Equilibrium Securitization with Diverse Beliefs

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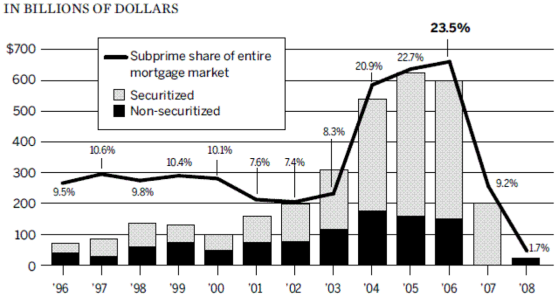
May 29, 2018

Subprime mortgage crisis

Securitization: pooling and tranching.
Senior tranche of pooled subprime mortgages thought safe
Post crisis: correlation underestimated so not actually safe

Subprime Mortgage Originations

In 2006, $600 billion of subprime loans were originated, most of which were securitized. That year, subprime lending accounted for 23.5% of all mortgage originations.



NOTE: Percent securitized is defined as subprime securities issued divided by originations in a given year. In 2007, securities issued exceeded originations.

SOURCE: Inside Mortgage Finance

Motivation

Question:

- Why and how to securitize assets when investors have diverse beliefs?
- What are the consequences of securitization?

Approach:

- Optimal security design with heterogeneous beliefs

Overview of Model

- GE model with
  - a risk free asset called cash and
  - a risky asset (later, collection of risky assets)
- Heterogeneous beliefs about asset's payoff
  - e.g. traders agree on mean but not correlation
- Intermediaries
  - purchase assets
  - issue monotone securities backed by the risky asset

Results

- Simple graphical method to characterize securities sold
- When risk-neutral agents disagree about distribution:
  - Tranching emerges as optimal securitization
  - Traders sort amongst tranches according to
    - misperceptions of correlation
    - value of liquidity
  - Asset price rises above expected value
  - Asset price increases in amount of disagreement

Results

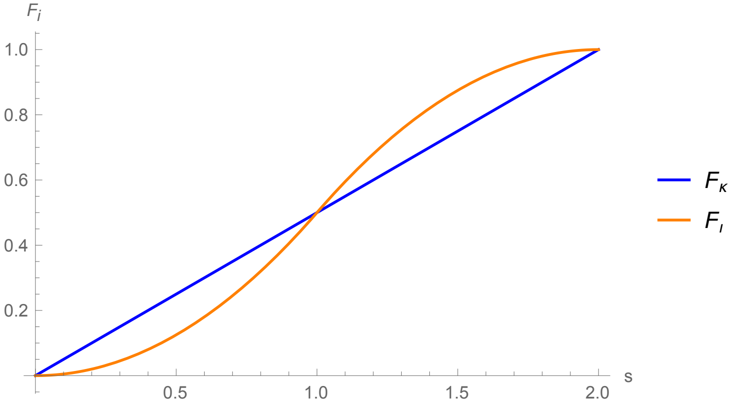
- Incentive for intermediary to pool assets and then tranche the pool when traders disagree about their correlation
  - pooling creates "complexity" and increases revenue by inducing disagreement (cf Ghent et al., 2017)
- Partial equilibrium with risk aversion: very similar to risk-neutral
- General equilibrium with risk aversion:
  - when same beliefs but different tastes, no tranching and sorting
    - (without background risk)
  - Speculation vs. Risk-sharing (vs. Adverse selection)

Setup

- Two period exchange economy
- One representative intermediary (issuer)
- N types of traders of equal measure
- Assets
  - Risky asset pays s in state $s \in S \equiv [0, \bar{s}]$
  - Safe asset (cash) pays 1 in each state
- Type i trader:
  - belief about risky asset payoff: CDF F_i
    F_i is non-atomic, support in S
  - continuously differentiable, concave, and strictly increasing utility index u_i with first derivative bounded
  - endowment:
    e_i^c units of cash, e_i^a assets, θ_i share of the intermediary

Example:

- $N = 2$
- Asset is bundle of two mortgages, each distributed $U[0,1]$
- Trader κ (blue) thinks perfectly correlated
- Trader ι (orange) thinks independent



Model

- The intermediary issues securities backed by the risky asset
- Set of **securities** is

$$\mathcal{B} = \{\phi : S \to \mathbb{R}_+ | \phi \text{ is increasing}\}$$

  - ϕ returns $\phi(s)$ in state s
  - increasing: moral hazard
  - Let $\mathcal{M}(\mathcal{B})$ be set of finite, Borel measures on $\mathcal{B}$ (positive)
- Purchases a_0 units of asset, sells $\mu_0 \in \mathcal{M}(\mathcal{B})$ securities
- **Feasibility** of the securities sold:

$$\int_{\mathcal{B}} \phi(s) d\mu_0 \leq s a_0$$

Model

- **Competitive markets** for the asset and each security
  - Cash numeraire: $p_c = 1$
  - Price of the risky asset: p
  - Price of security ϕ: $q(\phi)$
    - $q : \mathcal{B} \to \mathbb{R}_+$ is price function
- **No short selling**

Issuer's problem

Issuer chooses measure $\mu_0 \in \mathcal{M}(\mathcal{B})$ to maximize profit

$$\pi = \int_{\mathcal{B}} q d\mu_0 - pa_0$$

subject to

$$\int_{\mathcal{B}} \phi(s) d\mu_0 \leq sa_0 \, \forall s \in S$$

and non-negativity

Trader i's problem

Choose cash c_i, asset a_i and security purchases $\mu_i \in \mathcal{M}(\mathcal{B})$ to maximize utility

$$\max_{a_i,\mu_i,c_i} \mathbb{E}_i \left[u_i \left(s a_i + \int_{\mathcal{B}} \phi(s) d\mu_i + c_i \right) \right],$$

subject to

$$p a_i + \int_{\mathcal{B}} q d\mu_i + c_i \leq e_i^c + p e_i^a + \theta_i \pi$$

and non-negativity

- No short selling: $\mu_i \geq 0$

Equilibrium

An ***equilibrium*** for the economy $(F_i, e_i^a, e_i^c, \theta_i)_{i=1}^N$ is an allocation $(a_i, c_i, \mu_i)_{i=1}^N$, π, μ_0 and price vector (p, q) so that

- The intermediary and traders solve their problems
- The allocation is feasible:

$$\sum_{i=0}^{N} a_i \leq \sum_{i=1}^{N} e_i^a$$

$$\sum_{i=1}^{N} c_i \leq \sum_{i=1}^{N} e_i^c$$

$$\sum_{i=1}^{N} \mu_i = \mu_0$$

Proposition

An equilibrium exists.

Security pricing

Start with risk neutral traders: $u_i(x) = x$ for all i and all x

- **Obs. 1:** constant marginal value of cash

$$V_i(w; p, q) = v_i w$$

$$w = pe_i^a + e_i^c + \theta_i \pi.$$

v_i : trader i's marginal return on wealth

  - v_i is implicitly a function of p and q
  - call $v = (v_1, \ldots, v_N)$ the "return vector"

- **Obs. 2:** equilibrium price of security ϕ,

$$q(\phi) \geq \max_i \mathbb{E}_i \left[\frac{1}{v_i} \phi(s) \right],$$

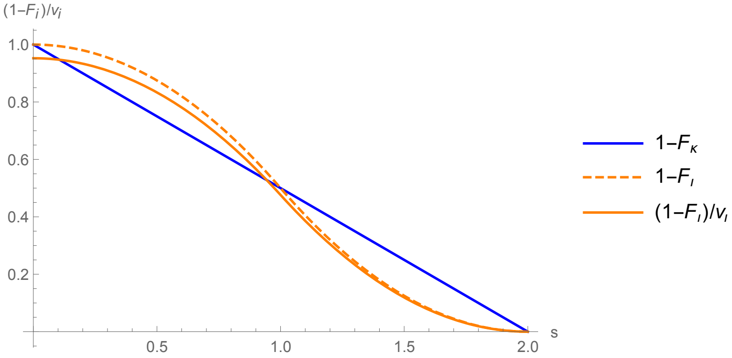
with equality whenever $\mu_0(\{\phi\}) > 0$

Intermediary's securitization decision

- We can write
$$\phi(s) = \int_S \chi_{[x,\bar{s}]}(s) d\phi(x)$$
(Lebesgue-Stieltjes measure; χ_E is indicator of E)
- By Fubini's Theorem, we have that
$$\mathbb{E}_i\left[\frac{1}{v_i}\phi(s)\right] = \int_S \frac{1}{v_i}\left[1 - F_i(x)\right] d\phi(x)$$
- Maximal revenue of the intermediary, as a function of v, is
$$r(v) = \int_0^{\bar{s}} \max_k v_k^{-1}\left[1 - F_k(x)\right] dx$$
per unit of asset securitized

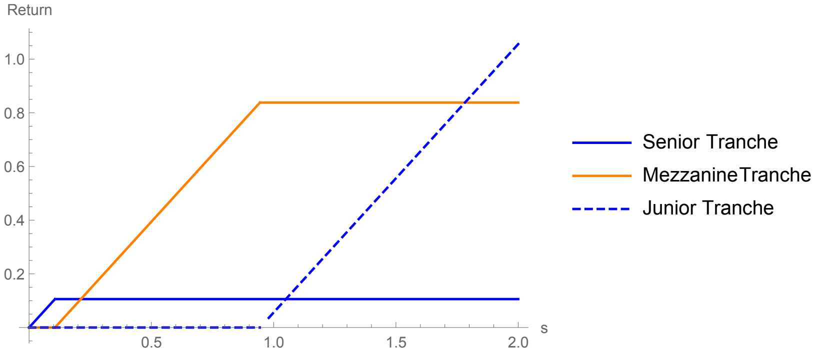
Intermediary's securitization decision

- Simple method for solving problem:
  1. Plot Inverse CDFs, adjusted for value of cash
  2. Maximum revenue is area below upper-envelope
  3. Find tranches corresponding to upper-envelope



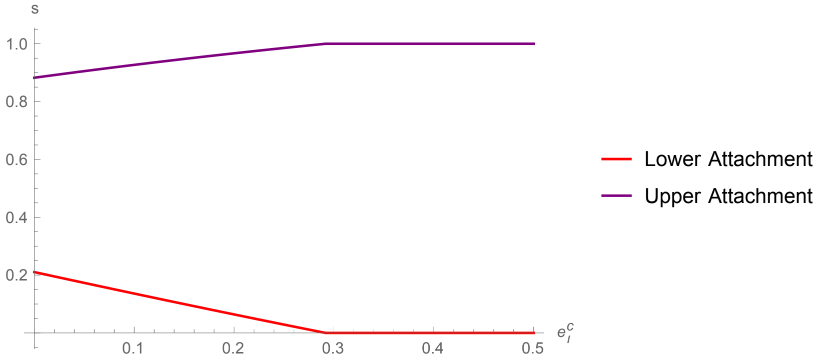
Intermediary's securitization decision

- Simple method for solving problem:
  1. Plot Inverse CDFs, adjusted for rate of return
  2. Maximum revenue is area under upper-envelope
  3. Find tranches corresponding to upper-envelope



Equilibrium

Equilibrium with endowments $e^a_\iota = e^a_\kappa = \frac{1}{2}$, $e^c_\kappa = 1$ and $e^c_\iota > 0$:



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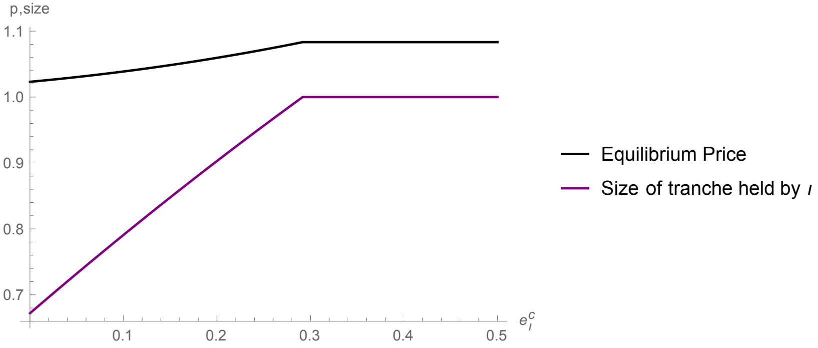

Interlude: Assumption

Assumption : Finite Crossing

For distinct traders i, j and any number $k > 0$,

$$1 - F_i(x) = k\,[1 - F_j(x)]$$

for at most finitely many $x \in [0, \bar{s}]$

- Finite Crossing implied by any of the following (among others):
  - Strict MLRP
  - Finite (or single) Crossing of Hazard Rates
  - Each F_i analytic on $(0, \bar{s})$

Equilibrium properties

Proposition

With risk neutrality:

- *Equilibrium utility and price are unique,* $p = r(\hat{v})$

In addition, with Finite Crossing:

- *equilibrium consumption is state-by-state unique and*
- *equilibrium supply of securities can equal a finite set of tranches*

- Tranche promises cash flow of asset above a but below b
    - Security $\phi_{[a,b]}$ with $\phi_{[a,b]}(0) = 0$, slope 1 on interval $[a, b]$, and slope 0 otherwise

Equilibrium properties: tranching

With Finite Crossing, tranching and sorting equilibrium exists:

Definition

An equilibrium is a ***tranching and sorting equilibrium*** if

1. only tranches are sold
2. each tranche is targeted at a particular trader.

proper if at least two are sold and none can be combined

Formally, there are intervals $\{[a_i, b_i] : i = 1, \ldots, m\}$ with $a_1 = 0$, $b_m = \bar{s}$, $a_i < b_i$, and $a_{i+1} = b_i$ such that $\hat{\mu}_0 \left(\left\{ \phi_{[a_i, b_i]} : i = 1, \ldots, m \right\}^c \right) = 0$ and $\hat{\mu}_i(\{\phi_{[a_j, b_j]}\}) > 0$ implies that $\hat{\mu}_k(\{\phi_{[a_j, b_j]}\}) = 0$ for all $k \neq i$.
Proper if $m \geq 2$ and $\hat{\mu}_i(\{\phi_{[a_j, b_j]}\}) > 0$ implies $\hat{\mu}_i(\{\phi_{[a_{j+1}, b_{j+1}]}\}) = \hat{\mu}_i(\{\phi_{[a_{j-1}, b_{j-1}]}\}) = 0$.

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Equilibrium properties: prices

Proposition

With risk neutrality, if $\mathbb{E}_i[s] = m$ for $i = 1, \ldots N$, then in any equilibrium, $\hat{p} \geq m$.
Under Finite Crossing, this is strict.

- Similar to Harrison-Kreps and Fostel-Geanakoplos

Equilibrium properties: changes in beliefs

Proposition

Let e_1^c, e_2^c be sufficiently large and $\mathbb{E}_{F_1}[s] = \mathbb{E}_{\tilde{F}_1}[s]$.
With risk neutrality, replacing Trader 1's beliefs F_1 with $\tilde{F}_1$ ***increases the equilibrium price*** *if and only if*

$$\int_0^{\bar{s}} |\tilde{F}_1(x) - F_2(x)| dx \geq \int_0^{\bar{s}} |F_1(x) - F_2(x)| dx.$$

- increasing disagreement increases price
- correct notion of disagreement is L_1-norm between CDFs

Equilibrium properties: asset holding

Proposition

Consider a tranching and sorting equilibrium where Trader i holds the senior tranche. With risk neutrality and Finite crossing:
If the cash endowment of Trader i is ***increases*** *by Δ, then Trader i's equilibrium allocation of cash* ***increases*** *by Δ*

- Generically, trader who holds senior tranche also holds cash
- Misidentification of risk preference from equilibrium demand
- Intuition:
    - Difference in beliefs about low returns is small
    - Demanded rate of return alone fixes WTP for senior tranche
    - Cash gives lowest rate, so this trader also holds cash

Risk Aversion

- Trader i maximizes

$$\mathbb{E}_i \left[u_i \left(s a_i + \int_{\mathcal{B}} \phi(s) \mu_i(d\phi) + c_i \right) \right],$$

where utility index $u_i : \mathbb{R}_+ \to \mathbb{R}$ is

  - strictly increasing
  - continuously differentiable with bounded derivative
  - weakly (or strictly) concave

- What do optimal securities look like?

Skip to Pooling

Security Pricing

- Two important endogenous variables:
  - $w(s) = (w_1(s), ..., w_2(s))$, the state-by-state *wealth function*
  - $v = (v_1, \ldots, v_N)$, the *return vector* (on ex ante wealth)
    - Lagrange multiplier on Budget Constraint
- For any security ϕ, we must have that

$$\int_0^{\bar{s}} u_i(w_i(s) + \epsilon\phi(s))dF_i - v_i q(\phi)\epsilon \leq \int_0^{\bar{s}} u_i(w_i(s))dF_i$$

  for infinitessimal $\epsilon > 0$ (also $\epsilon \leq 0$ if $\mu_i(\{\phi\}) > 0$)

- Dividing by ϵ and letting $\epsilon \to 0$, we have

$$\frac{1}{v_i} \int_S u_i'(w_i(s))\phi(s)dF_i \leq q(\phi)$$

  with equality for $\hat{\mu}_0$-a.e ϕ

Security Pricing

- Rewriting:

$$q(\phi) = \frac{1}{v_i}\int_S u_i'(w_i(s))\phi(s)dF_i$$
$$= \max_i \left\{ \frac{1}{\tilde{v}_i}\left(Cov_i\left[\phi, \frac{u_i' \circ w_i}{\mathbb{E}_i\left[u_i' \circ w_i\right]}\right] + \mathbb{E}_i\left[\phi\right]\right)\right\}$$

where $\tilde{v}_i = \frac{v_i}{\mathbb{E}_i\left[u_i' \circ w_i\right]}$ is opportunity cost in terms of cash

- Incentive to take advantage of disagreement
- But blunted by desire to share risks (Cov_i term negative)

Intermediary's securitization decision

- Find optimal securities in similar way to before: let

$$G_i(x|w) = \int_x^{\bar{s}} u_i'(w_i(s))dF_i(s)$$

  - $u_i'(x) = 1$ implies $G_i(x|w) = 1 - F_i(x)$
- For right-continuous ϕ, use Fubini to show

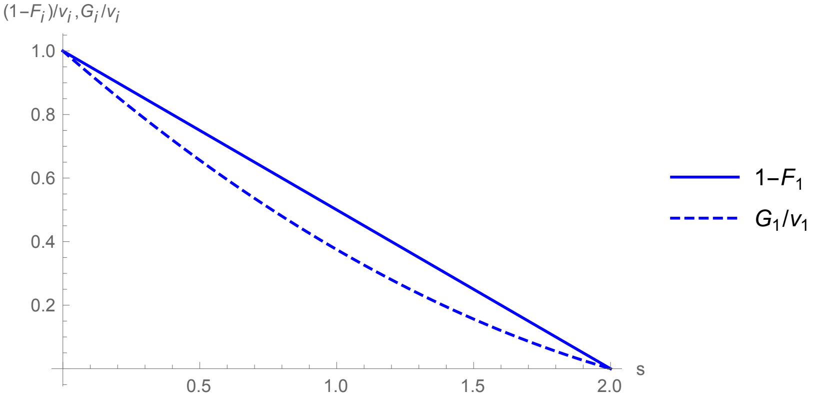
$$\int_{x\in S} u_i'(w_i(x))\phi(x)dF_i(x) = \int_{x\in S} G_i(x|w_i)d\phi(x)$$

- So for μ_0-a.e. $\phi \in \mathcal{B}$ we must have

$$q(\phi) = \max_k \frac{1}{v_k} \int_S G_k(x|w)d\phi(x)$$

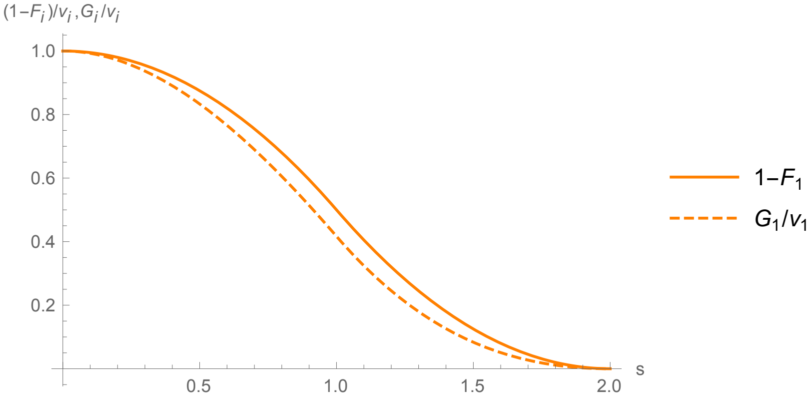
- Can solve issuer's problem as before, when we replace $v_i^{-1}[1 - F_i(x)]$ with $v_i^{-1} G_i(x)$

Intermediary's securitization decision



- $u_i(x) = x - \frac{1}{8}x^2$ and $w_i(s) = 1 + s$ for $i = 1, 2$
- same beliefs as before

Intermediary's securitization decision



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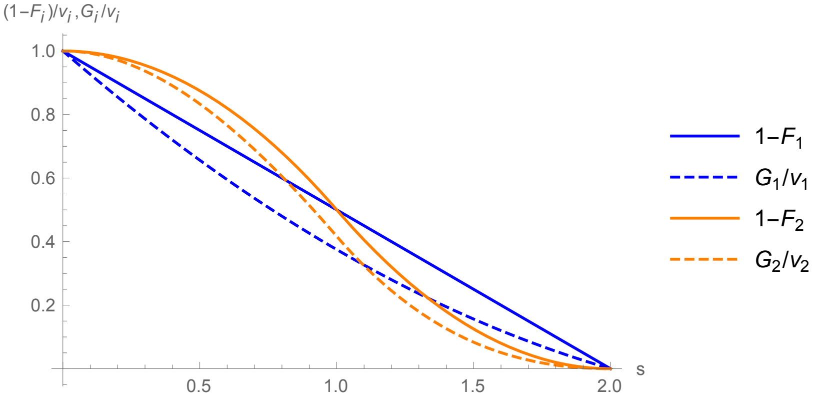

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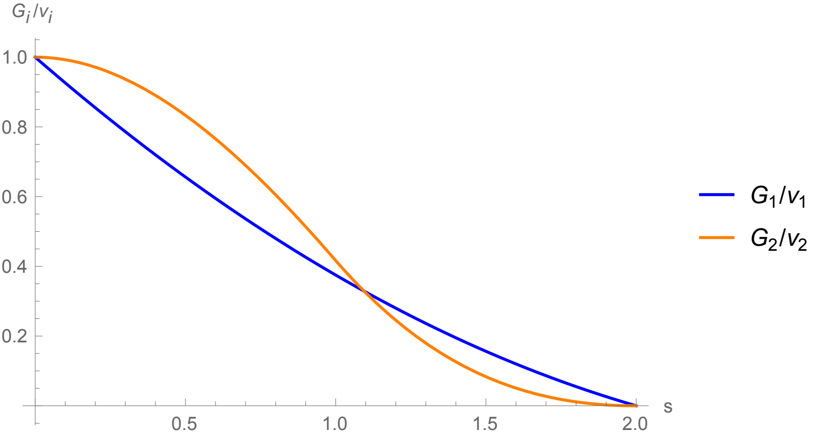

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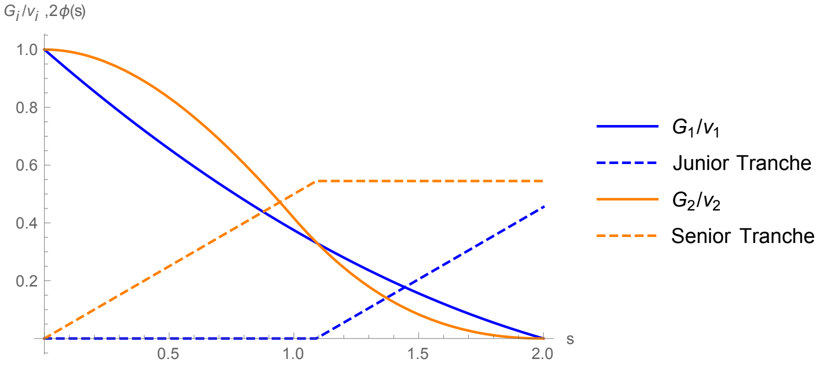

- $u_i(x) = x - \frac{1}{8}x^2$ and $w_i(s) = 1 + s$ for $i = 1, 2$
- same beliefs as before
- **NB:** Not equilibrium – $w_i(s)$ does not match up

Intermediary's securitization decision

Proposition

If the equilibrium return vector is $\hat{v}$ and state-by-state wealth is $\hat{w}$, then the issuer obtains revenue

$$\int_0^{\bar{s}} \max_k \hat{v}_k^{-1} G_k(x|\hat{w})dx.$$

- Generalizes previous result since $G_k(x|\hat{w}) = 1 - F_k(x)$ with risk neutrality

Diverse beliefs vs diverse tastes

Proposition

For traders with strictly concave utility indices and homogeneous, full-support beliefs, no proper tranching and sorting equilibrium exists when endowments are large enough that all traders hold cash.

- Diverse tastes alone does not generate tranching
- Optimal securitization does not allocate risky tranches to those most willing to bear it

Diverse beliefs vs diverse tastes

- Suppose each Trader i has a CARA utility with index α_i
- Equilibrium with large enough cash endowments and same beliefs:
  - Trader i purchases $\frac{\alpha_i^{-1}}{\sum_{k=1}^{N} \alpha_k^{-1}}$ units of the asset and no securities
  - The equilibruim asset price is

$$\hat{p} = \frac{\int_0^{\bar{s}} s \exp\left[-\left(\sum_{k=1}^{N} \alpha_k^{-1}\right)^{-1} s\right] dF(s)}{\int_0^{\bar{s}} \exp\left[-\left(\sum_{k=1}^{N} \alpha_k^{-1}\right)^{-1} s\right] dF(s)}$$

  - Tranching has no value since

$$\frac{1}{v_i} G_i(x|w_i(s)) = \frac{\int_x^{\bar{s}} \exp\left[-\left(\sum_{k=1}^{N} \alpha_k^{-1}\right)^{-1} s\right] dF(s)}{\int_0^{\bar{s}} \exp\left[-\left(\sum_{k=1}^{N} \alpha_k^{-1}\right)^{-1} s\right] dF(s)}$$

Setup with two types of risky assets

- Two traders
- Two types of risky assets, with payoffs $s_1 \in S_1$ and $s_2 \in S_2$
  - e.g. mortgages, credit card debt, auto-loans
- Endowment of type i trader: e_i^c, e_i^1, e_i^2
- Intermediary
  - purchases some amount of each asset
  - sells securities backed by return of entire pool

Setup with two types of risky assets

- Trader i thinks S_1, S_2 independent with probability ρ_i; otherwise, perfectly correlated
- Each asset has same marginal density f
  - Same marginal beliefs
  - No role for securities backed by single asset
- f differentiable, log-concave and symmetric about its mean
  - Applies to the uniform, normal, logistic or truncated normal distributions

Intermediary's problem

- The intermediary purchases a_0^j units of asset j
  - $h = h(a_0) \equiv \frac{a_0^1}{a_0^1 + a_0^2}$: proportion of type 1 asset in his pool
- Same set of contracts as before, but ϕ returns $\phi(s_h)$ where

$$s_h = hs_1 + (1-h)s_2$$

  - Write $F^h(\cdot)$ for CDF of s_h
- Maximizes profit

$$\max_{a_0,\mu_0} \left[\int_{\mathcal{B}} q_{h(a_0)}(\phi) d\mu_0 - p_1 a_0^1 - p_2 a_0^2 \right]$$

subject to

$$\int_{\mathcal{B}} \phi(s) d\mu_0 \leq (a_0^1 + a_0^2)s, \ \forall s \in [0, \bar{s}]$$

Pooling characterization

Proposition

If each e^i_c is large enough, then there exist an equilibrium where all assets are pooled and the price of both assets exceeds their mean. More formally: $\hat{a}^j_o = e^j_1 + e^j_2$ for $j = 1, 2$,

$$\hat{h} = \frac{\sum_{i=1}^{2} e_i^1}{\sum_{i=1}^{2} e_i^1 + \sum_{i=1}^{2} e_i^2}$$

$$\hat{p}_1 = R(\hat{h}) + (1 - \hat{h})R'(\hat{h})$$

$$\hat{p}_2 = R(\hat{h}) - \hat{h}R'(\hat{h})$$

for $R(h) = \int_S \max_k (1 - F_k^h(x))dx$

- Pooling and tranching allow traders to bet on correlation
  - (the correlation trade)
- Drives up asset price – sell asset and use to speculate
- "complexity" causes disagreement but does not deceive traders

Conclusion and Related Literature

- Collateralized loans with diverse beliefs:

  Simsek (2013a), Geanakoplos and Zame (1997/2014), Geanakoplos (2001/03), Fostel and Geanakoplos (2015), Gong and Phelan (2016), Toda (2015), ...
  - Existing literature focuses on
    - a particular structure for possible securities and
    - optimists vs pessimists: first moment heterogeneity
- Optimal security design – Allen and Gale (1988)
  - with diverse beliefs: Germaise (2001), Simsek (2013b), Ortner-Schmalz (2016)
  - under adverse selection: Dang, Gorton and Holmstrom (2015); DeMarzo-Duffie (1999); Fahri and Tirole (2015)
- Correlation misperception: Ellis and Piccione (2017)