Correlation Concern[1]

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[1] also bits of "Correlation Misperception in Choice", joint with Michele Piccione

Motivation

- Correlation is everywhere
  - **NB:** correlation shorthand for joint distribution
- Correlation is hard to estimate
- Correlation misperception has large costs
  - Subprime MBS losses larger than $450 billion
- How is choice affected when agents aware of complex connections in environment and concerned they may get it wrong?

Motivation

1. An individual may choose an index fund over a comparable collection stocks because she understands each of their individual distributions but not their correlation with each other
2. A financial institution may choose a suboptimal loan portfolio in order to pass a stress test that ensures it is not subject to too much systematic risk
3. A principal may design a contract to ensure that it is robust to the agent's knowledge of the correlation between the payoff she offers, the agent's own information, and the private information and actions of other agents

Objectives

- This paper: axiomatic model of agent who recognizes risks may be correlated and concerned she does not know exactly how
- Ellis and Piccione (2017; EP) introduced framework for studying misperception of correlation
  - e.g. S&P 500 index fund not indifferent to underlying 500 stocks
- EP: agents assign probability to all possible joint distributions and maximize expected utility
  - Appropriate when agents don't realize correlation uncertain
    - ⋆ e.g. MBS prior to 2007: $>$ \$600B securitized per year
  - Less appropriate when realize correlated but don't know degree
    - ⋆ e.g. MBS after 2008: $<$ \$20B issued per year, "none" securitized

Behavior of Interest, Part i

- EP thought experiment: choice between
  1. \$100 for sure ($\langle 100 \rangle$), and
  2. the combination of b_C and b_F ($\langle b_C, b_F \rangle$), where

$$b_C = \begin{cases} \$100 & \text{if high temp. here tomorrow } \geq 20^\circ C \\ \$0 & \textit{otherwise} \end{cases}$$

and

$$b_F = \begin{cases} \$100 & \text{if high temp. here tomorrow } < 68^\circ F \\ \$0 & \textit{otherwise} \end{cases}$$

- \$100 strictly preferred to having **both** b_C and b_F, i.e. $\langle 100 \rangle \succ \langle b_C, b_F \rangle$
- If portfolio reduced to act on temp., then must be indifferent

Behavior of Interest, Part ii

- Another experiment: Choose between
  1. bet b such that
     $$b = \begin{cases} \$100 & \text{if high temp. here tomorrow } \geq 20^\circ C \\ -\$100 & \textit{otherwise} \end{cases}$$
  2. the combination of bets b_C and $-b_F$, where
     $$-b_F = \begin{cases} -\$100 & \text{if high temp. here tomorrow } < 68^\circ F \\ \$0 & \textit{otherwise} \end{cases}$$
- Interested in an agent who expresses:
  - $\langle b \rangle \succ \langle b_C, -b_F \rangle$,
  - $\langle 100 \rangle \succ \langle b_C, b_F \rangle$, and
  - $\langle 0 \rangle \sim \langle b_F, -b_F \rangle$

Behavior of interest, Part ii

- Behavior of interest:
  - $\langle 100 \rangle \succ \langle b_C, b_F \rangle$, and $[\langle 100 \rangle \approx \langle b_C + b_F \rangle]$
  - $\langle b \rangle \succ \langle b_C, -b_F \rangle$ $[\langle b \rangle \approx \langle b_C - b_F \rangle]$
- Prefers "simple" version to "complex" version
- Reasonable if thinks $68^\circ F \approx 20^\circ C$ but not sure and knows not sure
- In EP with strictly concave u, impossible for both to hold
  - Either indifferent or underestimates the correlation between one pair
  - But then overestimates the correlation of the other pair

Modeling the Behavior of Interest, part i

- $\langle 100 \rangle$ yields \$100 for sure, but $\langle b_C, b_F \rangle$ yields either \$100, \$200 or \$0 depending on temperature and whether $20^\circ C \gtrless 68^\circ F$
- EP introduce **Weak Monotonicity** axiom to allow such preference
- Allows Monotonicity violations "caused" by correlation misperception
- Idea: if $\langle b, c \rangle$ always yields a better outcome than $\langle a \rangle$ for **every possible joint distribution** over a, b, c, then $\langle b, c \rangle \succsim \langle a \rangle$
  - $\min_\omega b(\omega) + \min_\omega c(\omega) \geq \max_\omega a(\omega) \implies \langle b, c \rangle \succsim \langle a \rangle$
  - Implies $\langle 50, 50 \rangle \sim \langle 100 \rangle \succ \langle 49, 50 \rangle$ but **does not** imply $\langle 100 \rangle \sim \langle b_C, b_F \rangle$

Modeling the Behavior of Interest, part ii

- Ceteris paribis, DM prefers uncorrelated prospects
- Capture this using lotteries and independence axiom logic
- Propose a "negative uncorrelated independence axiom"

$$\underbrace{\langle a, b\rangle}_{\text{correlated}} \succsim \underbrace{\langle c\rangle}_{\text{uncorrelated}}$$
$$\Longrightarrow \underbrace{\left(\frac{1}{2}, \langle a, b\rangle; \frac{1}{2}, r\right)}_{\text{correlated}} \succsim \underbrace{\left(\frac{1}{2}, \langle c\rangle; \frac{1}{2}, r\right)}_{\text{correlated}}$$

- With other standard axioms, there exists a representation where DM
  - considers a **set of possible correlations** in mind
  - evaluates each profile according to **worst** one

Evidence

- Experimental evidence, behavior of interest: Epstein and Halevy (2017)
- Experimental evidence, correlation misperception: Enke and Zimmerman (2017), Eyster and Weiszacker (2011), Rubinstein and Salant (2015), Hossain and Okui (2018)
- Indirect evidence from fiancial markets:
  - under-diversification and limited participation (Jiang and Tian, 2016; Liu and Zeng, 2016; Huang et al (2017))
  - comovement and correlated trading patterns (Jiang and Tian, 2016)

Related Theory Literature

- "Sophisticated" correlation neglect of Levy-Razin (2016) and Laokunakorn-Levy-Razin (2017)
- Epstein-Halevy (2017,19)
- Epstein and Seo (2010, 15) have similar representation, with objective state space and standard AA environment
  - focus on learning and indistinguishable but not identical experiments
- "Robustness" in Mechanism Design: Carroll (2017), Li (2017)

Framework

- Exogenous state space Ω describing true relationship between actions
  - e.g. payoff structure in a financial market
- An exogenous set $X = \mathbb{R}$ of consequences
- A set $\mathcal{A}$ of actions, mappings from Ω to X
  - e.g. security or behavioral strategy
  - at least one action corresponding to any Savage Act
  - but maybe more than one per Savage Act
- The set of all **action profiles** $\mathcal{F}$ over $\mathcal{A}$
  - "multi-sets" of actions (order unimportant and repetition allowed)
  - Take actions $a_1, a_2, ..., a_n$: $\langle a_1, a_2, ..., a_n \rangle = \langle a_i \rangle_{i=1}^n$
  - Profile $\langle a_1, ..., a_n \rangle$ yields $\sum_{i=1}^n a_i(\omega)$ in state ω
- Preference $\succsim$ on $\Delta\mathcal{F}$, the set of lotteries over $\mathcal{F}$
- Anscombe-Aumann (1963) / Fishburn (1970) with lotteries first

Axioms

Basic Axioms

1. (WO) $\succsim$ is complete and transitive
2. (C) The upper and lower contour sets of $\succsim$ are closed, in an appropriate toplogy ▸ details

- Weak* convergence for lotteries over constants
- Convergence in probability otherwise

Axiom: Weak Monotonicity (Ellis and Piccione, 2017)

- $\langle 100 \rangle \succ \langle b_C, b_F \rangle$ violates "Monotonicity"
- Want to allow violations of Monotonicity but only those attributable to misperception of correlation
- Idea: if $\langle b, c \rangle$ always yields a better outcome than $\langle a \rangle$ for **every possible joint distribution** over a, b, c, then $\langle b, c \rangle \succsim \langle a \rangle$
  - $\min_\omega b(\omega) + \min_\omega c(\omega) \geq \max_\omega a(\omega) \implies \langle b, c \rangle \succsim \langle a \rangle$
  - Implies $\langle 50, 50 \rangle \sim \langle 100 \rangle \succ \langle 49, 50 \rangle$
  - **Does not** imply $\langle 100 \rangle \sim \langle b_C, b_F \rangle$

Axiom: Weak Monotonicity (Ellis and Piccione, 2017)

- Formalize this notion, taking into account lotteries and that each action perfectly correlated with itself
- **Plausible realization for p and q** is a vector

$$\vec{x} \in range(b_1) \times \cdots \times range(b_n)$$

where $b_1, \ldots, b_n$ are the actions in some possible profile
- assigns an individually possible outcome to each action
- For a plausible realization $\vec{x}$ of p and q, p **induces the lottery**

$$p_{\vec{x}} \equiv \left(p(\langle a_i \rangle_{i=1}^n), \langle \sum_{i=1}^n x^{a_i} \rangle \right)_{p(\langle a_i \rangle)>0}$$

Axiom: Weak Monotonicity (WM)

For any $p, q \in \Delta\mathcal{F}$:
if $p_{\vec{x}} \succsim q_{\vec{x}}$ for every plausible realization $\vec{x}$ of p and q, then $p \succsim q$.

Axiom

Axiom: Action Monotonicity (AM)

For any $a, b \in \mathcal{A}$, if $a(\omega) \geq b(\omega)$ for all $\omega \in \Omega$, then $(1, \langle a \rangle) \succsim (1, \langle b \rangle)$.

- Usual monotonicity holds when comparing alternatives where correlation does not matter
- "Simple" objects do not confuse the DM

Behavior of Interest

Negative Uncorrelated Independence (NUI)

For any $\alpha \in (0, 1]$:

$$\underbrace{\langle a, b\rangle}_{\text{correlated}} \succsim \underbrace{\langle c\rangle}_{\text{uncorrelated}}$$

$$\implies \underbrace{(1/2)\langle a, b\rangle + (1/2)r}_{\text{correlated}} \succsim \underbrace{(1/2)\langle c\rangle + (1/2)r}_{\text{correlated}}$$

$$\underbrace{\langle a, b\rangle}_{\text{correlated}} \succsim \underbrace{\langle c\rangle}_{\text{uncorrelated}}$$

$$\implies \underbrace{\alpha\langle a, b\rangle + (1-\alpha)r}_{\text{correlated}} \succsim \underbrace{\alpha\langle c\rangle + (1-\alpha)r}_{\text{correlated}}$$

$$\underbrace{\langle a, b\rangle}_{\text{correlated}} \succsim \underbrace{q}_{\text{uncorrelated}}$$

Behavior of interest

- dislikes correlation but nonetheless prefers correlated option
- mixing adds correlation to previously-uncorrelated, less-preferred option
- so DM should not switch preference
- Closely related to
  - Default to Certainty (Gilboa et al., 2010) and
  - Negative Certainty IA (Dillenberger, 2010 and Cerreia-Vioglio et al., 2015)

Correlation Concern Representation

- As in EP, uncertainty beyond that captured by Ω relevant
- Represent by expanding the "dimension" of uncertainty
- Cartesian product of Ω one convenient way to model this
- $\Omega^{\mathcal{A}}$ captures all possible joint distributions between actions
  - One dimension per action
  - DM may think she gets $b_F(\tau_F) + b_C(\tau_C)$ from $\langle b_C, b_F \rangle$ for each τ_F, τ_C
  - Probabilities on $\Omega^{\mathcal{A}}$ assign likelihoods to these events

Correlation Concern Representation

A preference $\succsim$ has a **Correlation Concern Representation** if there exists

- continuous, increasing utility index $u : X \to \mathbb{R}$,
- set of finitely additive probability measures Π on $\Omega^{\mathcal{A}}$
  - (with appopriate σ-algebra)
- and a "marginal" probability measure q on Ω

such that for any $p', q' \in \Delta\mathcal{F}$, $p' \succsim q \iff V(p') \geq V(q')$ where

$$V(p) = \min_{\pi \in \Pi} \int_{\Omega^{\mathcal{A}}} \mathbb{E}_{p(\langle a_i \rangle)} \left[u \left(\sum_{i=1}^{n} a_i(\omega^{a_i}) \right) \right] d\pi$$

and the marginals agree with q:

for any $a \in \mathcal{A}$ and all $\pi \in \Pi$, $\int u(a(\omega^a))d\pi = \int u(a(\omega))dq$

Correlation Concern Representation

Theorem

A preference $\succsim$ satisfies Weak Order, Continuity, Weak Monotonicity, Action Monotonicity, and Negative Uncorrelated Independence
if and only if
it has a Correlation Concern Representation

Correlation Concern Representation

- DM acts a standard subjective expected utility maximizer when dealing with single action profiles
(Action Monotonicity and Negative Uncorrelated Independence)
- Misunderstanding of correlation captured when cannot represent with set of priors Π such that $\pi(\omega^a \neq \omega^b) = 0$ for every $\pi \in \Pi$ and $a, b \in \mathcal{A}$
(Failure of Monotonicity)
- DM does not think that "implausible" outcomes can occur
i.e. $\langle b_C, b_F \rangle$ yielding \$300 or $-$\$400
(Weak Monotonicity)
- Recognition and dislike that does not understand correlation captured by Π not a singleton and the min operator
(Negative Uncorrelated Independence)

Correlation Concern Representation

- q interpreted as "true" distribution
- Levy-Razin type representation: uncertain about relationship between signals
- Symmetry axioms imply Epstein-Seo style representation
- Contrast: no ambiguity about marginal distribution

Correlation Concern Representation

- Allows for behavior in examples
- Features (or bugs):
  - Strict preference for randomization
  - State space not unique, nor priors
  - High dimensional state space: flexible but hard to work with
- Soon: special case that addresses last two points

Correlation Concern Representation

Strict generalization of EP:

Corollary (Ellis and Piccione, 2017)

The preference $\succsim$ satisfies Weak Order, Continuity, Weak Monotonicity and Independence
if and only if
it has a Probabilistic Correlation Representation, i.e. a representation where Π is a singleton.

Proof

- NUI implies independence for singleton profiles
  - Yields utility index
- Map profiles to acts on product state sapce
  - Weak Montonicity insures well-defined
  - Preference defined on convex subset of acts
- Construct utility function via certainty equivalent
  - Linear when mixing with actions, so HOD(1)
  - NUI implies superlinear when mixing portfolios
- Extend function to all bounded, measurable functions while maintaining above properties
- Apply Gilboa-Schmeidler (1989) with slight tweak to get set of priors that agree on valuations of actions

Rich Correlation Concern Representation

Story of Representation:

- DM divides assets into subsets that are easy to understand
  - Such a subset of assets called an "understanding class"
- DM reduces any portfolio of assets in same class to act
  - $\mathcal{U}$ is the set of such classes
  - e.g. $\mathcal{U} = \{B_C, B_F\}$ where B_C are actions understood in terms of Celsius and B_F are actions understood in terms of Fahrenheit
  - Every action belongs to a class and each class is "rich"
  - Coarsest: $C' \in \mathcal{U}'$ implies $\exists C \in \mathcal{U}$ with $C' \subseteq C$
- Correctly evaluates within class, uncertain about correlation across classes

Non-singularity and understanding

- Informally, $\succsim$ **understands** $C \subseteq \mathcal{A}$ when "Monotonicity holds for C"
- $\succsim$ **understands** $C \subseteq \mathcal{A}$ if for any $p, q \in \Delta\mathcal{F}$, $p \succsim q$ whenever $p_{\vec{x}} \succsim q_{\vec{x}}$ for all C-synchronous plausible realizations $\vec{x}$ of p and q.
  - A plausible realization is C-synchronous if there is a single state that generating the realizations of all the actions belonging to C
- A set $B \subset \mathcal{A}$ is **rich** if for any function $f : \Omega \to X$, there exists $a \in B$ with $a(\omega) = f(\omega)$ for all $\omega \in \Omega$.

Asssumption

The environment satisfies **Non-Singularity** if each $a \in \mathcal{A}$ belongs to a rich, understood subset of actions.

Rich Correlation Concern Representation

$\succsim$ has a **Rich Correlation Concern Representation** if there exist:

- a Correlation Cover $\mathcal{U} \subset 2^A$ where each $C \in \mathcal{U}$ is rich
- a continuous, increasing utility index u
- a "marginal" probability measure q over Ω
- a set of priors Π over $\Omega^{\mathcal{U}}$ where
  - $\pi \in \Pi$ implies $marg_C \pi = q$ for all $C \in \mathcal{U}$

so that V represents $\succsim$ and

$$V(p) = \min_{\pi \in \Pi} \int_{\Omega^{\mathcal{U}}} \mathbb{E}_{p(\langle a_i \rangle)} \left[u \left(\sum_{i=1}^{n} a_i(\omega^{C(a_i)}) \right) \right] d\pi$$

for any $C : \mathcal{A} \to \mathcal{U}$ with $a \in C(a)$ for all $a \in \mathcal{A}$

Rich Correlation Concern Representation

Theorem

Under Non-Singularity:

a preference $\succsim$ satisfies Weak Order, Continuity, Weak Monotonicity, Action Monotonicity, and Negative Uncorrelated Independence
if and only if
it has a **Rich** Correlation Concern Representation.

- Under non-degeneracy, the utility index u is affinely unique, and the marginal probability measure q is unique
- There is a **unique, coarsest** rich correlation cover $\mathcal{U}$, and for this $\mathcal{U}$, Π is unique whenever u is **not** a polynomial

Rich Correlation Concern Representation

- Behavior interpreted similarly to non-rich case
- The Correlation Cover $\mathcal{U}$, representing the connections that are understood by DM, is revealed from DM's behavior
- As in EP, risk attitude affects whether set of priors is unique
- Marginal distribution unique, so equivalent representation as a family of copulas (if $\Omega \subset \mathbb{R}$)

Rich Correlation Concern Representation

Axiom: Correlation Aversion, CA

For any $a \in \mathcal{A}$ and $q \in \Delta\mathcal{F}$,

$$a(\omega) \succsim \Big(q(\langle a_i \rangle), \sum a_i(\omega)\Big)_{\langle a_i \rangle \in q}$$

for all $\omega \in \Omega$ implies $a \succsim q$.

- Implies Action Monotonicity

Rich Correlation Concern Representation

Corollary

Under Non-Singularity,
the preference $\succsim$ satisfies Weak Order, Weak Monotonicity, Negative Uncorrelated Independence, and Correlation Aversion
if and only if
it has a Rich Correlation Concern Representation where q itself belongs to Π
i.e. there exists $\pi^* \in \Pi$ where $\pi^*(E) = q(\bigcap_{C \in \mathcal{U}} E_C)$ for any event E

- Interpretation: q is true distribution and benchmark
- DM thinks it is "approximately" correct but unsure

Alternative approach

- Li (2017) proposes a robustness criterion for mechanisms called "Obviously Strategy Proof"
- An **obviously dominant strategy** has higher utility in worst possible history for every strategy of others than any other strategy has in its best possible history and strategy for others
  - "Dynamic" Weak Monotonicity
- OSP if each player has obviously dominant strategy
- incomplete ranking

Alternative approach

- Suggests alternative approach: relax completeness
  - Incomparable rather than strict preference in thought experiment
  - Analogus to Bewley in ambiguity aversion literature
- Correlation does not affect actions
- So $\succsim$ may be incomplete, but only on lotteries that contain proper profiles of actions

Axioms

- Preorder (PO): $\succsim$ is reflexive and transitive
- Action Complete (AC): $\succsim$ is complete on lotteries over actions
- Independence (IA): $p \succsim q$ and $\alpha \in (0, 1]$ implies $\alpha p + (1 - \alpha)r \succsim \alpha q + (1 - \alpha)r$

Alternative approach

Theorem

The following are equivalent:

- The preference $\succsim$ satisfies Preorder, Continuity, Action Completeness, Independence and Weak Monotonicity, and
- There exists a set of priors Π as in first representation such that for any $p, q \in \Delta\mathcal{F}$, $p \succsim q$ if and only if

$$\int_{\Omega^{\mathcal{A}}} \mathbb{E}_{p(\langle a_i \rangle)} \left[u \left(\sum_{i=1}^{n} a_i \left(\omega^{a_i} \right) \right) \right] d\pi$$

$$\geq \int_{\Omega^{\mathcal{A}}} \mathbb{E}_{q(\langle a_i \rangle)} \left[u \left(\sum_{i=1}^{n} a_i \left(\omega^{a_i} \right) \right) \right] d\pi$$

for all $\pi \in \Pi$.

Thank you.

Topology

- Identifying $X = \{\langle x \rangle : x \in X\}$
- Endow $\mathcal{F}$ with the metric d that is discrete on $\mathcal{F} \setminus X$ and agrees with the usual metric on X.
- Endow $\Delta\mathcal{F}$ with the weak* topology using this metric on X
- Formally for any profiles $\langle a_i \rangle_{i=1}^{n} \neq \langle b_i \rangle_{i=1}^{n'}$:
  1. $d(\langle a_i \rangle_{i=1}^{n}, \langle b_i \rangle_{i=1}^{n'}) = 1$ whenever n or n' exceeds 1
  2. $d(\langle a_1 \rangle, \langle b_1 \rangle) = 1$ whenever a_1 or b_1 is not constant
  3. $d(\langle x \rangle, \langle y \rangle) = |x - y|$ for $x, y \in X$
- According to d, a sequence of profiles converges only if it is eventually constant or every profile therein is a single action that gives a constant outcome

Return