ for each $(z,z') \in Z \times Z$ since $\gamma(z,z') = 0$ if $\lambda(z,z') = 0$ and $u(z)+n(z,z')w(z,z')-s(z,z') = 0$ if $\lambda(z,z') = 1$. Likewise, $\gamma(z,\emptyset)(u(z) - U_s(z)) = 0$ for each $z \in Z$ since $\gamma(z,\emptyset) = 0$ if $\lambda(z,\emptyset) = 0$ and $u(z) - U_s(z) = 0$ if $\lambda(z,\emptyset) = 1$. Hence, the objective function (3) of OP is zero at (γ, u).

A.4 Details for Example 2

Without the pay-cap constraint, we can solve the surplus maximization problem and its dual; variable for the case without pay-cap will be denoted with a superscript $*$ and those of the pay-cap case without superscript. Let γ^* denote the marginal of μ^* on $Z \times Z_\emptyset$; then γ^* is obtained by solving the following linear programming problem:

$$\max_{\gamma} \sum_{z \in Z} \sum_{z' \in Z_\emptyset} z n(z') \gamma(z, z')$$

subject to:

$$\sum_{z' \in Z_\emptyset} \gamma(z, z') + \sum_{\hat{z} \in Z} n(z) \gamma(\hat{z}, z) = \nu(z) \text{ for each } z \in Z,$$

$$\gamma(z, z') \geq 0 \text{ for each } z \in Z, z' \in Z_\emptyset.$$

The earnings function $u^* : Z_\emptyset \to \mathbb{R}$ that gives the value $u^*(z)$ of each knowledge level is, in the discrete case, a vector $u^* = (u^*(z))_{z \in Z_\emptyset}$ and is computed by solving the following linear programming problem:

$$\min_{u} \sum_{z \in Z} u(z) \nu(z)$$

subject to:

$$u(z) + n(z') u(z') \geq z n(z') \text{ for each } z \in Z, z' \in Z_\emptyset$$

$$u(\emptyset) = 0.$$

We obtain

$$\begin{aligned}
&\text{supp}(\mu^*) = \{(0.9, 0, u^*(0)), (0.9, 0.1, u^*(0.1)), (0.1, 0, u^*(0))\},\\
&\mu^*(0.9, 0, u^*(0)) = 0.07,\\
&\mu^*(0.9, 0.1, u^*(0.1)) = 0.93 \text{ and}\\
&\mu^*(0.1, 0, u^*(0)) = 0.43, \text{ with}\\
&u^*(0) = 0.0379,\\
&u^*(0.1) = 0.1241,\\
&u^*(0.9) = 1.7241.
\end{aligned}$$

With pay-cap constraint: I'll use $c^*(z) = u^*(z)$ for $z \in \{0, 0.1\}$. We have that $u^*(0.9) = (0.9 - u^*(z))n(z)$ for each $z \in \{0, 0.1\}$, hence

$$u^*(z) = \bar{z} - \frac{u^*(\bar{z})}{n(z)}$$

for each $z \in \{0, 0.1\}$. The ratio

$$\frac{c^*(z)}{u^*(\bar{z})} = \frac{\bar{z}}{u^*(\bar{z})} - \frac{1}{n(z)}$$

is increasing in $z \in \{0, \tilde{z}\}$, hence we can focus on type 0. The pay-cap constraint is satisfied if $\xi \geq 45.45$, hence nothing changes in this case. Suppose that $\xi < 45.45$.

Claim 1 *If $\xi > 1.81$, it cannot be that $\gamma(0.9, \emptyset) > 0$ and $\gamma(0.1, \emptyset) > 0$.*

Proof. Suppose that $\gamma(0.9, \emptyset) > 0$ and $\gamma(0.1, \emptyset) > 0$. Then type $\bar{z}$ can hire $\tilde{z}$ at wage $\frac{\bar{z}n(\tilde{z})}{\xi + n(\tilde{z})} = 0.49 > \tilde{z}$ to get $\frac{\xi \bar{z} n(\tilde{z})}{\xi + n(\tilde{z})} > \bar{z}$ since $\xi > 1.81$ — contradiction. ■

Claim 2 *$\gamma(0.9, 0.1) > 0$ if $\xi > 1.81$.*

Proof. If $\gamma(0.9, 0.1) = 0$, then there are self-employed of type $\bar{z}$ and $\tilde{z}$ since $1 = \nu(\bar{z}) > \frac{\nu(0)}{n(0)} = 1/2$ and $2.5 = \nu(\tilde{z}) > \frac{\nu(0)}{n(0)} = 1/2$. But this contradicts Claim 1. ■

Claim 3 *$\gamma(0.9, \emptyset) = 0$ if $\xi > 1.81$.*

Proof. Suppose that $\gamma(0.9, \emptyset) > 0$. Since $\gamma(0.9, 0.1) > 0$ by Claim 2, $\bar{z}$ is both manager and self-employed. Thus, $\bar{z} = (\bar{z} - c(\tilde{z}))n(\tilde{z})$ and, hence,

$$c(\tilde{z}) = \bar{z}\left(1 - \frac{1}{n(\tilde{z})}\right) = 0.495.$$

Note that $\frac{u(\bar{z})}{c(\tilde{z})} = \frac{\bar{z}}{c(\tilde{z})} = 1.81 < \xi$.

Since $\gamma(0.9, \emptyset) > 0$, it follows that $\gamma(\tilde{z}, \emptyset) = 0$. Since $2.5 = \nu(\tilde{z}) > n(\tilde{z})\nu(\bar{z}) = 2.2$, it follows that $\gamma(\tilde{z}, 0) > 0$. Since $\frac{u(\bar{z})}{c(\tilde{z})} < \xi$, it follows that type $\tilde{z}$ is indifferent to being a worker or manager. This implies that

$$c(\tilde{z}) = (\tilde{z} - \tilde{c}(0))n(0) \Leftrightarrow \tilde{c}(0) = \tilde{z} - \frac{c(\tilde{z})}{n(0)} = -0.1475 < 0,$$

a contradiction. ■

Claim 4 *$\gamma(0.1, \emptyset) = 0$ if $\xi > 2$.*

Proof. Suppose that $\gamma(0.1, \emptyset) > 0$. Consider first the case that $\gamma(0.9, 0) > 0$ and recall that $\gamma(0.9, 0.1) > 0$ by Claim 2. Managers earn the same, hence type $\bar{z}$ has the same earnings $u(\bar{z})$. Hence,

$$c(z) = \bar{z} - \frac{u(\bar{z})}{n(z)}.$$

Then $c(0.1) > c(0)$ and $\xi c(0) \geq u(\bar{z})$ implies that $\xi c(0.1) > u(\bar{z})$. Since $\xi c^*(0) < u^*(\bar{z}) = (\bar{z} - c^*(0))n(0)$, it must be that $c(0) > c^*(0)$. Hence, $u(\bar{z}) < u^*(\bar{z})$ and $c(\tilde{z}) > c^*(\tilde{z})$. Since a type $\tilde{z}$ earns $\tilde{z} < c^*(\tilde{z}) < c(\tilde{z})$ as self-employed, any manager of type $\bar{z}$ hiring $\tilde{z}$ at wage $c(\tilde{z})$ can attract those self-employed at $c(\tilde{z}) - \varepsilon$ while still satisfying the pay-cap constraint — contradiction.

Consider next the case where $\gamma(0.9, 0) = 0$ and recall that $\gamma(0.9, \emptyset) = 0$ by Claim 3. Then $\gamma(0.9, 0.1) = \nu(0.9) = 1$ and $2.5 = \nu(0.1) = n(0.1) + \gamma(0.1, \emptyset) + \gamma(0.1, 0)$ implies that $\gamma(0.1, 0) < 2.5 - n(0.1) < 0.3$. Then $1 = \nu(0) = \gamma(0, \emptyset) + n(0)\gamma(0.1, 0)$ implies that $\gamma(0, \emptyset) > 0$. Hence, a self-employed of type $\tilde{z}$ can hire a self-employed of type 0 by paying him $\frac{\tilde{z}n(0)}{\xi + n(0)} > 0$ to earn

$$\frac{\xi \tilde{z} n(0)}{\xi + n(0)} > \tilde{z} \Leftrightarrow \xi > 2,$$

a contradiction to stability. ■

Claim 5 *$\gamma(0.1, 0) > 0$ if $\xi > 2$.*

Proof. We have that $\gamma(0.9, 0.1) \leq \nu(0.9) = 1$ and, hence, by Claim 4,

$$2.5 = \nu(0.1) = n(0.1)\gamma(0.9, 0.1) + \gamma(0.1, 0).$$

Thus, $\gamma(0.1, 0) > 0$. ■

So far: if $\xi > 2$, then

$$(0.9, 0.1), (0.1, 0) \in \text{supp}(\gamma) \text{ and } (0.9, \emptyset), (0.1, \emptyset) \notin \text{supp}(\gamma).$$

Let $\xi > 2$ and suppose that $\gamma(0.9, 0) = 0$. Then $\gamma(0.9, 0.1) = \nu(0.9) = 1$, $\gamma(0.1, 0) = \nu(0.1) - n(0.1) = 2.5 - 0.(2) = 0.2(7)$ and $\gamma(0, \emptyset) = \nu(0) - n(0)\gamma(0.1, 0) = 0.(4) > 0$.

Claim 6 *If $\xi > 1.81$, $\gamma(0.1, 0) > 0$ and $\gamma(0.9, 0) = 0$, then $\xi\tilde{c}(0) = u(0.1)$.*

Proof. Suppose that $\xi\tilde{c}(0) > u(0.1)$. We have that $\gamma(0.9, \emptyset) = 0$ since $\xi > 1.81$; this together with $\gamma(0.9, 0) = 0$ implies that $\gamma(0.9, 0.1) = 1$, $\gamma(0.1, 0) \leq 0.27$ and $\gamma(0, \emptyset) > 0$. Since $\gamma(0, \emptyset) > 0$, those of type 0 are indifferent between being workers and self-employed, hence $\tilde{c}(0) = 0$. Thus, $0 = \xi\tilde{c}(0) > u(0.1)$, a contradiction to $u(0.1) \geq 0.1$. ■

Thus, if $\xi > 2$ and $\gamma(0.9, 0) = 0$, then

$$\tilde{c}(0) = \frac{\tilde{z}n(0)}{\xi + n(0)},$$
$$u(0.1) = \xi\tilde{c}(0).$$

Claim 7 *If $\xi > 9.82$ and $\gamma(0.9, 0) = 0$, then $\xi c(0.1) > u(0.9)$.*

Proof. If $\xi c(0.1) = u(0.9)$, then

$$c(0.1) = \frac{\bar{z}n(\tilde{z})}{\xi + n(\tilde{z})},$$
$$u(0.9) = \xi c(0.1).$$

Then $u(0.1) > c(0.1)$ provided that $\xi > 9.82$, a contradiction. ■

Claim 8 *If $\xi > 16.36$, then $\gamma(0.9, 0) > 0$.*

Proof. Suppose that $\gamma(0.9, 0) = 0$. Then $\xi c(0.1) > u(0.9)$ by Claim 7. This implies that $c(0.1) = u(0.1)$ and $u(0.9) = (0.9 - c(0.1))n(0.1)$. Note that $\bar{c}(0) = \frac{\bar{z}n(0)}{\xi + n(0)}$ is such that $\bar{c}(0) > \tilde{c}(0)$ and the pay-cap constraint is satisfied when a manager of type $\bar{z}$ offers $\bar{c}(0)$ to a type 0. The type $\bar{z}$ manager earns $\frac{\xi \bar{z} n(0)}{\xi + n(0)}$ which exceeds $u(\bar{z})$ if $\xi > 16.36$. ■

Claim 9 *If $\gamma(0.9, 0) > 0$, then $\xi \geq 16.36$, $\xi c(0) = u(\bar{z})$ and*

$$
\begin{aligned}
c(0) &= \frac{\bar{z}n(0)}{\xi + n(0)}, \\
u(\bar{z}) &= \xi c(0), \\
c(\tilde{z}) &= u(\tilde{z}) = \bar{z} - \frac{u(\bar{z})}{n(\tilde{z})}, \\
\tilde{c}(0) &= \tilde{z} - \frac{u(\tilde{z})}{n(0)}.
\end{aligned}
$$

Moreover, $\xi \tilde{c}(0) = u(\tilde{z}) \iff \xi = 16.36$.

Proof. First suppose that $\gamma(0.9, 0) > 0$ and $\gamma(0.9, 0.1) > 0$. Since type $\bar{z}$ must be indifferent between hiring 0 and $\tilde{z}$, we have $c(z) = \bar{z} - \frac{u(\bar{z})}{n(z)}$ for each $z \in \{0, \tilde{z}\}$, where $c(z)$ is the wage that type $\bar{z}$ pays to $z \in \{0, \tilde{z}\}$.

If $\gamma(0.1, 0) = 0$, then $\gamma(0.1, \emptyset) > 0$ and the first part of the proof of Claim 4 implies a contradiction; thus $\gamma(0.1, 0) > 0$. Since $\xi c(0) \geq u(\bar{z})$ and $c(\tilde{z}) > c(0)$, $\xi c(\tilde{z}) > u(\bar{z})$. Then $\tilde{z}$ must be indifferent between being manager and worker; letting $\tilde{c}(0)$ be the wage that type $\tilde{z}$ pays to type 0, this implies $\tilde{c}(0) = \tilde{z} - \frac{c(\tilde{z})}{n(0)}$.

If $\xi c(0) > u(\bar{z})$, then $\xi c(\tilde{z}) > u(\bar{z})$ and $c(0) = \tilde{c}(0)$. Thus, no pay caps are binding and hence $(\gamma, u) = (\gamma^*, u^*)$, a contradiction since (γ^*, u^*) satisfies the pay cap constraint only when $\xi \geq 45.45$.

Thus, we may assume that $\xi c(0) = u(\bar{z})$. Then

$$\begin{aligned} c(0) &= \frac{\bar{z}n(0)}{\xi + n(0)}, \\ u(\bar{z}) &= \xi c(0), \\ c(\tilde{z}) &= u(\tilde{z}) = \bar{z} - \frac{u(\bar{z})}{n(\tilde{z})}, \\ \tilde{c}(0) &= \tilde{z} - \frac{u(\tilde{z})}{n(0)}, \end{aligned}$$

which implies that $\xi \geq 16.36$ since $\xi \geq 16.36 \iff \xi\tilde{c}(0) \geq u(\tilde{z})$.

Finally, suppose that $\gamma(0.9, 0.1) = 0$ which, by Claim 2, implies $\xi \leq 1.81$. Since $\frac{\xi\bar{z}n(0)}{\xi+n(0)} \geq \bar{z} \iff \xi \geq 2$, we have $\gamma(0.9, 0) = 0$. ■

We can use the previous claims to characterize the stable matching for all ξ.

Lemma 12 *If* $16.36 < \xi < 45.45$*, then there is a unique stable matching* μ *defined by*

$$\begin{aligned} \mu(0.9, 0.1, c(0.1)) &= 0.93, \\ \mu(0.9, 0, c(0)) &= 0.07, \\ \mu(0.1, 0, \tilde{c}(0)) &= 0.43, \end{aligned}$$

with

$$\begin{aligned} c(0) &= \frac{\bar{z}n(0)}{\xi + n(0)}, \\ u(\bar{z}) &= \xi c(0), \\ c(\tilde{z}) &= u(\tilde{z}) = \bar{z} - \frac{u(\bar{z})}{n(\tilde{z})}, \\ \tilde{c}(0) &= \tilde{z} - \frac{u(\tilde{z})}{n(0)}. \end{aligned}$$

Proof. It follows from $\xi > 16.36$ and Claim 8 that $\gamma(0.9, 0) > 0$.

Claim 9 then implies $\xi\tilde{c}(0) > u(\tilde{z})$, $\xi c(0) = u(\bar{z})$ and

$$\begin{aligned} c(0) &= \frac{\bar{z}n(0)}{\xi + n(0)}, \\ u(\bar{z}) &= \xi c(0), \\ c(\tilde{z}) &= u(\tilde{z}) = \bar{z} - \frac{u(\bar{z})}{n(\tilde{z})}, \\ \tilde{c}(0) &= \tilde{z} - \frac{u(\tilde{z})}{n(0)}. \end{aligned}$$

It must also be the case that $\gamma(0, \emptyset) = 0$ since otherwise $\tilde{c}(0) = 0$ and, hence, $0 = \xi\tilde{c}(0) > u(\tilde{z}) \geq \tilde{z} > 0$, a contradiction. Then

$$\text{supp}(\gamma) = \{(0.9, 0), (0.9, 0.1), (0.1, 0)\}$$

and obtain the values of γ by solving

$$\begin{aligned} 1 &= \gamma(0.9, 0) + \gamma(0.9, 0.1) \\ 2.5 &= n(0.1)\gamma(0.9, 0.1) + \gamma(0.1, 0) \\ 1 &= n(0)(\gamma(0.9, 0) + \gamma(0.1, 0)). \end{aligned}$$

Thus,

$$\begin{aligned} \gamma(0.9, 0.1) &= 0.93, \\ \gamma(0.9, 0) &= 0.07, \\ \gamma(0.1, 0) &= 0.43. \end{aligned}$$

■

Thus, provided that $\xi > 16.36$, the assignment γ is the same as in the no pay-cap case (but here there is no extensive margin) and the earnings change to make the pay-cap of the best and worse type bind.

Lemma 13 *If $\xi = 16.36$, there is a continuum of stable matchings indexed by $\alpha \in [0, 0.07]$:*

$$\begin{aligned}
&\mu(0.9, 0, c(0)) = \alpha,\\
&\mu(0.9, 0.1, c(0.1)) = 1 - \alpha,\\
&\mu(0.1, 0, \tilde{c}(0)) = \nu(0.1) - n(0.1)(1 - \alpha),\\
&\mu(0, \emptyset, 0) = 1 - n(0)(\alpha + \nu(0.1) - n(0.1)(1 - \alpha)),
\end{aligned}$$

with

$$\begin{aligned}
c(0) &= \frac{\bar{z}n(0)}{\xi + n(0)},\\
\tilde{c}(0) &= \frac{\tilde{z}n(0)}{\xi + n(0)},\\
c(0.1) &= u(0.1) = \xi\tilde{c}(0),\\
u(0.9) &= (0.9 - c(0.1))n(0.1).
\end{aligned}$$

Proof. It follows by Claims 1–5 that

$$(0.9, 0.1), (0.1, 0) \in \text{supp}(\gamma) \text{ and } (0.9, \emptyset), (0.1, \emptyset) \notin \text{supp}(\gamma).$$

Let $\gamma(0.9, 0) = \alpha$. Then $\gamma(0, \emptyset) \geq 0$ implies that $\alpha \leq 0.07$. If $\alpha = 0$, Claim 6 implies that

$$\begin{aligned}
\tilde{c}(0) &= \frac{\tilde{z}n(0)}{\xi + n(0)},\\
u(0.1) &= \xi\tilde{c}(0)
\end{aligned}$$

and Claim 7 that $\xi c(0.1) > u(0.9)$. Thus, $c(0.1) = u(0.1)$ and $u(0.9) = (0.9 - c(0.1))n(0.1)$.

If $\alpha > 0$, then Claim 9 implies that $\xi\tilde{c}(0) = u(\tilde{z})$ and $\xi c(0) = u(\bar{z})$. Then $\xi c(0.1) > u(\bar{z})$ and, hence, $c(0.1) = u(0.1)$ and $u(0.9) = (0.9 - c(0.1))n(0.1)$. ■

Lemma 14 *If* $9.82 < \xi < 16.36$*, there is a unique stable matching* μ *defined by*

$$\begin{aligned}
&\mu(0.9, 0.1, c(0.1)) = 1,\\
&\mu(0.1, 0, \tilde{c}(0)) = 0.27,\\
&\mu(0, \emptyset, 0) = 0.44,
\end{aligned}$$

with

$$\begin{aligned}
&\tilde{c}(0) = \frac{\tilde{z}n(0)}{\xi + n(0)},\\
&c(0.1) = u(0.1) = \xi\tilde{c}(0),\\
&u(0.9) = (0.9 - c(0.1))n(0.1).
\end{aligned}$$

Proof. Claim 9 implies that $\gamma(0.9, 0) = 0$. Claim 6 implies that

$$\begin{aligned}
&\tilde{c}(0) = \frac{\tilde{z}n(0)}{\xi + n(0)},\\
&u(0.1) = \xi\tilde{c}(0)
\end{aligned}$$

and Claim 7 that $\xi c(0.1) > u(0.9)$. Thus, $c(0.1) = u(0.1)$ and $u(0.9) = (0.9 - c(0.1))n(0.1)$.

■

Lemma 15 *If* $2 < \xi \leq 9.82$*, there is a unique stable matching* μ *defined by*

$$\begin{aligned}
&\mu(0.9, 0.1, c(0.1)) = 1,\\
&\mu(0.1, 0, \tilde{c}(0)) = 0.27,\\
&\mu(0, \emptyset, 0) = 0.44,
\end{aligned}$$

with

$$\tilde{c}(0) = \frac{\tilde{z}n(0)}{\xi + n(0)},$$
$$u(0.1) = \xi\tilde{c}(0),$$
$$c(0.1) = \frac{\bar{z}n(\tilde{z})}{\xi + n(\tilde{z})},$$
$$u(0.9) = \xi c(0.1).$$

Proof. Claim 9 implies that $\gamma(0.9, 0) = 0$. Claim 6 implies that

$$\tilde{c}(0) = \frac{\tilde{z}n(0)}{\xi + n(0)},$$
$$u(0.1) = \xi\tilde{c}(0).$$

If $\xi c(0.1) > u(0.9)$, then $c(0.1) = u(0.1) \leq \frac{\bar{z}n(\tilde{z})}{\xi+n(\tilde{z})}$ since $\xi \leq 9.82$. Hence, $\xi c(0.1) \leq u(0.9)$, contradiction. Thus, $\xi c(0.1) = u(0.9)$, hence,

$$c(0.1) = u\frac{\bar{z}n(\tilde{z})}{\xi + n(\tilde{z})},$$
$$u(0.9) = \xi c(0.1).$$

■

Lemma 16 *If $\xi = 2$, then there is a continuum of stable matchings indexed by $\alpha \in [0, 0.27]$:*

$$\mu(0.9, 0.1, c(0.1)) = 1,$$
$$\mu(0.1, 0, \tilde{c}(0)) = \alpha,$$
$$\mu(0, \emptyset, 0) = 1 - 2\alpha,$$

with

$$\tilde{c}(0) = \frac{\tilde{z}n(0)}{\xi + n(0)},$$
$$u(0.1) = \xi\tilde{c}(0),$$
$$c(0.1) = \frac{\bar{z}n(\tilde{z})}{\xi + n(\tilde{z})},$$
$$u(0.9) = \xi c(0.1).$$

Proof. Claim 9 implies that $\gamma(0.9, 0) = 0$. Claims 3 and 2 imply that $\gamma(0.9, \emptyset) = 0$ and $\gamma(0.9, 0.1) > 0$. Thus, $\gamma(0.9, 0.1) = \nu(0.9) = 1$.

Let $\alpha = \gamma(0.1, 0)$. Then $\gamma(0.1, \emptyset) = 0.27 - \alpha \geq 0$ implies that $\alpha \leq 0.27$. Furthermore, $1 = \nu(0) = 2\alpha + \gamma(0, \emptyset)$ implies that $\gamma(0, \emptyset) = 1 - 2\alpha$.

We have that $\xi c(0.1) = u(0.9)$ and, hence,

$$c(0.1) = u\frac{\bar{z}n(\tilde{z})}{\xi + n(\tilde{z})},$$
$$u(0.9) = \xi c(0.1).$$

Suppose not; then $\xi c(0.1) > u(0.9)$. If $\alpha < 0.27$, then $\gamma(0.1, \emptyset) > 0$ and $c(0.1) = 0.1$, $u(0.9) = (0.9 - 0.1)n(0.1)$ and $\xi c(0.1) < u(0.9)$ since $\xi < 17.77$, a contradiction. If $\alpha = 0.27$, then $c(0.1) = u(0.1) \leq \frac{\bar{z}n(\tilde{z})}{\xi + n(\tilde{z})}$ since $\xi \leq 9.82$. Hence, $\xi c(0.1) \leq u(0.9)$, contradiction.

Finally, if $\alpha > 0$, Claim 6 implies that

$$\tilde{c}(0) = \frac{\tilde{z}n(0)}{\xi + n(0)},$$
$$u(0.1) = \xi\tilde{c}(0).$$

■

Lemma 17 *If* $1.81 < \xi < 2$*, there is a unique stable matching* μ *defined by*

$$\mu(0.9, 0.1, c(0.1)) = 1,$$
$$\mu(0.1, \emptyset, 0) = 0.27,$$
$$\mu(0, \emptyset, 0) = 1,$$

with

$$c(0.1) = \frac{\bar{z}n(\tilde{z})}{\xi + n(\tilde{z})},$$
$$u(0.9) = \xi c(0.1).$$

Proof. Claim 9 implies that $\gamma(0.9, 0) = 0$. Claims 3 and 2 imply that $\gamma(0.9, \emptyset) = 0$ and $\gamma(0.9, 0.1) > 0$. Thus, $\gamma(0.9, 0.1) = \nu(0.9) = 1$.

Suppose that $\gamma(0.1, 0) > 0$. Then $\xi\tilde{c}(0) \geq u(0.1)$ and, hence, $u(0.1) \leq \frac{\xi\tilde{z}n(0)}{\xi+n(0)} < \tilde{z}$ since $\xi < 2$, a contradiction. Thus, $\gamma(0.1, 0) = 0$. Hence, $\gamma(0.1, \emptyset) = \nu(0.1) - n(0.1)\gamma(0.9, 0.1) = 0.27$ and $\gamma(0, \emptyset) = \nu(0) = 1$.

If $\xi c(0.1) > u(0.9)$, then $c(0.1) = 0.1$, $u(0.9) = (0.9 - 0.1)n(0.1)$ and $\xi c(0.1) < u(0.9)$ since $\xi < 17.77$, a contradiction. Thus, $\xi c(0.1) = u(0.9)$, hence,

$$c(0.1) = u\frac{\bar{z}n(\tilde{z})}{\xi + n(\tilde{z})},$$
$$u(0.9) = \xi c(0.1).$$

■

Lemma 18 *If* $\xi = 1.81$*, then there is a continuum of stable matchings indexed by* $\alpha \in [0, 1]$*:*

$$\mu(0.9, 0.1, c(0.1)) = \alpha,$$
$$\mu(0.1, \emptyset, 0) = 2.5 - n(\tilde{z})\alpha,$$
$$\mu(0, \emptyset, 0) = 1,$$

with

$$c(0.1) = \frac{\bar{z}n(\tilde{z})}{\xi + n(\tilde{z})},$$
$$u(0.9) = \xi c(0.1).$$

Proof. Claim 9 implies that $\gamma(0.9, 0) = 0$. Let $\alpha = \gamma(0.9, 0.1)$. Thus, $\alpha + \gamma(0.9, \emptyset) = \nu(0.9) = 1$, hence $\alpha \leq 1$.

Suppose that $\gamma(0.1, 0) > 0$. Then $\xi\tilde{c}(0) \geq u(0.1)$ and, hence, $u(0.1) \leq \frac{\xi\tilde{z}n(0)}{\xi+n(0)} < \tilde{z}$ since $\xi < 2$, a contradiction. Thus, $\gamma(0.1, 0) = 0$. Hence, $\gamma(0.1, \emptyset) = \nu(0.1) - n(0.1)\gamma(0.9, 0.1) = 2.5 - n(0.1)\alpha$ and $\gamma(0, \emptyset) = \nu(0) = 1$.

If $\xi c(0.1) > u(0.9)$ whenever $\alpha > 0$, then $c(0.1) = 0.1$, $u(0.9) = (0.9 - 0.1)n(0.1)$ and $\xi c(0.1) < u(0.9)$ since $\xi < 17.77$, a contradiction. Thus, $\xi c(0.1) = u(0.9)$, hence,

$$c(0.1) = u\frac{\bar{z}n(\tilde{z})}{\xi + n(\tilde{z})},$$
$$u(0.9) = \xi c(0.1).$$

■

Lemma 19 *If $\xi < 1.81$, there is a unique stable matching μ defined by*

$$\mu(0.9, \emptyset, 0) = 1,$$
$$\mu(0.1, \emptyset, 0) = 2.5,$$
$$\mu(0, \emptyset, 0) = 1.$$

Proof. Claim 9 implies that $\gamma(0.9, 0) = 0$ and the proof of Lemma 17 implies that $\gamma(0.1, 0) = 0$. Thus, $\gamma(0, \emptyset) = 1$.

Suppose that $\gamma(0.9, 0.1) > 0$. Then $\xi c(0.1) \geq u(0.9)$ and, hence, $u(0.9) \leq \frac{\xi\bar{z}n(0.1)}{\xi+n(0.1)} < \bar{z}$ since $\xi < 1.81$, a contradiction. Thus, $\gamma(0.9, 0.1) = 0$. Hence, $\gamma(0.1, \emptyset) = 2.5$ and $\gamma(0.9, \emptyset) = 1$.

■

A.5 Details for Example 3

Let $\hat{z}$ satisfy $\frac{\xi n(\hat{z})}{n(\hat{z})+\xi} = 1$, i.e. $\hat{z} = 3/8$. Note that the utility for any individual z from hiring $\hat{z}$ at the pay cap is $\frac{\xi z n(\hat{z})}{\xi + n(\hat{z})} = z$. Moreover, for each $z' < \hat{z}$,

$$\frac{\xi z n(z')}{\xi + n(z')} < z$$

since $x \mapsto \frac{n(x)}{n(x)+\xi}$ is strictly increasing. Hence, it follows that for each z, self-employment is better than hiring any individual $z' < \hat{z}$. We conjecture that individuals $[0, \hat{z}]$ are self-employed and solve for the stable outcome of the economy with only individuals in $[\hat{z}, 1]$, ignoring pay caps. We will check ex-post that the pay caps are satisfied.

For the economy with only individuals $[\hat{z}, 1]$, as in Garicano and Rossi-Hansberg (2004), there exists (z_1, z_2) such that individuals in $[\hat{z}, z_1]$ are workers, those in $[z_1, z_2]$ are self-employed, those in $[z_2, 1]$ are managers, the assignment between managers and workers is given by $\phi : [z_2, 1] \to [\hat{z}, z_1]$, the wage of the workers is given by $c : W \to [0, 1]$, and the following differential equations corresponding to feasibility and the first order condition of the managerial optimization problem hold:

$$\begin{aligned} \phi'(z) &= n(\phi(z)) \text{ for each } z \in [z_2, 1], \text{ and} \\ c'(z) &= \frac{\phi^{-1}(z) - c(z)}{1 - z} \text{ for each } z \in [\hat{z}, z_1]. \end{aligned}$$

Solving these differential equations with boundary conditions $\phi(1) = z_1$ and $c(z_1) = z_1$ yields:

$$\begin{aligned} \phi(z) &= 1 - \sqrt{\kappa(z_1) - 2.5z} \text{ for each } z \in [z_2, 1], \\ c(z) &= \frac{\kappa(z_1)}{2.5} + (1 - z)\left(\frac{2.5z_1 - \kappa(z_1) + (1 - z_1)(z_1 - z)}{2.5(1 - z_1)}\right) \text{ for each } z \in [\hat{z}, z_1], \end{aligned}$$

where, for convenience, $\kappa(z_1) = \frac{1}{2}(2z_1^2 - 4z_1 + 7)$. Furthermore, we must have $\phi(z_2) = \hat{z}$ and $c(\hat{z}) = \frac{z_2 n(\hat{z})}{\xi + n(\hat{z})}$ since z_2 must be indifferent between hiring $\hat{z}$ and being self-employed; thus, we

can solve:

$$c(\hat{z}) = \frac{\kappa(z_1)}{2.5} + (1-\hat{z})\left(\frac{2.5z_1 - \kappa(z_1) + (1-z_1)(z_1-\hat{z})}{2.5(1-z_1)}\right) = \frac{z_2 n(\hat{z})}{\xi + n(\hat{z})}$$
$$\phi(z_2) = 1 - \sqrt{\kappa(z_1) - 2.5z_2} = \hat{z}$$

to get $z_1 = \frac{1}{8}(-2+\sqrt{65}) \approx 0.7578$ and $z_2 = \frac{1}{8}(15-\sqrt{65}) \approx 0.8672$.

It can be checked by direct calculation that:

$$\frac{\phi^{-1}(z')n(z')}{\xi + n(z')} \leq c(z') \text{ for all } z' \in [\hat{z}, z_1],$$

with equality for $z' = \hat{z}$. Thus, the pay cap is satisfied.

It follows by Theorem 6 in CLT that

$$u(z) + n(z') \max\left\{u(z'), \frac{zn(z')}{\xi + n(z')}\right\} \geq zn(z') \text{ and } u(z) \geq z \tag{32}$$

holds for each $(z, z') \in (\hat{z}, 1] \times (\hat{z}, 1]$. Since $u(z) = z$ for each $z \in [0, \hat{z}]$ and $\frac{\xi n(z')}{\xi + n(z')} \leq 1$ for each $z' \in [0, \hat{z}]$, (32) holds for each $(z, z') \in [0,1] \times [0, \hat{z}]$ since $u(z) \geq z \geq \frac{\xi z n(z')}{\xi + n(z')} = zn(z') - n(z')\frac{zn(z')}{\xi + n(z')}$. Finally, (32) holds for each $(z, z') \in [0, \hat{z}] \times (\hat{z}, 1]$ since $u(z') \geq z' > z$ implies that $(z - u(z'))n(z') < 0 \leq u(z)$.

A.6 Proof of Theorem 4

Theorem 4 follows from Lemmas 20 and 21 below. Important inequalities in the problem of determining whether there are firms are:

$$\begin{aligned}
&\xi > \frac{n(\bar{z})}{n(\bar{z})-1} \Leftrightarrow \xi > \frac{1}{1-h+h\bar{z}},\\
&\xi \leq \frac{n(0)}{n(0)-1} \Leftrightarrow \xi \leq \frac{1}{1-h},\\
&1-\bar{z}h < \xi(1-h) \Leftrightarrow \xi > \frac{1-\bar{z}h}{1-h},\\
&\frac{1}{1-h+h\bar{z}} < \frac{1-\bar{z}h}{1-h} \Leftrightarrow \bar{z} < 1, \text{ and}\\
&\frac{1-\bar{z}h}{1-h} < \frac{1}{1-h} \Leftrightarrow \bar{z}h > 0;
\end{aligned}$$

hence,

$$\xi \in \left[1, \frac{1}{1-h+h\bar{z}}\right] \cup \left(\frac{1}{1-h+h\bar{z}}, \frac{1-\bar{z}h}{1-h}\right] \cup \left(\frac{1-\bar{z}h}{1-h}, \frac{1}{1-h}\right] \cup \left(\frac{1}{1-h}, \infty\right).$$

Everyone is self-employed in the first and second of these intervals, and there are managers and workers in the third and forth.

Lemma 20 *If $\xi \leq \frac{1-\bar{z}h}{1-h}$, then $\mu = \nu \circ \sigma^{-1}$, where $\sigma(z) = (z, \emptyset, 0)$ for each $z \in Z$.*

Proof. Consider first the case where $\xi \leq \frac{1}{1-h+h\bar{z}}$. As noted, $\xi \leq \frac{1}{1-h+h\bar{z}}$ is equivalent to $\xi \leq \frac{n(\bar{z})}{n(\bar{z})-1}$. Also, note that $\xi \leq \frac{n(\bar{z})}{n(\bar{z})-1}$ is equivalent to $\frac{\xi n(\bar{z})}{\xi+n(\bar{z})} \leq 1$.

Suppose that there is $(z, z', w) \in \text{supp}(\mu)$ such that $z' \in Z$. Then $w \geq \frac{zn(z')}{\xi+n(z')}$ and, hence,

$$U_z(m, z', w) = (z-w)n(z') \leq \frac{\xi z n(z')}{\xi + n(z')} \leq \xi z \frac{n(\bar{z})}{\xi + n(\bar{z})} \leq z = U_s(z)$$

since $n \mapsto \frac{n}{\xi+n}$ and $z' \mapsto n(z')$ are strictly increasing. It then also follows that $U_z(m, z', w) < U_s(z)$ unless $\xi = \frac{n(\bar{z})}{n(\bar{z})-1}$, $z' = \bar{z}$ and $w = \frac{zn(\bar{z})}{\xi+n(\bar{z})}$. But then

$$w = \frac{zn(\bar{z})}{\xi + n(\bar{z})} < \frac{\xi z n(\bar{z})}{\xi + n(\bar{z})} = z \leq \bar{z} = U_s(\bar{z}),$$

contradicting the stability of μ.

Consider next the case where $\frac{1}{1-h+h\bar{z}} < \xi \leq \frac{1-\bar{z}h}{1-h}$. As noted, $\xi \leq \frac{1-\bar{z}h}{1-h}$ is equivalent to $1-\bar{z}h \geq \xi(1-h)$ and $\xi > \frac{1}{1-h+h\bar{z}}$ is equivalent to $\xi > \frac{n(\bar{z})}{n(\bar{z})-1}$.

We have that $\xi \leq \frac{n(0)}{n(0)-1}$ since $\xi \leq \frac{1-\bar{z}h}{1-h} < \frac{1}{1-h}$. Let $z^* \in [0,\bar{z})$ be such that $\xi = \frac{n(z^*)}{n(z^*)-1}$. Then $z^* = \frac{1-\xi(1-h)}{h\xi}$.

Suppose that there is $(z,z',w) \in \text{supp}(\mu)$ such that $z' \in Z$. Then

$$\begin{aligned}
&\frac{zn(z')}{\xi + n(z')} \leq w, \\
&z' \leq w, \text{ and} \\
&(z-w)n(z') \geq z \Leftrightarrow w \leq z\frac{n(z')-1}{n(z')}.
\end{aligned}$$

Then $\frac{zn(z')}{\xi+n(z')} \leq z\frac{n(z')-1}{n(z')}$ which holds if and only if $\xi \geq \frac{n(z')}{n(z')-1}$. Since $\xi = \frac{n(z^*)}{n(z^*)-1}$ and $x \mapsto \frac{n(x)}{n(x)-1}$ is strictly decreasing, it follows that $z' \geq z^*$.

Furthermore, $z' \leq z\frac{n(z')-1}{n(z')} \Leftrightarrow z'\frac{n(z')}{n(z')-1} \leq z$. Since $x \mapsto x\frac{n(x)}{n(x)-1}$ is strictly increasing, it follows that $z^*\frac{n(z^*)}{n(z^*)-1} \leq \bar{z}$ and, in fact, $z^*\frac{n(z^*)}{n(z^*)-1} < \bar{z}$ unless $(z,z') = (\bar{z},z^*)$.[11] If $(z,z') \neq (\bar{z},z^*)$, then

$$z^*\frac{n(z^*)}{n(z^*)-1} < \bar{z} \Leftrightarrow 1-\bar{z}h < \xi(1-h) \Leftrightarrow \frac{1-\bar{z}h}{1-h} < \xi;$$

since $\frac{1-\bar{z}h}{1-h} \geq \xi$, it follows that $(z,z') = (\bar{z},z^*)$.

This then shows that $\text{supp}(\mu) \cap (Z \times X) \subseteq \{\bar{z}\} \times X$. Thus, $\mu(Z \times X) = \mu(\text{supp}(\mu) \cap (Z \times X)) \leq \mu(\{\bar{z}\} \times X) \leq \nu(\{\bar{z}\}) = 0$. Hence, $Z \times \{(\emptyset,0)\}$ has full μ-measure and is closed, thus $\text{supp}(\mu) \subseteq Z \times \{(\emptyset,0)\}$.

Hence, for each Borel $B \subseteq Z \times X_\emptyset$,

$$\begin{aligned}
&\mu(B) = \mu(\text{supp}(\mu) \cap B) = \mu(B \cap (Z \times (X_\emptyset \setminus X))) = \mu(\{z \in Z : \sigma(z) \in B\} \times \{(\emptyset,0)\}) \\
&= \mu(\sigma^{-1}(B) \times X) + \mu(\sigma^{-1}(B) \times \{(\emptyset,1,0)\}) + \int_{Z\times X} 1_{\sigma^{-1}(B)}(z')n\text{d}\mu(z,z',n,w) \\
&= \nu(\sigma^{-1}(B)) = \nu \circ \sigma^{-1}(B).
\end{aligned}$$

Thus, it follows that $\mu = \nu \circ \sigma^{-1}$. ■

[11] Letting $f(x) = x\frac{n(x)}{n(x)-1}$, then $f(x) = \frac{x}{1-h+hx}$ and $f'(x) > 0$ if and only if $1-h > 0$, which holds.

Lemma 21 *If* $\xi > \frac{1-\bar{z}h}{1-h}$, *then* $\mu_{Z\times Z_\emptyset}(Z \times Z) > 0$.

Proof. If not, then $\text{supp}(\mu) = Z \times \{(\emptyset, 0)\}$ since this set is closed and has full μ-measure.

As noted, $\xi > \frac{1-\bar{z}h}{1-h}$ is equivalent to $1-\bar{z}h < \xi(1-h)$ and implies that $\xi > \frac{n(\bar{z})}{n(\bar{z})-1}$. Consider first the case where $\xi > \frac{n(0)}{n(0)-1} \Leftrightarrow \xi > \frac{1}{1-h}$. Then set $w = \frac{\bar{z}n(0)}{\xi+n(0)} > 0$. Since $U_z(0) = 0$, it follows that $(0, w) \in T^m_{\bar{z}}(\mu)$ and $\xi w = (\bar{z} - w)n(0)$. Then $U_{\bar{z}}(m, 0, w) = \frac{\xi\bar{z}n(0)}{\xi+n(0)} > \bar{z} = U_s(\bar{z})$ since inequality is equivalent to $\xi > \frac{n(0)}{n(0)-1}$. But $U_{\bar{z}}(m, 0, w) > U_s(\bar{z})$ contradicts the stability of μ.

Then assume $\xi \leq \frac{n(0)}{n(0)-1}$ and let $n^* \in [0, \bar{z})$ be such that $\xi = \frac{n(z^*)}{n(z^*)-1}$. Then $z^* = \frac{1-\xi(1-h)}{h\xi}$.

It follows that $z^* < \bar{z}\frac{n(z^*)-1}{n(z^*)}$ since this inequality is equivalent to

$$\frac{z^*n(z^*)}{n(z^*)-1} < \bar{z} \Leftrightarrow z^*\xi < \bar{z} \Leftrightarrow \frac{1-\xi(1-h)}{h} < \bar{z} \Leftrightarrow 1-\bar{z}h < \xi(1-h)$$

and the last inequality holds.

Then let $\varepsilon > 0$ be such that $z^* + \varepsilon < \bar{z}\frac{n(z^*+\varepsilon)-1}{n(z^*+\varepsilon)}$. We have that

$$\frac{\bar{z}n(z^*+\varepsilon)}{\xi + n(z^*+\varepsilon)} < \bar{z}\frac{n(z^*+\varepsilon)-1}{n(z^*+\varepsilon)}$$

since this inequality is equivalent to $\xi > \frac{n(z^*+\varepsilon)}{n(z^*+\varepsilon)-1}$ which holds since $z \mapsto \frac{n(z)}{n(z)-1}$ is strictly decreasing and, hence, $\xi = \frac{n(z^*)}{n(z^*)-1} > \frac{n(z^*+\varepsilon)}{n(z^*+\varepsilon)-1}$.

Thus, let

$$w \in \left(\max\left\{z^*+\varepsilon, \frac{\bar{z}n(z^*+\varepsilon)}{\xi + n(z^*+\varepsilon)}\right\}, \bar{z}\frac{n(z^*+\varepsilon)-1}{n(z^*+\varepsilon)}\right).$$

Since $w > z^* + \varepsilon$ and $w > \frac{\bar{z}n(z^*+\varepsilon)}{\xi+n(z^*+\varepsilon)}$, it follows that $(z^*+\varepsilon, w) \in T^m_{\bar{z}}(\mu)$. Since $w < \bar{z}\frac{n(z^*+\varepsilon)-1}{n(z^*+\varepsilon)}$, it follows that

$$U_{\bar{z}}(m, z^*+\varepsilon, w) = (\bar{z} - w)n(z^*+\varepsilon) > \bar{z} = U_s(\bar{z}).$$

But this is a contradiction to the stability of μ. ■