On Dynamic Consistency in Ambiguous Games

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question

- Static, incomplete information games where players
  1. have Maxmin Expected Utility preferences (Gilboa-Schmeidler, 1989)
  2. have a common set of priors
  3. obey a consequentialist, strategy-independent update rule
- Under what circumstances will Dynamic Consistency hold?
- What assumptions on set of priors ("beliefs") suffice for DC?

answer

- Under minor technical conditions, only if no ambiguity about signals
  - At least two types for each player
  - Full support (ex-ante)
- applies to most mechanism design settings with at least two agents
- applies to most models of ambiguity-sensitivity
- if only uncertainty (for agents) is about their types, then either:
  - No ambiguity,
  - No DC, or
  - Different ex-ante behavior

positive implications

- Renegotiation, even with complete contracts
- Ex-ante vs Ex-post participation
- Non-equivalence of normal vs. strategic form of game

normative / modeling implications

- Interpretation of "non-Bayesian" equilibrium
  - Is it caused by ambiguity, failure of DC, failure of consequentialism, etc.
  - All 3 EU properties important for some classic results, e.g. on speculative trade
- Lack of "common beliefs"
- Ex-ante vs Ex-post welfare
- Conditional independence impossible

Potential solutions

- Sophistication (Siniscalchi, 2011)
- Non-consequentialist updating (Hanany and Klibanoff, 2007)
- Allow distinct ex-ante behavior
- Get rid of ambiguity
- Take interim behavior as given

literature

- Dynamic consistency:
  - Epstein-Lebreton (1993): no restrictions on information structure and DC implies probabilistic sophistication
  - In games, not every information structure relevant
  - Follow Epstein and Schneider (2003), Maccheroni et al. (2006), Klibanoff et al. (2009); only require DC w.r.t. one info. structure
  - Alternative approaches: Hanany and Klibanoff (2007), Siniscalchi (2011), Hanany, Klibanoff and Mukerji (2016)
- Formulations of Incomplete Information games w/ ambiguity: Lo (1999), Kajii and Ui (2005), Azrieli and Teper (2011)
- Results apply to (discretized versions of): Salo-Weber (1995), Lo (1998), Bose et al (2006), Chen et al (2007), Bose and Daripa (2009), Bodoh-Creed (2012), Bose-Renou (2014), Ellis (2015), others ?

Illustration of setting

- Two players with two types each (R and B)
- Player i's type drawn from Ellsberg urn i with
  - 100 total balls each
  - all balls either R or B
- Other details of the game abstracted away
- Concreteness: both players have Gilboa-Schmeidler preferences
- Assumption: symmetric information (and taste for ambiguity) so sets of priors are the same
- Allow utility index to differ across players
- Which common sets of priors make both players DC?

Model

- Players: $I = \{1, ..., n\}$ $(n \geq 2)$
- Types: T_0 be a finite set of states of nature
- Types: T_i be a finite set of types of player i
  - $\# T_i \geq 2$ for all $i > 0$
- $T = T_0 \times T_1 \times ... \times T_n$
  - $T_{-i} = T_0 \times ... \times T_{i-1} \times T_{i+1} \times ... \times T_n$
  - (t_i, t_{-i}) defined in usual way
- Information: Player i learns her type before choosing strategy
- Outcomes: convex set X
- Strategy profiles form subset of acts, $f : T \to X$
- Player i's ex-ante preference: $\succeq_0^i$ (over acts)
- Player i's preference conditional on $t_i = t$: $\succeq_t^i$

Assumptions

1. Consequentialism:
   If $f(t_i, t_{-i}) = g(t_i, t_{-i})$ for all $t_{-i} \in T_{-i}$, then $f \sim^i_{t_i} g$
2. DC for i:
   If $f \succeq^i_{t_i} g$ for all $t_i \in T_i$, then $f \succeq^i_0 g$ (strict if $\succ^i_{t_i}$ for some t_i)
3. Full support: If $f(t) \succeq^i_0 g(t)$ for all t, then $f \succeq^i_0 g$. Moreover, if there exists t' so that $f(t') \succ^i_0 g(t)$, then $f \succ^i_0 g$.
   - Epstein-Schneider: still role for ambiguity given these three assumptions
4. Common ex-ante behavior: There exists an interval $B \subseteq \mathbb{R}$, a family of utility indexes u_i with range B, and a function $U_0 : (B^T) \to \mathbb{R}$ such that

$$f \succeq^i_0 g \iff U_0(u_i \circ f) \geq U_0(u_i \circ g)$$

Scope

Model includes:

- EU: $U_0(f) = \int fd\pi$, Savage (1954)
- MEU: $U_0(f) = \min_{\pi \in \Pi} \int fd\pi$, Gilboa-Schmeidler (1989)
- Variational: $U_0(f) = \min_{\pi \in \Delta\Omega}[\int fd\pi + c(\pi)]$, $c : \Delta\Omega \to [0, \infty]$, Maccheroni et al (2006)
- Smooth: $U_0(f) = \int \phi(\int fd\pi)\mu(d\pi)$, $\mu \in \Delta(\Delta\Omega)$, Klibanoff et al (2005)
- most others you can name

Result

Theorem

Players satisfy common ex-ante behavior, full support, consequentialism, and DC for each i if and only if

$$U_0(f) = \sum_{t_{-0} \in T_{-0}} U_t(f(\cdot, t_{-0}))$$

(perhaps after a normalization), where each of the $U_t : B^{T_0} \to \mathbb{R}$ *is a (strictly) monotone, continuous function.*

Result

Corollary

If players satisfy Assumptions 1-4 and U_0 is MEU with set of priors Π, then for any $t \in T_{-0}$ and any $\pi, \pi' \in \Pi$, $\pi(T_0 \times \{t\}) = \pi'(T_0 \times \{t\})$.

- Expected utility when uncertainty only involves signals

Corollary

Under Assumptions 1-4, if T_0 is a singleton, then $U_0(\cdot)$ is expected utility, after a normalization.

- If types/signals are only uncertainty, then no role for ambiguity

Idea of proof

- Similar idea to Epstein and Seo (2011) but different construction
- Key tool: Gorman (1968)'s overlapping theorem: intersection and union of overlapping, separable events are also separable
  - Separable: Savage's P2 holds for event
  - i.e. $fEh \succeq_0 gEh$ if and only if $fEh' \succeq_0 gEh'$
- If DC and Consequentialism hold, event $\{t \in T : t_i = \tau\}$ is separable for any i and any $\tau \in T_i$
- Use this to show $T_0 \times \{t_1\} \times ... \times \{t_n\}$ separable
- Given at least 3 pairs of types, Gorman's theorem implies U_0 has additive structure

Idea of proof

- In example, DC $+$C imply the following are separable:
    1. $\{R_I\} \times \{B_{II}, R_{II}\}$,
    2. $\{B_I\} \times \{B_{II}, R_{II}\}$
    3. $\{B_I, R_I\} \times \{R_{II}\}$
    4. $\{B_I, R_I\} \times \{B_{II}\}$

  all separable
- Taking intersection, the following are also separable:
    1. $\{(R_I, B_{II})\}$
    2. $\{(B_I, B_{II})\}$
    3. $\{(R_I, R_{II})\}$
    4. $\{(B_I, R_{II})\}$
- Classic representation result: U_0 must be additive

Extension

- All assumptions important for result
- Common ex-ante behavior is less compelling than common priors
  - Common "perception" of uncertainty (good)
  - Common "attitude" towards uncertainty (not so good)
- Remainder of talk: relax common ex-ante behavior
- Focus on α-MEU model with heterogeneous α but same priors

Extension

Definition

A preference relation $\succeq$ has an (α, C)-MEU representation if for any acts f, g, $f \succsim g \iff V(f) \geq V(g)$ where

$$V(h) = \alpha \min_{p \in C} \int u \circ h dp + (1 - \alpha_i) \max_{p \in C} \int u \circ h dp.$$

Definition (α_i-MEU with set of priors C)

There exist sets $C_t \subset \Delta\Omega$ and an $\alpha_i \in [0, 1] \setminus \frac{1}{2}$ for every $i \in I$ such that for all $i \in I$:
(i) $\succeq_0^i$ has a minimal (α_i, C)-MEU representation, and
(ii) $\succeq_t^i$ has a minimal (α_i, C_t)-MEU representation for each $t \in T_i$.

- **NB:** $\alpha_i = \frac{1}{2}$ ruled out & $\alpha_i = 0$ for all i is earlier assumption

Extension

Corollary

If players satisfy DC for each i, Consequentialism, Full-Support, and α_i-MEU with set of priors C,
then for any $t \in T_{-0}$ and any $\pi, \pi' \in C$, $\pi(T_0 \times \{t\}) = \pi'(T_0 \times \{t\})$.

- "Beliefs" the same but taste for ambiguity different
- Same conclusion as main Theorem
- Suggests (but does not prove) that same perception of ambiguity drives result
- Have attempted same result with KMM but assumptions