SUPPLEMENT TO "FOUNDATIONS FOR OPTIMAL INATTENTION"

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1. Identification

In the text, an example is given showing that the prior is not unique in general. This example is a special case where the likelihoods of certain events do not affect the DM's choices. The likelihood of E is *not* decision relevant if the DM's choice when a state in E obtains does not depend on the consequence that any available act takes on E^c and vice versa. To state the formal definition, first let $BEg = \{fEg : f \in B\}$ for any $E \in \Sigma$, any $g \in \mathcal{F}$, and all $B \in K(\mathcal{F})$.

Definition A1. The *likelihood of E is not decision-relevant* if for any $g \in \mathcal{F}$ and any $B \in K(\mathcal{F})$ with a unique optimal partition,[1] $f \in c(B|\omega) \iff fEg \in c(BEg|\omega)$ for all $\omega \in E$ and $f' \in c(B|\omega') \iff f'E^cg \in c(BE^cg|\omega')$ for all $\omega' \in E^c$. Otherwise, *the likelihood of E is decision-relevant.*

This definition is very strong. If the likelihood of E is not decision-relevant, then the cost of paying attention to information about the states in E is unaffected by the information to which she pays attention about the states in E^c and vice versa. Moreover, if f is available and an act is introduced that improves f on E^c but disimproves f on E, then the new act is never chosen for any state in E.

Say that the *likelihoods of all events are decision-relevant* if the likelihood of E is decision-relevant for all E except $\emptyset$ and Ω.

Theorem A1. *If both $(u, \pi, \gamma, \hat{P})$ and $(u', \pi', \gamma', \hat{Q})$ represent $c(\cdot)$, then*

1. *$supp(\gamma) = supp(\gamma')$,*
2. *$\hat{Q}(B) = \hat{P}(B)$ for all $B \in K(\mathcal{F})$ whenever $\hat{P}$ and $\hat{Q}$ are canonical,*
3. *$\pi = \pi'$ whenever the likelihoods of all events are decision-relevant, and*
4. *there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ so that $u'(x) = \alpha u(x) + \beta$ for all $x \in X$ and $\gamma'(Q) = \alpha\gamma(Q)$ for all $Q \in \mathbb{P}$ whenever $\pi = \pi'$.*

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[1]From behavior: any B for which there exists an $\epsilon > 0$ so that

$$c(B|\omega) \neq c(B|\omega') \iff c(B'|\omega) \neq c(B'|\omega')$$

for all B' with $d(B, B') < \epsilon$.

The proof is similar to that in Ellis [2017], except more care must be taken for uniqueness of the prior.

Proof. I show only that if all events are decision relevant, then $\pi_1 = \pi_2$, as the remainder is as in Theorem 3 of Ellis [2017]. Say that Q *is exposed* if $\gamma_1(Q) < \infty$ and any Q' strictly finer than Q has $\gamma_1(Q') > \gamma_1(Q)$.

Lemma A1. *If Q is exposed, then $\pi_1(\cdot|E) = \pi_2(\cdot|E)$ for each $E \in Q$.*

Proof. Follows from very small stakes betting. □

Lemma A2. *If $\pi_1(\cdot|E) = \pi_2(\cdot|E)$, $\pi_1(\cdot|E') = \pi_2(\cdot|E')$ and $E \cap E' \neq \emptyset$ then $\pi_1(\cdot|E \cup E') = \pi_2(\cdot|E \cup E')$.*

Proof. Follows from Bayes rule using

$$\frac{\pi_1(E \cap E'|E \cup E')}{\pi_1(E|E \cup E')} = \pi_1(E \cap E'|E) = \pi_2(E \cap E'|E) = \frac{\pi_2(E \cap E'|E \cup E')}{\pi_2(E|E \cup E')}$$

and that $\pi_1(E \cup E'|E \cup E') = \pi_2(E \cup E'|E \cup E') = 1$. □

Let $\mathbb{Q}_0$ be the finest common coarsening of the set of exposed partitions, noting $\gamma(\mathbb{Q}_0) = 0$. By Lemma A2, if $E \in \mathbb{Q}_0$ then $\pi_1(\cdot|E) = \pi_2(\cdot|E)$. Now, introduce some notation. For any partition Q and event $E \in \mathbb{Q}_k$, define

$$Q^k_E = \{F \cap E : F \in Q\} \cup \{F \cap E^c : F \in \mathbb{Q}_k\}.$$

Note that $\pi_1(F) = \pi_2(F)$ for all $F \in \mathbb{Q}_0$.

Lemma A3. *Suppose $\gamma_1(\mathbb{Q}_k) = 0$ and $\pi_1(\cdot|F) = \pi_2(\cdot|F)$ for all $F \in \mathbb{Q}_k$. If there exists an exposed Q such that $\sum_{F \in \mathbb{Q}_k} \gamma_1(Q^k_F) \neq \gamma_1(Q)$, then there exists $\mathbb{Q}_{k+1}$, strictly coarser than $\mathbb{Q}_k$ and thus having $\gamma_1(\mathbb{Q}_{k+1}) = 0$, with $\pi_1(\cdot|F) = \pi_2(\cdot|F)$ for all $F \in \mathbb{Q}_{k+1}$.*

Proof. Label the given $\mathbb{Q}_k = \{F_1, F_2, ..., F_n\}$ and suppose Q is such that $\sum_i \gamma_1(Q^k_{F_i}) \neq \gamma_1(Q)$. Let Q be one of the Q satisfying the hypothesis with the most cells in $\mathbb{Q}_k$; there are at least two indexes (WLOG, $1, 2$) such that $Q^k_{F_i} \neq \mathbb{Q}_k$. Otherwise, all but at most one equal $\mathbb{Q}_k$, so for $i > 1$, $Q^k_{F_u} = \mathbb{Q}_k$ and $\gamma(Q^k_{F_i}) = 0$; this implies $Q = Q_{F_1}$ and thus $\gamma_1(Q) = \sum_{F \in \mathbb{Q}_k} \gamma_1(Q^k_F)$, a contradiction.

Denote $\alpha_1 = 1$ and $\alpha_2 = \alpha$ and identify $x \in \mathbb{R}$ with $x \in X$ such that $u_1(x) = x$. Let $Q^{i'} = \{E \cap F^c_{i'} : E \in Q\} \cup \{F_{i'}\}$ for $i' = 1, 2$. I want to write $\gamma_j(Q) - \gamma_j(Q^1)$, $\gamma_j(Q) - \gamma_j(Q^2)$, $\gamma_j(Q^k_{F_1})$, and $\gamma_j(Q^k_{F_2})$ as a function of $\pi_j(F_1)$ and $\pi_j(F_2)$.

To write $\gamma_j(Q^k_{F_{i'}})$ as a function of $\pi_j(F_{i'})$, follow the construction on p. 37 of Ellis [2017] with

$$B^{F_{i'}}_x = \{0Fy(x) : F \in \mathbb{Q}_k\} \cup \{xF \cap F_{i'}y(x) : F \in Q^k_{F_{i'}}\}$$

replacing B_x. Note that each E_i defined in Eq. (5) is a subset F_j, so that

$$\begin{aligned}\gamma_j(Q^k_{F_{i'}}) - \gamma_j(\mathbb{Q}_k) &= \sum_i [\pi_j(E_i) - \pi_j(E_{i-1})]\alpha_j x_i^{F_{i'}} \\ &= \alpha_j \pi_j(F_{i'})\{\sum_i [\pi_1(E_i|F_{i'}) - \pi_1(E_{i-1}|F_{i'})] x_i^{F_{i'}}\} \equiv \alpha_j \pi_j(F_{i'}) z_{F_{i'}}.\end{aligned}$$

To write $\gamma_j(Q) - \gamma_j(Q^{i'})$, and follow the construction starting with

$$B_x^{i'} = \{0Fy(x) : F \in Q^{i'}\} \cup \{x[F \cap F_{i'}]y(x) : F \in Q\}.$$

As above, the E_i in the construction (Eq. (5)) is a subset $F_{i'}$, so

$$\begin{aligned}\gamma_j(Q) - \gamma_j(Q^{i'}) &= \sum_i [\pi_j(E_i) - \pi_j(E_{i-1})]\alpha_j x_i^{i'} \\ &= \alpha_j \pi_j(F_{i'})\{\sum_i [\pi_1(E_i|F_{i'}) - \pi_1(E_{i-1}|F_{i'})] u_j(x_i^{i'})\} \equiv \alpha_j \pi_j(F_{i'}) z_{i'}\end{aligned}$$

Because Q^i has one more cell in $\mathbb{Q}^k$ than Q for $i = 1, 2$,

$$\begin{aligned}\gamma_j(Q) &= \gamma_j(Q) - \gamma_j(Q^i) + \gamma_j(Q^i) - \gamma_j(\mathbb{Q}_k) \\ &= \gamma_j(Q) - \gamma_j(Q^i) + \gamma_j(Q^k_{F_{3-i}}) + \sum_{j' \neq 1,2} \gamma_j(Q^k_{F_{j'}}) \\ &= \alpha_j \pi_j(F_i) z_i + \alpha_j \pi_j(F_{3-i}) z_{F_{3-i}} + \sum_{j' \neq 1,2} \gamma_j(Q^k_{F_{j'}}).\end{aligned}$$

Since $\gamma_j(Q) \neq \sum_{j=1}^n \gamma_j(Q^k_{F_j})$, $\alpha_j \pi_j(F_i) z_i \neq \alpha_j \pi_j(F_1) z_{F_i}$. Since the above must hold for both $i = 1, 2$,

$$\alpha_j \pi_j(F_1)(z_1 - z_{F_1}) = \alpha_j \pi_j(F_2)(z_2 - z_{F_2}),$$

implying that

$$\frac{\pi_1(F_2)}{\pi_1(F_1)} = \frac{\pi_2(F_2)}{\pi_2(F_1)}$$

and thus $\pi_1(\cdot|F_1 \cup F_2) = \pi_2(\cdot|F_1 \cup F_2)$. Hence, $\pi_2(\cdot|F) = \pi_1(\cdot|F)$ for all $F \in \mathbb{Q}_{k+1} = \{E_1 \cup E_2, E_3, ...E_n\}$. □

Lemma A4. *For an event E and $j \in \{1, 2\}$, if $\gamma_j(Q_E) + \gamma_j(Q_{E^c}) = \gamma_j(Q)$ all Q, where $Q_E = \{E' \cap E : E' \in Q\} \cup \{E^c\}$ and $Q_{E^c} = \{E' \cap E^c : E' \in Q\} \cup \{E\}$ if and only if E is not decision-relevant.*

Proof. Follows from noting that

$$\int u \circ f d\pi - \gamma(Q) = \pi(E) \int u \circ f d\pi(\cdot|E) - \gamma(Q_E) + \pi(E^c) \int u \circ f d\pi(\cdot|E^c) - \gamma(Q_{E^c})$$

for any Q and f. □

Apply Lemma A3 successively to complete the proof. □

2. An Approach Using Ex Ante Preference

In this section, I revisit the ex ante approach to identifying conditional preference. My goal is twofold. First, I provide three assumptions on ex ante preference that allow elicitation of a consistent conditional choice correspondence satisfying INRA. Second, I provide a more robust but more complication identification strategy for conditional preference. Since it relies only on ex ante choice, the conclusion is similar to de Olivera et al. [2016], though the assumptions that lead to the conclusion differ substantially. Finally, I show that this more robust identification allows me to use results in Ellis [2017] to prove a representation theorem.

As in the text, the DM chooses a menu by maximizing a complete and transitive binary relation $\succsim$ over $K(\mathcal{F})$, with $\succ$ denoting strict preference and $\sim$ denoting indifference. Throughout, I adopt the customary abuse of notation by writing $f \in \mathcal{F}$ instead of $\{f\} \in K(\mathcal{F})$ when it will cause no confusion. The other primitives of the model remain as in the main text. Concepts introduced in the main text are discussed only there.

2.1. Elicitation.

We impose three properties of $\succsim$. Recall that for $E \subseteq \Omega$, fEg is the act that agrees with f on E and g otherwise.

Axiom A1. If for every $\omega' \in \Omega$ there is $f \in A \bigcap B$ such that $f \bigcup B_{\omega'} \sim B$, then $A \succsim B$.

Axiom A2. For any $\omega' \in \Omega$: if $f, g \in B$ and $B \sim f \bigcup B_\omega \sim g \bigcup B_{\omega'}$, then

$$x_{B'}\{\omega\}g \bigcup B'_{\omega'} \sim B'$$

where $B' = f \bigcup B_\omega$.

Axiom A3. For any $\omega \in \Omega$ and $B \in K(\mathcal{F})$, there exists $f \in B$ such that $\{f\} \bigcup B_\omega \sim B$.

Recall the definition of anticipated choice from the text. For a menu B and state ω, define

$$B_\omega = \{x_B\{\omega\}f : f \in B\}$$

where $x_B \in X$ is chosen such that $\{f(\omega')\} \succ \{x_B\}$ for all $f \in B$ and $\omega' \in \Omega$.

Definition A2. The *anticipated choice from the menu B in the state ω* is the set

$$c^A(B|\omega) = \left\{ f \in B : \{f\} \bigcup B_\omega \sim B \right\}.$$

As in the main text, the preference $\succsim$ relates the to the conditional choice correspondence $c(\cdot)$ as follows.

Definition A3. The pair $(\succsim, c)$ is *consistent* when for any $A, B \in K(\mathcal{F})$, if $c(B|\omega') \bigcap A \neq \emptyset$ for each $\omega' \in \Omega$, then $A \succsim B$; and if there exists ω^* such that $c(A|\omega^*) \bigcap B = \emptyset$ and

$$c(A|\omega) \neq c(A|\omega') \iff c(B|\omega) \neq c(B|\omega')$$

for every $\omega, \omega' \in \Omega$, then $A \succ B$.

Proposition A1. *If $\succsim$ satisfies Axioms A1, A2 and A3, then $(\succsim, c^A)$ satisfies Consistency and $c^A(\cdot)$ satisfies INRA.*

Recall that INRA is implied by WARP. The result thus provides a general method for inferring ex post conditional choices from ex ante choice of menu, under the assumption that information is partitional and ex ante and ex post choices are consistent. This applies to models that take a family of preferences $\{\succeq_E\}_{E \in P}$ as a primitive but that are not dynamically consistent, provided that P is a partition of Ω.

Proof. Define $c^A(\cdot)$ as above; by Axiom A3, it is non-empty. We show first that $(\succsim, c^A)$ are consistent. Suppose $c^A(B|\omega) \bigcap A \neq \emptyset$ for all ω. With standard slight abuse of notation, for any act f write f instead of $\{f\}$ when it will not cause confusion. Then for every ω, there exists $f_\omega \in c^A(B|\omega) \bigcap A$. Then $f_\omega \bigcup B_\omega \sim B$ for all ω, and by Axiom A1, $A \succsim B$.

Now, suppose that in addition $c^A(A|\omega^*) \bigcap B = \emptyset$. Pick any $g \in c^A(A|\omega^*)$, and observe $A \sim g \bigcup A_{\omega^*} \succ f_{\omega^*} \bigcup A_{\omega^*}$ since $f_{\omega^*} \notin c^A(A|\omega^*)$. Let $f'_\omega = x_{\{f\} \bigcup B_\omega}\{\omega\} f_\omega$ for each ω. Obviously $f_{\omega^*} \in f_{\omega^*} \bigcup A_{\omega^*}$ and $B_{\omega^*} \bigcup f_{\omega^*} \sim B$ by definition. For $\omega \neq \omega^*$,

$$f'_\omega \bigcup [B_{\omega^*} \bigcup f_{\omega^*}]_\omega \sim B_{\omega^*} \bigcup f_{\omega^*} \sim B$$

by Axiom A2 and $f'_\omega \in A_{\omega^*}$. Applying Axiom A1, $A_{\omega^*} \bigcup f^*_\omega \succsim B_{\omega^*} \bigcup f_{\omega^*}$. Transitivity then implies $A \succ B$.

To see that c^A satisfies INRA, fix arbitrary $A \subset B$ with $c^A(B|\omega') \bigcap A \neq \emptyset$ for all ω'. By Axiom A1, $A \sim B$. If $f \in c^A(A|\omega^*)$, then $A \sim f \bigcup A_{\omega^*}$. Then, $B \succsim f \bigcup B_{\omega^*} \succsim f \bigcup A_{\omega^*} \succsim A \sim B$, so transitivity implies $f \in c^A(B|\omega^*)$. Now, suppose $g \in c^A(B|\omega^*) \bigcap A$. As above, for every ω, let $f_\omega \in c^A(B|\omega) \bigcap A$ and $f'_\omega = x_{\{f\} \bigcup B_\omega}\{\omega\} f_\omega$. For $\omega \neq \omega^*$, $f'_\omega \bigcup [B_{\omega^*} \bigcup f_{\omega^*}]_\omega \sim B_{\omega^*} \bigcup f_{\omega^*} \sim B$ by Axiom A2 and $f'_\omega \in A_{\omega^*}$. Then since $[g \bigcup B_{\omega^*}]_{\omega^*} \bigcup g = g \bigcup B_{\omega^*}$, $g \bigcup A_{\omega^*} \succsim g \bigcup B_{\omega^*}$ by Axiom A1. Conclude $g \bigcup A_{\omega^*} \sim A$ from transitivity, so $g \in c^A(A|\omega^*)$. Hence, $c^A(A|\omega^*) = c^A(B|\omega^*) \bigcap A$ and INRA is satisfied. ☐

Remark. *Axiom A2 implies that $A \succ B$ whenever $c^A(B|\omega') \bigcap A \neq \emptyset$ for all ω' and $c^A(A|\omega^*) \bigcap B = \emptyset$ for some ω^*. This may cause problems when c^A disagrees with c, such as when there are menus with multiple optimal information partitions. It is easy to see that this can be rectified by weakening the axiom to apply only to $B \in K(\mathcal{F})$ with the property that there exists $\epsilon > 0$ so that*

$$c^A(B|\omega) \neq c^A(B|\omega') \iff c^A(B'|\omega) \neq c^A(B'|\omega')$$

for every $\omega, \omega' \in \Omega$ and every $B' \in K(\mathcal{F})$ with $d(B', B) < \epsilon$. Then, $(\succsim, c^A)$ satisfies consistency, and c^A satisfies INRA for any B as above.

2.2. **Robust Approach and Representation.** I impose the following property on $\succsim$ in what follows.

Axiom A4 (Well Behaved)**.** The preference relation $\succsim$ is *Well Behaved* if it is complete, transitive and continuous: for any $A, B, D \in K(\mathcal{F})$, the sets $\{\lambda \in [0,1] : \lambda A + (1-\lambda)D \succsim B\}$ and $\{\lambda \in [0,1] : B \succsim \lambda A + (1-\lambda)D\}$ are closed.

A preference $\succsim$ has an *OIR* (u, π, γ) if it is represented by a function $V : K(\mathcal{F}) \to \mathbb{R}$ where

$$V(B) = \max_{Q \in \mathbb{P}} V(B|Q)$$

and

$$V(B|Q) = \sum_{E \in Q} \pi(E) \max_{f \in B} \int u \circ f d\pi(\cdot|E) - \gamma(Q),$$

where u, π, and γ are as in Definition 1 of Ellis [2017].

As shown in the main text, c^A agrees with c for any menu with a unique optimal partition. When it does not, c^A typically disagrees c. For instance, let $\Omega = \{1, 2\}$, $f_i \in \mathcal{F}$ give a utility of 1 in state i and -1 otherwise, $0 \in X$ give a constant utility of zero, and the prior π assign both states equal probability. If the cost of the partition $Q = \{\{1\}, \{2\}\}$ equals 1, then $c^A(\{f_1, f_2, 0\}|i) = \{f_i, 0\}$, while either $c(\{f_1, f_2, 0\}|i) = \{0\}$ for $i = 1, 2$ or $c(\{f_1, f_2, 0\}|i) = \{f_i\}$ for $i = 1, 2$.

I now turn to a more robust way of deriving conditional choices. c^A fails to identify the conditional choices from B only if there is more than one set of optimal conditional choices. I can identify all of these sets as follows.

Definition A4. The function $c_B^P : \Omega \to K(\mathcal{F})$ describes *potential conditional choices from B for the partition P* if for every $E \in P$ there exists $f_E \in B$ such that $B_P = \{f_E E x_B : E \in P\} \sim B$ and

$$c_B^P(\omega) = \left\{ g \in B : gP(\omega)x_B \bigcup \left[B_P \setminus f_{P(\omega)}P(\omega)x_B \right] \sim B \right\}.$$

The interpretation of $c_B^P(\omega)$ is similar to $c^A(B|\omega)$. Conditional choices exist for P only if there exists $\{f_E\}_{E \in P}$ as above such that the resulting B_P is as good as B. Since B_P is clearly no better than B, this means the DM thinks that she could not better when facing B then by paying attention to P (or something finer) and choosing $f_{P(\omega)}$ in state ω. If $f_{P(\omega)}P(\omega)x_B$ can be replaced in B_P by $gP(\omega)x_B$ without making the DM worse off, then g must be at least as good as $f_{P(\omega)}$ given $P(\omega)$. Thus the DM thinks that she could pay attention to P when facing B, and if she were to do so, that she would be willing to choose g in state ω.

Proposition A2. *Assume $(\succsim, c)$ is consistent. If $c(\cdot)$ has an optimal in attention representation and $\succsim$ is Well Behaved, then there are potential conditional choices c_B^P with $c_B^P(\cdot) = c(B|\cdot)$. If B has a unique optimal partition, then the only potential conditional choices from B equal $c^A(B|\cdot)$.*

Proof. Suppose $c(\cdot)$ has an optimal inattention representation $(u, \pi, \gamma, \hat{P})$ and let $V(B|Q)$ and $V(B)$ as above be defined using its parameters. Fix B and let Q be so that $V(B) = V(B|Q)$.

For each $E \in Q$, pick $f_E \in \arg\max_{f \in B} \int u \circ f d\pi(\cdot|E)$. Set $B_Q = \{f_E E x_B : E \in Q\}$ and $B_\epsilon = (1-\epsilon)B + \epsilon\{x_B\}$ where x_B is from Definition A2 for all $\epsilon > 0$. Observe

$$V(B) = V(B_Q) = V(B_Q \bigcup B_\epsilon) > V(B_Q \bigcup B_\epsilon | Q')$$

for any $Q' \not\gg Q$ and all ϵ, and that if $\hat{P}(B_Q \cup B_\epsilon) \gg Q$, then the DM chooses only items from B_Q. Thus

$$c(B_Q \bigcup B_\epsilon | \omega) \bigcap B_Q \neq \emptyset$$

for all ω. By consistency and transitivity, $B_Q \succsim B_\epsilon$. Letting $\epsilon \to 0$ gives $B_Q \succsim B$ by Well Behaved. It is easy to show $B \succsim B_Q$ and thus $B_Q \sim B$. Since $V(B|\hat{P}(B)) = V(B)$, the above holds, in particular, for $Q\hat{P}(B)$.

Let $B' = fP(\omega)x_B \bigcup\{f_E E x_B : E \in P \setminus P(\omega)\}$ for $P = \hat{P}(B)$. If $f \in c(B|\omega)$, then the arguments above with B' replacing B_Q give $B' \sim B$, so $f \in c_B^P(\omega)$. If $f \notin c(B|\omega)$, then there exists $g \in B$ with $\int u \circ g d\pi(\cdot|Q(\omega)) > \int u \circ f d\pi(\cdot|Q(\omega))$. But then

$$B \succsim gQ(\omega)x_B \bigcup \{f_E E x_B : E \in Q \setminus Q(\omega)\} \succ B'$$

by consistency and transitivity. Hence $f \notin c_B^P(\omega)$.

When Q is the unique optimal information, the conclusion follows from the first part and Theorem 6 of Ellis [2017]. $\square$

Corollary A1. *Under the assumptions of Proposition A2, almost all menus have unique potential conditional choices.*

Definition A5. The *potential choice correspondence* for $\succsim$ is

$$PC = B \mapsto \{G | G \text{ describes potential conditional choices from } B\}.$$

The relation $\succsim$ admits a *consistent selection* $\tilde{c} : K(\mathcal{F}) \times \Omega \to K(\mathcal{F})$ if $\tilde{c}(B|\cdot) \in PC(B)$ for all $B \in K(\mathcal{F})$ and the pair $(\succsim, \tilde{c})$ is consistent.

The final assumption on $\succsim$ handles cases where likelihoods are not decision relevant.[2]

Axiom A5 (Ex Ante SEU)**.** Restricted to singletons, $\succsim$ admits an SEU representation. That is, for any $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1]$, $f \succsim g \iff \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$, and if $f(\omega) \succsim g(\omega)$ for all $\omega \in \Omega$, then $f \succsim g$, strictly whenever $f(\omega) \succ g(\omega)$ for some ω.

[2]To see why Ex Ante SEU is not implied by the other axioms when not all events are decision relevant, let $\Omega = \{1,2\}$ and $\gamma(\{\{1\},\{2\}\}) = 0$. Suppose the preference $\succsim$ is represented by

$$V(B) = \max_{Q \in \mathbb{P}} \min_{\pi \in \Pi} \left[\sum_{E \in Q} \pi(E) \max_{f \in B} \int u \circ f d\pi(\cdot|E) - \gamma(Q) \right]$$

where Π is a set of full-support probability measures with at least two elements. Since the anticipated choice from B is the same as it would be were Π a singleton, one can easily verify that $\succsim$ satisfies all the below properties except Ex Ante SEU.

To keep the proof short, I focus on the constrained attention special case characterized in Corollary 2. The corresponding result for the fully general optimal inattention model also is true but requires a substantially more complicated construction. See footnote 3.

Theorem A2. *The preference $\succsim$ has an OIR (u, π, γ) with $\gamma(Q) \in \{0, \infty\}$ for all $Q \in \mathbb{P}$ if and only if $\succsim$ is Well Behaved, Ex Ante SEU and admits a consistent selection $\tilde{c}$ satisfying INRA, Strong ACI, Monotonicity, SC, Continuity, and Unboundedness.*

Proof. For necessity, observe that the potential choice correspondence is well-defined and agrees with the representation by the argument in Proposition A2. Thus necessity is easily verified by defining $\hat{P}$ as in Theorem 2 of Ellis [2017] and choosing the corresponding selection from the potential choice correspondence.

For sufficiency, assume $\succsim$ is Well Behaved, Ex Ante SEU and admits a consistent selection $\tilde{c}$ satisfying INRA, Strong ACI, Monotonicity, SC, Continuity and Unboundedness. By Corollary 2 of Ellis [2017], $\tilde{c}$ has an Optimal Inattention representation $(\tilde{u}, \tilde{\pi}, \tilde{\gamma}, \hat{P})$ with $\tilde{\gamma}(Q) \in \{0, \infty\}$.

Fix any $B \in K(\mathcal{F})$ and let $Q = \hat{P}(B)$. For every $E \in Q$, choose $\hat{f}_E \in \arg\max_B \int \tilde{u} \circ f d\tilde{\pi}(\cdot|E)$ and set $f_E = [\hat{f}_E]Ex_B$. Letting $A = \{f_E : E \in Q\}$, consistency, INRA and Monotonicity give that $B \sim A$. Let $g \in \mathcal{F}$ be such that $g(\omega) = f_{Q(\omega)}(\omega)$, observe that for all $\omega \in \Omega$ and $\epsilon \in (0,1)$,

$$g = \tilde{c}(g \bigcup (1-\epsilon)A + \epsilon x_B | \omega)$$

and

$$f_{Q(\omega)} = \tilde{c}(A \bigcup (1-\epsilon)g + \epsilon x_B | \omega).$$

By consistency and continuity, $g \sim A$ and thus $g \sim B$.[3]

For any $B \in K(\mathcal{F})$, let g_B be this corresponding act. Thus, for any $A, B \in K(\mathcal{F})$, $B \succsim A$ if and only if $g_B \succsim g_A$. Since $\succsim$ has an expected utility representation with probability π and utility u, $B \succsim A$ if and only if $\int u \circ g_B d\pi \geq \int u \circ g_A d\pi$. Consistency implies that $u = \tilde{u}$ (after an affine transformation) and that $\pi(\cdot|E) = \tilde{\pi}(\cdot|E)$ for any decision relevant E. While π may not agree with $\tilde{\pi}$ on every event, the events on which they differ do not affect which partition is optimal (see Lemma A4). Thus,

$$\int u \circ g_B d\pi = \max_{Q \in \mathbb{P}} \sum_{E \in Q} \pi(E) \max_{f \in B} \int u \circ f d\pi(\cdot|E) - \tilde{\gamma}(Q),$$

completing the sufficiency proof. □

[3]This step fails with a more general cost function. This issue can be overcome by considering a sequence of choices based on the construction in Theorem 3 of Ellis [2017]. In the interest of brevity, the construction is not reproduced here.

3. An approach using plans

Let $\mathcal{F}$ be the set of acts, with $X \subset \mathcal{F}$ the subset of constant acts. Let $\mathcal{F}^\Omega$ be the set of plans: functions from Ω to $\mathcal{F}$. $\succeq$ is a binary relation defined on $\mathcal{F}^\Omega$. Let $\sigma(F)$ be the sigma algebra generated by the plan F. Identify $\mathcal{F}$ with the subset of plans that call for choosing the same acts in every state.

Theaxioms are as follows.

Axiom A6 (Order)**.** $\succeq$ is complete and transitive

Axiom A7 (ACI)**.** For $\alpha \in (0, 1]$, $F, G \in \mathcal{F}^\Omega$ and $f, g \in \mathcal{F}$:
If $\alpha F + (1-\alpha)f \succeq \alpha G + (1-\alpha)f$, then $\alpha F + (1-\alpha)g \succeq \alpha G + (1-\alpha)g$.

Axiom A8 (Monotonicity)**.** If $\sigma(G) \subset \sigma(F)$ and $F(\omega)(\omega) \succeq G(\omega)(\omega)$ for all $\omega \in \Omega$, then $F \succeq G$. If $\forall x \in X$, $x \succ F$ and $x \succ G$, then $F \sim G$.

Axiom A9 (Continuity)**.** For any $F, G, H, J \in \mathcal{F}^\Omega$ so that for every $\alpha \in [0, 1]$ $\sigma(\alpha F + (1-\alpha)G) = \sigma(F)$, the sets $\{\lambda \in [0, 1] : \lambda F + (1-\lambda)G \succeq H\}$ and $\{\lambda \in [0, 1] : H \succeq \lambda F + (1-\lambda)G\}$ are closed.

Axiom A10 (Unboundedness)**.** There exists $x, y \in X$ with $x \succ y$ so that for any $\alpha \in (0, 1]$, there exists $z, z' \in X$ so that $y \succ \alpha x + (1-\alpha)z$ and $\alpha y + (1-\alpha)z' \succ x$.

It should be noted that Monotonicity and Continuity are restricted to apply only to certain plans.

Definition A6. The preference $\succeq$ has an optimal attention representation if there is an unbounded $u(\cdot)$, finitely-additive $\pi(\cdot)$ and information cost function $\gamma : \{\sigma(Q) : Q \in \mathbb{P}\} \to [0, \infty]$ where $\gamma(\{\emptyset, \Omega\}) = 0$ and $\mathcal{A} \subset \mathcal{B}$ implies $\gamma(\mathcal{A}) \leq \gamma(\mathcal{B})$ so that $F \succeq G \iff V(F) \geq V(G)$ where

$$V(F) = \int u\left(F(\omega)(\omega)\right)\pi(d\omega) - \gamma(\sigma(F)).$$

Theorem A3. *The preference $\succeq$ has a optimal attention representation if and only if $\succeq$ satisfies Order, Monotonicity, Continuity, Unboundedness and ACI.*

Proof. On $\mathcal{F}$, $\succeq$ has an expected utility represenation with unbounded $u(\cdot)$, i.e. for $f, g \in \mathcal{F}$

$$f \succeq g \iff \int u \circ f d\pi(\cdot) \geq \int u \circ g d\pi(\cdot),$$

since the axioms imply Herstein-Milnor's when restricted to $\mathcal{F}$, and Anscombe-Aumann's Monotonicity allows interpretation as SEU. Denote by 0 the element of X with $u(0) = 0$.

Define $\mathcal{H} = \{F \in \mathcal{F}^\Omega : \exists x \in X \, s.t. \, F \succeq x\}$. If $F \in \mathcal{H}$, then $\exists x_F \in \mathcal{F}$ so that $x_F \sim F$ by Order, Continuity and Monotonicity. Define $V(F) = \int u \circ x_F d\pi$ for all $F \in \mathcal{H}$ and $V(F) = -\infty$ if $F \notin \mathcal{H}$. Using Transitivity and Monotonicity, $F \succeq G \iff V(F) \geq V(G)$.

Lemma A5. *If $F \in \mathcal{H}$ and $\sigma(F) = \sigma(G)$, then $G \in \mathcal{H}$.*

Proof. If $F \in \mathcal{H}$, then $F \succeq x$ some $x \in X$. Take $F' \in \mathcal{F}^\Omega$ and $x' \in X$ s.t. $u \circ F' = 2u \circ F$, $\sigma(F') = \sigma(F)$ and $u(x') = 0$. Then $\frac{1}{2}F' + \frac{1}{2}x' \sim F$ by Monotonicity. Further, $\exists z \in X$ s.t. $\frac{1}{2}z + \frac{1}{2}x' \sim \frac{1}{2}F' + \frac{1}{2}x'$ by Continuity and Order. Now take $f \in \mathcal{F}$ so that $u \circ f = 2(u \circ G^* - u \circ F^*)$. By Weak-ACI, $\frac{1}{2}F' + \frac{1}{2}f \sim \frac{1}{2}z + \frac{1}{2}f$. By Monotonicity, $G \sim \frac{1}{2}F' + \frac{1}{2}f \sim \frac{1}{2}z + \frac{1}{2}f \succeq z'$, where $z' = \min_{\succeq}\{\frac{1}{2}z + \frac{1}{2}f(\omega) : \omega \in \Omega\}$, which is an element of X. Consequently, $G \in \mathcal{H}$. □

Now, fix any Q such that there exists $F \in \mathcal{H}$ with $\sigma(Q) = \sigma(F)$. Let $0_Q \in \mathcal{F}^\Omega$ be such that $0_Q(\omega) = xQ(\omega)y$ where $u(x) = 0$ and $u(y) = -1$. Since $\sigma(0_Q) = \sigma(F)$, $0_Q \in \mathcal{H}$. Let $x_Q \in X$ be such that $\frac{1}{2}x_Q + \frac{1}{2}0 \sim \frac{1}{2}0_Q + \frac{1}{2}0$. For any $F \in \mathcal{H}$, let $F^* \in \mathcal{F}$ be such that $u(F^*(\omega)) = 2u(F(\omega)(\omega))$, and observe that

$$F \sim \frac{1}{2}0_Q + \frac{1}{2}F^* \sim \frac{1}{2}x_Q + \frac{1}{2}F^*.$$

Thus, $F \succsim G$ if and only if $F^* \succsim G^*$ when $\sigma(F) = \sigma(G) = \sigma(Q)$.

Defining x_Q as above for all such Q, set $\gamma(\sigma(Q)) = -\frac{1}{2}u(x_Q)$. For $F \in \mathcal{H}$, $V(F) = V(\frac{1}{2}x_Q + \frac{1}{2}F^*)$. But this is just

$$V(F) = \frac{1}{2}u(x_Q) + \frac{1}{2}\int 2u \circ F(\omega)(\omega)\pi(d\omega),$$

establishing the claim on $\mathcal{H}$. Letting $\gamma(Q) = \infty$ for any other Q completes the proof. □

4. Counter-Examples

To show my characterization is tight, I provide models that satisfy some but not all of my axioms. An alternative model of particular interest is the *inattention* model. An inattentive DM maximizes expected utility conditional on her subjective information, but her subjective information is not necessarily optimal. Although she has stable tastes and beliefs, the information to which she pays attention varies with the problem in a general manner. Formally, $c(\cdot)$ has an *inattention representation* if Equation (2) holds for all problems B and states ω but the source of $\hat{P}(\cdot)$ is left unspecified.

Corollary A2. *If $c(\cdot)$ has an inattention representation, then $c(\cdot)$ satisfies Monotonicity, SC, Continuity (i), and Unboundedness.*

In particular, an inattentive DM's choices may violate INRA, ACI or Continuity (ii). Consequently, these three axioms reflect the optimality of her subjective information. They capture her reaction to her attention constraint but not that she exhibits inattention in the first place.[4] The proof follows that of Theorem 2. In light of this result, I defer counterexamples for Monotonicity, SC, Continuity, and Unboundedness to the Supplementary Appendix.

For the following, set $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $P = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}$ and $X = \Delta\mathbb{R}$. To economize on space, I write (a, b, c) for an act that gives a for sure in state ω_1, b for sure in state ω_2 and c for sure in state ω_3 and $c(B|\cdot) = (c(B|\omega_1), c(B|\omega_2), c(B|\omega_3))$.

4.1. **All but INRA.** INRA reflects that the DM's subjective information is optimal. If her subjective information cannot be represented as maximizing behavior, then the DM's choices violate INRA. In this section I consider a DM who pays attention to the "worst" rather than the best and show INRA fails.

Formally, suppose that $\mathbb{P}^* = \{Q \ll P : \#Q \leq 2\}$, $u(x) = x$, $\pi(\omega) = \frac{1}{3}$, $\gamma(Q) = 0$ if $Q \in \mathbb{P}^*$ and $\gamma(Q) = \infty$ otherwise, and that

$$\hat{P}(B) = \max_{>} \arg \min_{Q \in \mathbb{P}^*} \sum_{E \in Q} \pi(E)[\min_{f \in B} \int u \circ f d\pi(\cdot|Q(\omega))]$$

for every B, and Equation (2) holds for each B and ω. This $c(\cdot)$ violates INRA. Take f, g, h, j so that $f = (3, 1, 2)$, $g = (1, 3, 1)$, $h = (1, 0, 0)$ and $j = (0, 1, 1)$. If $B = \{f, g, h, j\}$ and $A = \{f, g\}$, then $\hat{P}(B) = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$ while $\hat{P}(A) = \{\{\omega_2\}, \{\omega_1, \omega_3\}\}$. Consequently, $c(B|\cdot) = (\{f\}, \{g\}, \{g\})$ and $c(A|\cdot) = (\{f\}, \{g\}, \{f\})$, contradicting INRA.[5] Equation (2) implies that Subjective Consequentialism and Monotonicity hold. To see why ACI holds, note that

$$\min_{f \in \alpha B + (1-\alpha)\{g\}} \int u \circ f d\pi(\cdot|E) = \alpha \min_{f \in B} \int u \circ f d\pi(\cdot|E) + (1 - \alpha) \int u \circ g d\pi(\cdot|E)$$

for any B, g and E. This implies that $\hat{P}(\alpha B + (1 - \alpha)\{g\}) = \hat{P}(B)$, and Equation (2) gives that ACI holds.

4.2. **All but ACI.** ACI reflects that the DM has an additive cost function attention and that her underlying ex-ante preference is expected utility. It may be violated when the cost function is multiplicative; that is, if $\rho(Q) \in [0, 1] \cup \{\infty\}$ for all $Q \in \mathbb{P}$,

$$\hat{P}(B) \in \arg \max_{\{Q:\rho(Q)<\infty\}} \rho(Q) \sum_{E \in Q} \pi(E)[\max_{f \in B} \int u \circ f d\pi(\cdot|E)] \tag{1}$$

for every problem B, and Equation (2) holds for every problem B and state ω, then the DM violates ACI but satisfy the remaining axioms. Suppose $\rho(P) = \frac{1}{2}$, $\rho(\{\Omega\}) = 1$, $Q \notin \{P, \{\Omega\}\} \implies \rho(Q) = -\infty$, $u(x) = x$ and $\pi(\omega) = \frac{1}{3}$ for every ω.

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[4]A characterization of the inattention model is available as supplementary material.

[5]Similar choices occur for any B' with $d(B', B) < \epsilon$ for ϵ suitably small.

Consider $B = \{(2,0,0),(0,2,0),(\epsilon,0,2)\}$ where $1 > \epsilon > 0$ and $\frac{1}{2}B + \frac{1}{2}\{0,0,0\}$. Note that

$$\hat{P}(\frac{1}{2}B + \frac{1}{2}\{(0,0,0)\}) = P$$

because $\frac{1}{2}1 > \frac{1}{3}1 + \frac{1}{6}\epsilon$. Therefore, $c(\frac{1}{2}B + \frac{1}{2}\{(0,0,0)\}|\omega_1) = \{\frac{1}{2}(2,0,0) + \frac{1}{2}(0,0,0)\}$. However,

$$\hat{P}(\frac{1}{2}B + \frac{1}{2}\{(2,2,2)\}) = \{\Omega\}$$

because $\frac{4}{3} + \frac{1}{3}\epsilon > \frac{1}{2}2$. Therefore, $c(\frac{1}{2}B + \frac{1}{2}\{(2,2,2)\}|\omega_1) = \{\frac{1}{2}(\epsilon,0,2) + \frac{1}{2}(2,2,2)\}$, violating ACI.

Additionally, if choice in each state maximizes the same Gilboa and Schmeidler [1989] preference, then $c(\cdot)$ satisfies all axioms except for ACI and violates ACI whenever the set of priors is not a singleton.

4.3. **All but Monotonicity.** Let $v(x,\omega_1) = x$ and $v(x,\omega_2) = v(x,\omega_3) = -x$. Define

$$c(B|\omega) = \arg\max_{f\in B} \sum_{\omega\in\Omega} v(f(\omega),\omega)$$

and note that $0 \in c(\{0,1\}|\omega)$ for all ω. Set $f = (1,0,0)$ and $B = \{f,0\}$. Since $\sum_{\omega\in\Omega} v(f(\omega),\omega) = 1$ and $\sum_{\omega\in\Omega} v(0,\omega) = 0$, $\{f\} = c(B|\omega)$. However, $0 \in c(\{0,f(\omega)\}|\omega)$ for all ω, so Monotonicity is contradicted. It is trivial to verify that the other axioms are satisfied.

4.4. **All but Subjective Consequentialism.** Return to the setup from the first two counter-examples. Set $\pi_1(\omega_1) = \pi_2(\omega_3) = \frac{1}{2}$, $\pi_1(\omega_2) = \pi_1(\omega_3) = \pi_2(\omega_1) = \pi_2(\omega_2) = \frac{1}{4}$, and $\pi_3 = \pi_2$. Suppose that

$$c(B|\omega_i) = \arg\max_{f\in B} \int u \circ f d\pi_i$$

and consider $f = (4,2,2)$, $g = (4,2,0)$, $h = (0,4,5)$ and $B = \{f,g,h\}$. By construction $c(B|\cdot) = (\{f\},\{h\},\{h\})$. Note that $\{\omega_1\} = \{\omega'' : c(B|\omega'') = c(B|\omega_1)\}$ and that $f(\omega_1) = g(\omega_1)$, a contradiction of subjective consequentialism. The other properties are trivial to verify.

4.5. **All but Continuity.** Lexicographic, rather than expected utility, conditional preference maximization provide a counter-example to continuity (i).

To show a counter example for Continuity (ii), take $\mathbb{P}^*$, $u(\cdot)$ and $\pi(\cdot)$ as in the first example. Write $P_i = \{\{\omega_i\},\{\omega_i\}^c\}$. For every problem B, define an ordering $>_B$ by $P_i >_B P_j$ if and only if

$$\max_{f\in B} u(f(\omega_i))) > \max_{f\in B} u(f(\omega_j))\, OR$$
$$[\max_{f\in B} u(f(\omega_i))) = \max_{f\in B} u(f(\omega_j))\, AND\, i < j]$$

Also, set every $P_i >_B \{\Omega\}$. For every problem B, take $\hat{P}(B) = \max_{>_B} \mathbb{P}^*$ and suppose Equation (2) holds.

Let $f = (\frac{-1}{3}, \frac{-1}{3}, \frac{-1}{3})$ and $x = (0,0,0)$. Thus $x \in c(\{x, f(\omega\}|\omega) = c(\{0, \frac{-1}{3}\}|\omega)$ and $f \notin c(\{x, f\}|\omega)$ for any ω. Now, I show that $f_1 = (1,-2,0)$ has $\{f_1\}\, IS\, \{x\}$. Clearl $\{f_1\}\, IS\, \{f_1, f_1\{1\}z, f_1\{2,3\}z\} = B_1$ where $z \in X$ has $u(z) = -10$. Setting $g_2 = (0, \frac{1}{2}, 0)$,

$$\hat{P}(\{f_1\{1\}z, f_1\{2,3\}z, g_2\{2\}z, g_2\{1,3\}z\} = B_2) = P_1$$

since $\max_{f \in B_2} u(f(\omega_1)) = 1$ and $\max_{f \in B_2} u(f(\omega_2)) = \frac{1}{2}$. Similarly, $\hat{P}(B_3 = \{g_2\{2\}z, g_2\{1,3\}z, x\}) = P_2$. Thus $\{f_1\}\, IS\, B_1\, IS\, B_2\, IS\, B_3\, IS\, \{x\}$. Similarly, $f_2 = (0,1,-2)$ and $f_3 = (-2,0,1)$ both have $\{f_2\}\, IS\, \{x\}$ and $\{f_3\}\, IS\, \{x\}$.

Fix y with $u(y) = 1$. For any $\epsilon > 0$ and any ω, $u(\epsilon y + (1-\epsilon) f(\omega)) > \frac{-1}{3}$, but $\sum_{i=1}^{3} \frac{1}{3} u(f_i(\omega)) = \frac{-1}{3}$, so there is no $\epsilon > 0$ and ω' for which $\epsilon y + (1-\epsilon) f(\omega')$ is revealed strictly preferred to $\sum_{i=1}^{3} \frac{1}{3} f_i(\omega')$.

4.6. **Optimal inattention versus inattention to alternatives.** I first construct an example compatible with optimal inattention but not inattention to alternatives, then one compatible with inattention to alternatives but not optimal inattention. For both, let $\Omega = \{a,b,c,d\}$, $P = \{\{a\},\{b\},\{c\},\{d\}\}$, π be such that $\pi(\omega) = \frac{1}{4}$ for every ω, and $\gamma(Q) < \infty$ only if $Q_1 \gg Q$ or $Q_2 \gg Q$ where $Q_1 = \{\{a\},\{b,c\},\{d\}\}$ and $Q_2 = \{\{a,d\},\{b\},\{c\}\}$.

Define acts x, y, z, w that give the utility values in the following table:

	a	b	c	d
$u \circ w$	6	6	6	4
$u \circ x$	8	9	0	0
$u \circ y$	0	0	0	16
$u \circ z$	2	0	9	0

If $\gamma(Q_1) = \gamma(Q_2) = 0$, then one can verify that $\hat{P}(\{x,y,z,w\}) = Q_1$, $\hat{P}(\{x,z,w\}) = Q_2$, $\hat{P}(\{x,y,z\}) = Q_2$, and $\hat{P}(\{y,z\}) = Q_1$, so $c(\{x,y,z,w\}|a) = \{x\}$, $c(\{x,z,w\}|a) = \{w\}$, $c(\{x,y,z\}|a) = \{y\}$, and $c(\{y,z\}) = \{z\}$. But then by Lemma 1 and Theorem 3 of Masatlioglu et al. [2012], xPy and yPx so $c(\cdot|a)$ cannot be a choice with limited attention.

Fix any $x', y', z' \in X$, i.e. all three are lotteries. Suppose $c(\cdot|a)$ is a choice with limited attention where $\Gamma(\{x',y',z'\}) = \{y',z'\}$, $\Gamma(\{x',y'\}) = \{x',y'\}$ and $x' \succ y' \succ z'$. Then $c(\{x',y',z'\}|a) = \{y'\}$ and $c(\{x',y'\}|a) = \{x'\}$. If $c(\cdot)$ has an optimal inattention representation, then $c(\{x',y',z'\}|a) = \{y'\}$ implies $u(y) > u(x)$ but $c(\{x',y'\}|a) = \{x'\}$ implies $u(x) > u(y)$, a contradiction.

5. Miscellany

In footnote 21, Ellis [2017] asserts "By assuming Strong ACI (below) or requiring that for any x, there exists $\epsilon > 0$ so that $c(B|\cdot)$ is constant when $d(B, \{x\}) < \epsilon$, one can relax

Continuity to the following:
For any $x, y \in X$ and $f, g \in \mathcal{F}$ where $y \succ^R x$, $x \in c(\{f, x\}|\omega')$ for all ω', and $f \notin c(\{f, x\}|\omega'')$ for some ω'': if $\{g\}$ IS $\{x\}$, then $\exists \epsilon > 0$ and ω^* such that $g(\omega^*) \succ^R \epsilon y + (1-\epsilon) f(\omega^*)$." To see why this is true, observe that Continuity (ii) is used only in the proof of Lemma 10.

With strong ACI, $f_1, f_2 \succeq 0$ if and only if $\alpha f_1 + (1-\alpha) f_2 \succeq \alpha 0 + (1-\alpha) f_2$, and $\alpha 0 + (1-\alpha) f_2 \succeq 0$ by strong ACI. Hence, a simple induction proof shows that when $f_1, ..., f_m \in K$, so is $\sum \alpha_i f_i$.

Now, assume that for any x, there exists $\epsilon > 0$ so that $c(B|\cdot)$ is constant when $d(B, \{x\}) < \epsilon$. Pick x such that $u(x) = 0$. There exists ϵ such that if $d(B, \{x\}) < \epsilon$, then $c(B|\cdot)$ is constant. So consider $K^* = B_\epsilon(x)$ and $K(K^*)$ the compact subsets of K^*. Restricted to this set, $c(\cdot)$ is constant and so its revealed preference relation is well-defined and satisfies the Herstein Milnor axioms and so has a subjective expected utility representation with prior π and utility u. I claim that $\{f\}$ IS $\{x\}$ if and only if $\int u \circ f d\pi \geq 0$. If the claim is true, then Lemma 10 immediately follows.

First, suppose $\{f\}$ IS $\{x\}$ but $0 > \int u \circ f d\pi$. Let $y \in B_\epsilon(x)$ be such that $\int u \circ f d\pi < u(y) < 0)$. Choosing n "large enough", $\{y\} = c(\{\frac{1}{n} f + \frac{n-1}{n} x, y\}|\omega)$ for all ω. Using Lemma 3 with Monotonicity, $\{y\}$ IS $\{f\}$, and since $\{f\}$ IS $\{x\}$, $\{y\} IS \{x\}$. But then by continuity, there exists ω^* such that $y(\omega^*) \succ^R (1-\epsilon) x(\omega^*) + \epsilon x'$, for $x' \succ^R x$. This is a contradiction, since $u(x), u(x') > u(y)$.

Now, suppose $\int u \circ f d\pi \geq 0$. Let $f_n = \frac{1}{n} f + \frac{n-1}{n} x$. Then for n large enough, $d(\{f_n, x\}) < \epsilon$. Hence $f_n \in c(\{f_n, x\}|\omega)$ for all ω, so $\{f_n\}$ IS $\{x\}$. Again using Lemma 3 with Monotonicity, $\{f\}$ IS $\{x\}$ (by iterating $\{\frac{1}{2} g_m + \frac{1}{2} f_n^*\}$ IS $\{\frac{1}{2} g_m + \frac{1}{2} x\}$ with $g_0 = x$ and f_n^* and g_m being any acts such that $2u \circ f_n^* = u \circ f_n$ and $u \circ g_{m+1} = u \circ g_m + u \circ f_n^*$).

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