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Mathematical Finance

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Mathematical finance is a relatively new field, with most observers attributing its birth in 1971 to the development by Black and Scholes of their celebrated formula for the price of a call option. They took the price of a stock to be geometric Brownian motion and used the economic concept of arbitrage to argue that the price of a call option is the solution of a certain partial differential equation. In subsequent years these ideas have been greatly extended and generalized by mathematicians, financial economists, and researchers in the finance industry. In particular, it was soon recognized that many of the powerful tools of stochastic calculus and martingale theory could be employed to derive results of both fundamental and practical importance. Research in this field is now rapidly accelerating as mathematicians interact with financial researchers and as the world's financial markets continue to grow and become more sophisticated.

The first conference devoted to mathematical finance was held at Cornell University during the summer of 1989. Prompted by its success, the Oberwolfach conference was developed and organized. Thirty-six scholars from thirteen countries attended. Most of these were mathematicians, but included were several financial economists plus two researchers from the financial industry. Twenty-nine papers were presented, divided among the following topics:

- fundamentals of arbitrage and martingale measures
- statistical estimation
- consumption, investment, and optimal choice
- options and futures
- term structure models
- insurance, risk and actuarial problems
- miscellaneous financial applications.

L. P. Foldes:

Martingale Conditions versus Programming Conditions for Portfolio Optimality

Brief review of a model of optimal saving and portfolio selections with general semimartingale investments over an infinite horizon in continuous time. Discussion of the relationship between conditions of optimality expressed in terms of martingale properties of shadow prices and those expressed as programming conditions. The use of integer-valued random measures and their predictable compensators to obtain programming conditions in the form of integral equations (or inequalities). Illustration of the procedure in the special case of logarithmic utility. Also, a simple proof of the existence of an optimum in this case.

In a longer, written, version, I would consider some additional points: Difficulties in calculating solutions in general, in particular difficulties arising from the interdependence of saving and portfolio decisions. Review of special assumptions which avoid some of the difficulties, distinguishing between cases with continuous processes and those with jumps.

For Oberwolfach
23-29/8/1992

ABSTRACT

Semimartingale Calculus in Portfolio Theory

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We review a model of optimal saving and portfolio choice over an infinite horizon in continuous time which the speaker has considered in several recent papers. In this model, the vector process representing returns to investments is a semimartingale (which may be neither special nor continuous, and may even have a countably infinite set of jumps during a finite interval of time). We call attention to some useful techniques, in particular:

(i) the use of variables discounted or compounded by suitable interest processes,

(ii) the use of mart-logs of returns and shadow prices, (the *mart-log* being defined for positive semimartingales as the inverse of the Doléans exponential), and

(iii) the representation of the jumps of the market process by means of an integer-valued random measure and the calculation of its compensator (Lévy system).

We set out general expressions for the return to a portfolio and for the equation of accumulation, and consider the relationship between conditions for optimality expressed in terms of martingale properties of shadow prices and those expressed as portfolio equations or programming conditions.

Special attention is paid to the case where welfare is represented by the time integral of the subjectively discounted expected values of log-consumption, and this is considered more fully than in previous work. The problem of optimal portfolio choice can be separated from the problem of optimal consumption, and the conditions for portfolio optimality take on a particularly simple form. Among other things, they illustrate clearly the differing significance for portfolio selection of market jumps at predictable and at totally inaccessible times. A relatively elementary proof of the existence of an optimum can be given in this case.

Basic Notation

$\mathcal{B}$ Borel
$\mathcal{H}$ progressive

$(\Omega, \mathcal{Q}, P)$, $(\mathcal{Q}_t)$, $\mathcal{T} = [0, \infty)$; $\mathcal{O}$ optional, $\mathcal{P}$ predictable

$\lambda = 1, \dots, \Lambda$ assets, securities

$X = (x^1, \dots, x^\lambda, \dots, x^\Lambda)$ vector semimartingale.

x log-return, values $x(\omega, t)$; $x(0) = 0$

$z = e^x$ return, positive ; $z(0) = 1$

$$\mathfrak{J}_T = \int_0^T \frac{dz(t)}{z(t-)} \qquad z = \mathcal{E}(\mathfrak{J}), \quad \mathfrak{J} = \mathcal{L}(z) \tag{1.1}$$

$$x^\lambda = M^\lambda + V^\lambda = M^{\lambda c} + M^{\lambda d} + V^{\lambda c} + V^{\lambda d}$$

$X = M + V$ etc.

If $\mathcal{J}^x$ is the integer-valued measure associated with the jumps of X, with compensating measure F^x one can set

$$\left.\begin{aligned} M_T^{\lambda d} &= \int_0^T \int_{|\xi| \le 1} \xi^\lambda (\mathcal{J}^x - F^x)(d\xi, dt), \\ V_T^{\lambda d} &= \int_0^T \int_{|\xi| > 1} \xi^\lambda \mathcal{J}^x(d\xi, dt) + \sum_{t \le T} \int_{|\xi| \le 1} \xi^\lambda F^x(d\xi \times \{t\}) \end{aligned}\right\} \tag{1.2}$$

here $|\xi|$ is the Euclidian norm of the vector ξ taking values in the 'space of jumps' Ξ, a copy of $\mathbb{R}^\Lambda$. $\mathcal{J}$ is the predictable support of μ^x.

Note that $F^x(d\xi \times \{t\}) = 0$ for $(\omega, t) \notin \mathcal{J}$, where $\mathcal{J}$ is the predictable support of μ^x.

π portfolio plan : predictable vector process, Π : all portfolio plans
values $\pi^\lambda(\omega, t)$, $\lambda = 1, \dots, \Lambda$, (1.3)
$\sum_\lambda \pi^\lambda(\omega, t) = 1$.

$\pi \ge 0$ if no short sales allowed, (assume this here).

x^π, z^π, $\mathfrak{J}^\pi$ refer to portfolio log-return etc. The basic definition is (1.4)

Portfolio equation

$$\boxed{\mathfrak{J}^\pi(T) = \int_0^T \pi^\lambda(t)\, d\mathfrak{J}^\lambda(t)} \tag{1.5}$$

Using $\mathfrak{J}_T^\lambda = \int_0^T \frac{dz^\lambda(t)}{z^\lambda(t-)} = x_T^\lambda + \frac{1}{2} \langle M^{\lambda c} \rangle_T + \sum_{t \le T} \left[e^{\Delta x^\lambda(t)} - 1 - \Delta x^\lambda(t) \right]$

one calculates

$$\left.\begin{aligned} x^\pi(T) = &\int_0^T \textstyle\sum_\lambda \pi_t^\lambda \, d M_t^{\lambda c} \\ &+ \int_0^T \textstyle\sum_\lambda \pi_t^\lambda \, d V_t^{\lambda c} + \frac{1}{2} \int_0^T \sum_\lambda \pi_t^\lambda \, d\langle M^{\lambda c} \rangle_t - \frac{1}{2} \int_0^T \sum_\lambda \sum_\varepsilon \pi_t^\lambda \pi_t^\varepsilon \, d\langle M^{\lambda c}, M^{\varepsilon c} \rangle_t \\ &+ \int_0^T \textstyle\sum_\lambda \pi_t^\lambda \, d M_t^{\lambda d} \\ &+ \sum_{t \le T} \left(\Delta x_t^\pi - \textstyle\sum_\lambda \pi_t^\lambda \Delta M_t^\lambda \right) \qquad \text{where } \Delta x_t^\pi = \ln\left(\sum_\lambda \pi_t^\lambda e^{\Delta x^\lambda(t)} \right) \end{aligned}\right\} \tag{1.6}$$

$\bar c, \bar k$ consumption, capital in 'natural' units, e.g. money.
these are essentially non-negative processes

K_0 initial capital.

Leaving aside any problems about dividing by zero, an obvious way to write the equation of accumulation in natural units – for given π and z^π – is

$$\int_0^\tau \frac{d\bar k(t)}{\bar k(t-)} = \int_0^\tau \frac{dz^\pi(t)}{z^\pi(t-)} - \int_0^\tau \frac{\bar c(t)}{\bar k(t-)}, \qquad \bar k(0) = K_0 > 0 \qquad (2.1)$$

where the progressive process $\bar c \ge 0$ is considered as the 'driving process' and the semimartingale $\bar k$ as the 'solution', which is required to be non-negative; (the solution of 2.1 is considered only up to its first arrival time at the zero level; thereafter we set $\bar k_t = 0$, $\bar c_t = 0$).

Alternatively, for given π and z^π, consumption and capital in π-standardised units are defined by

$$c(\omega,t) = \bar c(\omega,t)/z^\pi(\omega,t), \qquad k(\omega,t) = \bar k(\omega,t)/z^\pi(\omega,t) \qquad (2.2)$$

and then (2.1) reduces to

$$k(\tau) = K_0 - \int_0^\tau c(t)\,dt \qquad (2.3)$$

and the requirement that $k_\tau \ge 0$ for all τ yields the constraint

$$\int_0^\infty c(t)\,dt \le K_0 \qquad (2.4)$$

Thus we may define a feasible c-plan (consumption plan in π-standardised units) as a progressive, non-negative process satisfying (2.4) a.s., and the set $\mathcal{C}$ of such plans is the same whatever π. Once c and π are chosen, $\bar c$ is determined by (2.2).

The welfare functional is

$$E\int_0^\infty \bar u[\bar c(\omega,t);\omega,t]\,dt = E\int_0^\infty \bar u[c(\omega,t)z^\pi(\omega,t);\omega,t] = \varphi(c,\pi) \qquad (2.5)$$

(φ to be defined as a product Lebesgue integral and satisfy a finite sup condition.)
to be maximised by choosing $c \in \mathcal{C}$, $\pi \in \Pi$.
$\bar u(\cdot;\omega,t)$ has the usual properties of a utility function, in particular $\bar u' > 0$, $\bar u'' < 0$; it may take + or − values or both; $\bar u'(0;\omega,t) = \infty$. and $\bar u(\cdot;\cdot,\cdot)$ is $\mathcal{B}\times\mathcal{H}$ measurable. $\bar u(\gamma;\cdot,\cdot)$ is a progressive process for each γ,

The distinguished 'star' plan is written $(\bar c^*, \pi^*)$ or (c^*, π^*); we have $c^* > 0$. Write, for short,

$$v_t = v(\omega,t) = \bar u'[\bar c^*(\omega,t);\omega,t]. \qquad (2.6)$$

For any π, the shadow price process y^π is defined by

$$y^\pi(\omega,t) = v(\omega,t)\,z^\pi(\omega,t) = v_t\, e^{x^\pi(t)}; \text{ in particular } y^* = v z^* = v e^{x^*} \qquad (2.7)$$

Define $\eta^\pi_\tau = \int_0^\tau \frac{dy^\pi(t)}{y^\pi(t-)}$, $\eta^\lambda_\tau = \int_0^\tau \frac{dy^\lambda(t)}{y^\lambda(t-)}$; then $\boxed{\eta^\pi_\tau = \int_0^\tau \sum_\lambda \pi^\lambda_t\, d\eta^\lambda_t}$ (2.8)

for any π. Thus $\eta = \mathcal{L}(y/y_0)$, $y = y_0\,\mathcal{E}(\eta)$, in particular $\eta^*_\tau = \int_0^\tau \frac{dy^*(t)}{y^*(t-)}$ (2.9)
If y^π is a local mart / supermart, so is η^π.

<u>Example</u> Note $\ln z^{\pi} = x^{\pi}$

Let $\bar{u}(\bar{c}(\omega,t);\omega,t) = \ln \bar{c}(\omega,t)\cdot e^{-\alpha t} = (\ln c_t + x_t^{\pi})e^{-\alpha t}$ (3.1)

$$\varphi(c,\pi) = E\int_0^\infty (\ln c_t)e^{-\alpha t}dt + E\int_0^\infty x_t^{\pi} e^{-\alpha t}dt , \quad \alpha > 0 \tag{3.2}$$

In this case, if an optimal plan (c^*,π^*) exists, one can choose c^* non-random by (3.3)

$$\max \int_0^\infty (\ln c_t)e^{-\alpha t}dt \quad \text{subject to} \int_0^\infty c_t\,dt \le K_0, \quad c_t > 0$$

We have
$$\left.\begin{aligned} v_t &= \bar{u}_t'^{*} = (1/\bar{c}_t^*)e^{-\alpha t} = (1/c_t^*)e^{-x_t-\alpha t} \\ y_t^* &= \bar{u}_t' e^{x_t} = (1/c_t^*)e^{-\alpha t} = \text{constant} = 1/c_0^* = y_0^* (\text{deterministic martingale}) \end{aligned}\right\} \quad (3.4)$$

Choose c_0^* so that $K_0 = \int_0^\infty c_t^* dt = c_0^*\int_0^\infty e^{-\alpha t}dt = c_0^*/\alpha$. The integral $\int (\ln c_t^*)e^{-\alpha t}dt$ converges for $\alpha > 0$ and c_t^* is optimal.

We have: $\eta_t^* \equiv 0$ (3.5)

$$y_T^{\lambda} = y_T^* e^{x^{\lambda}(T) - x^*(T)} = y_0^* e^{x^{\lambda}(T) - x^*(T)}$$

$$\left.\begin{aligned} \eta_T^{\lambda} = \int_0^T \frac{dy^{\lambda}(t)}{y^{\lambda}(t-)} &= x_T^{\lambda} - x_T^* + \tfrac{1}{2}\langle x^{\lambda c} - x^{*c}, x^{\lambda c} - x^{*c}\rangle_T \\ &+ \textstyle\sum_{t\le T}\left(e^{\Delta x^{\lambda}(t) - \Delta x^*(t)} - 1 - \Delta x_t^{\lambda} - \Delta x_t^*\right) \end{aligned}\right\} \quad (3.6)$$

For $x^* = x^{\pi^*}$ - see (1.6) - (3.7)

$$\left.\begin{aligned} \text{write}\quad M_T^* &= \int_0^T \textstyle\sum_\lambda \pi^{\lambda} dM^{\lambda} \\ N_T^{\lambda} &= V_T^{\lambda c} + \tfrac{1}{2}\langle M^{\lambda c}, M^{\lambda c}\rangle_T + \langle (\ln v)^c, M^{\lambda c}\rangle_T \\ &= V_T^{\lambda c} + \tfrac{1}{2}\langle M^{\lambda c}, M^{\lambda c}\rangle_T - \int_0^T \textstyle\sum_\lambda \pi_t^{*\lambda} d\langle M^{\lambda c}, M^{\lambda c}\rangle_t \end{aligned}\right\} \quad (3.8)$$

$$N_T^* = \int_0^T \textstyle\sum \pi_t^{*\ell} dN_t^{\ell} , \quad N_T^{\delta\lambda} = N_T^{\lambda} - N_T^* \tag{3.9}$$

$$S_T^{\delta\lambda} = \textstyle\sum_{t\le T}\left[e^{\Delta x_t^{\lambda} - \Delta x_t^*} - 1 - \Delta M_t^{\lambda} + \Delta M_t^* \right]$$

Calculation gives (3.10)

$\eta^{\lambda} = [\eta^* + M^*] + [N^{\delta\lambda} + S^{\delta\lambda}]$ = local mart + process of finite variation

while η^{λ} is a supermart. Now $N^{\delta\lambda}$ is continuous, but $S^{\delta\lambda}$ need not be predictable. However, it follows from the above eq. that $S^{\delta\lambda}$ is locally of integrable variation, so has a compensator $\tilde{S}^{\delta\lambda}$. Now

$\eta^{\lambda} = [\eta^* + M^* + S^{\delta\lambda} - \tilde{S}^{\delta\lambda}] + [N^{\delta\lambda} + \tilde{S}^{\delta\lambda}]$ = local mart + <u>predictable</u> process of f.v. (3.11)

gives the canonical decomposition of η^{λ}, which I also write as $\mu^{\lambda} + \partial^{\lambda}$.

The "Kuhn-Tucker" condition for portfolio optimality is (informally)

$$\left.\begin{array}{l} \pi_t^{*\lambda} \ge 0, \quad d\partial^\lambda \le 0, \quad \pi_t^{*\lambda}\, d\partial_t^\lambda = 0 \\ \text{or, more correctly, } \int_0^T \pi_t^{*\lambda}\, d\partial_t^\lambda = 0 \text{ on } \mathcal{J}, \text{ a.s.} \end{array}\right\} \quad (4.1)$$

and of course $\sum_\lambda \pi_t^{*\lambda} = 1$

To calculate ∂^λ:

$$S_T^{\delta\lambda} = \sum_{t\le T}\Big[e^{\Delta x^\lambda - \Delta x^*} - 1 - \Delta x^\lambda + \Delta V^\lambda + \sum \pi_t^{*\ell}\Delta x^\ell - \sum \pi_t^\ell \Delta V^\ell\Big]$$

$$= \int_0^T\int_{\Xi}\Big[e^{\xi^\lambda}\big(\sum \pi_t^{*\ell} e^{\xi^\ell}\big)^{-1} - 1 - \xi^\lambda + \sum \pi_t^\ell \xi^\ell + I_{|\xi|>1}\big(\xi^\lambda - \sum_\ell \pi_t^{*\ell}\xi^\ell\big)\Big]\mathcal{F}(d\xi, dt)$$

$$+ \sum_{t\le T}\int_{|\xi|\le 1}\big(\xi^\lambda - \sum_\ell \pi_t^{*\ell}\xi^\ell\big) F(d\xi \times \{t\}) \qquad (4.2)$$

To obtain $\tilde{S}^{\delta\lambda}$, replace $\mathcal{F}$ by F and simplify:

$$\tilde{S}_T^{\delta\lambda} = \int_0^T\int_{\Xi}\Big[e^{\xi^\lambda}\big(\sum \pi_t^{*\ell} e^{\xi^\ell}\big)^{-1} - 1 - I_{|\xi|\le 1}\, I_{(\omega,t)\notin \mathcal{J}}\big(\xi^\lambda - \sum_\ell \pi_t^{*\ell}\xi^\ell\big)\Big]F(d\xi, dt) \qquad (4.3)$$

Now factorise F as $\quad F(d\xi, dt) = f_t(d\xi)\, dG(t), \quad G = G^c + G^d \qquad (4.4)$

and assume for simplicity that the following (a.e. defined, finite) derivatives exist on $\mathcal{J}$, a.s.: $\qquad (4.5)$

$$dV^{\lambda c}/dt = \dot{V}_t^\lambda, \quad d\langle M^{\lambda c}, M^{\ell c}\rangle/dt = \sigma_t^{\lambda\ell}; \quad dG^c/dt = g_t,$$

and note that G^d is concentrated on $\mathcal{J}$.

For brevity, suppose $\pi^* > 0$ always and write out the conditions for $\partial_t^\lambda = 0$ on $\mathcal{J}$.

For $(\omega, t) \in \mathcal{J}$ we have $\Delta G_t^d > 0$ by definition, so $0 = \Delta \partial_t^\lambda$ yields $\qquad (4.6)$

$$0 = \Delta \tilde{S}_t^{\delta\lambda} = \int_{\Xi}\Big[e^{\xi^\lambda}\big(\sum \pi_t^{*\ell} e^{\xi^\ell}\big)^{-1} - 1\Big] f_t(d\xi).$$

In $(\omega, t) \notin \mathcal{J}$, we have

$$0 = d\partial_t^\lambda/dt = \dot{N}_t^\lambda - \dot{N}_t^* + \int_{\Xi}\Big[e^{\xi^\lambda}\big(\sum \pi_t^{*\ell} e^{\xi^\ell}\big)^{-1} - 1 - I_{|\xi|\le 1}\big(\xi^\lambda - \sum \pi_t^{*\ell}\xi^\ell\big)\Big] f_t(d\xi)\, g(t) \qquad (4.7)$$

$$\text{where } \dot{N}_t^\lambda = \dot{V}_t^\lambda + \tfrac{1}{2}\sigma_t^{\lambda\lambda} - \sum_\ell \pi_t^{*\ell}\sigma_t^{\lambda\ell}, \quad \dot{N}_t^* = \sum_\ell \pi_t^{*\ell}\dot{N}_t^\ell. \qquad (4.8)$$

Note how the portfolio conditions for predictable jumps of X separate out, while those for totally inaccessible jumps combine with the continuous terms.

Suppose that the asset μ is riskless. It has at most predictable (in fact, fixed) jump times, and $\dot{N}^{\mu} = \dot{\nu}^{\mu}$. If X has no jumps at all, the classical equation

$$0 = \dot{\nu}^{\lambda} - \dot{\nu}^{\mu} + \tfrac{1}{2}\sigma^{\lambda\lambda} - \sum \pi_t^{*\ell}\sigma^{\lambda\ell} \tag{5.1}$$

is obtained. In general, subtracting the equations for μ from those for λ gives

$$0 = \int_{\Xi}\left[(e^{\xi^{\lambda}} - e^{\xi^{\mu}})\left(\textstyle\sum_{\ell}\pi_t^{*\ell}e^{\xi^{\ell}}\right)^{-1}\right]f_t(d\xi) \qquad (\omega,t)\in\mathcal{J} \tag{5.2}$$

$$0 = \dot{N}_t^{\lambda} - \dot{N}_t^{\mu} + \int_{\Xi}\left[e^{\xi^{\lambda}}\left(\textstyle\sum_{\ell}\pi_t^{*\ell}e^{\xi^{\ell}}\right)^{-1} - I_{|\xi|\le 1}\,\xi^{\lambda}\right]f_t(d\xi)\,g(t) \qquad (\omega,t)\notin\mathcal{J}. \tag{5.3}$$

Postscript: Existence of an optimal portfolio plan in the case of log-utility (sketch)

For given $(\omega, t) \in \mathcal{J}$, consider the problem of maximising

$$\int_{\Xi} \ln\left(\Sigma_1^n \check{\pi}^\lambda e^{\xi^\lambda}\right) f_t(d\xi) \quad \text{subject to} \quad \Sigma_1^n \check{\pi}^\lambda = 1, \quad \check{\pi}^\lambda \geq 0 \text{ each } \lambda \qquad \ldots (6.1)$$

$\check{\pi} = (\check{\pi}^1, \ldots, \check{\pi}^n)$ vector in $\mathbb{R}^n$

Under suitable conditions, e.g. if

$$\int_{\Xi} \max_{1 \leq \lambda \leq n} |\xi^\lambda| f_t(d\xi) < \infty \qquad \ldots (6.2)$$

the integral in (6.1) is a continuous function of $\check{\pi}$ on the simplex

$$\mathcal{S} = \{\check{\pi} \in \mathbb{R}^n : 0 \leq \pi^\lambda \leq 1 \text{ for } \lambda = 1, \ldots, n \text{ and } \Sigma_1 \check{\pi}^\lambda = 1\}, \qquad (6.3)$$

hence attains a max at some $\check{\pi}^* = \pi^*(\omega, t)$. One can differentiate under the integral sign in (1) to obtain first order necessary conditions for a max. Taking into account the concavity of the function, these conditions are also sufficient. If $\pi^{*\lambda}(\omega, t) > 0$ for all λ, the conditions are (4.6) and $\Sigma \pi^{*\lambda} = 1$, showing that these equations do have a solution. (If $\pi^{*\lambda} = 0$ for some indices, the corresponding 'Kuhn-Tucker' conditions for a constrained max are satisfied). The solution is unique if the integral in (6.1) is strictly concave on $\mathcal{S}$; it is sufficient if the linear subspace of $\mathbb{R}^n$ generated by the essential support of $f_t(\cdot)$ is the whole space (or at least contains a $(n-1)$-dimensional subspace $\mathcal{H}$ with $\underset{\sim}{1} \notin \mathcal{H}$).

In the same way, for $(\omega, t) \notin \mathcal{J}$, consider maximising

$$\Sigma_1^n \check{\pi}^\lambda\left(\dot{v}^\lambda + \tfrac{1}{2}\sigma^{\lambda\lambda} - \tfrac{1}{2}\Sigma_1^n \check{\pi}^\ell \sigma^{\lambda\ell}\right) + \int_{\Xi} \left\{\ln\left(\Sigma_1^n \check{\pi}^\lambda e^{\xi^\lambda}\right) - I_{|\xi| \leq 1} \Sigma_1^n \check{\pi}^\lambda \xi^\lambda\right\} f_t(d\xi) g(t) \qquad \ldots (6.4)$$

to show that the conditions (4.7), (or the corresponding K-T conditions), have a solution. (details omitted).

Then check that the collection $(\pi^*(\omega, t); \omega \in \Omega, t \in \mathcal{J})$ defines a predictable process.

To complete the proof of existence of an optimal portfolio plan, it remains to impose conditions ensuring that $E\int_0^\infty x_t^\pi e^{-\alpha t} dt$ is defined for all π and satisfies a finite supremum condition, and that

$$E\int_0^\infty |x_t^*| e^{-\alpha t} dt < \infty \quad \text{where} \quad x^* = x^{\pi^*} \qquad (6.5)$$

If X is a process with independent increments, a similar approach to existence works with a utility function of the form

$$(1-b)^{-1} (c)^{1-b} e^{-\alpha t} \qquad b > 0 \qquad b \neq 1. \qquad (6.6)$$

With general utility and general X, the problems of optimal portfolio choice and optimal saving cannot be separated.

Lucien Foldes
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For talk at Essex.

9.1.93

To show that an optimum exists for the problem

max $E\int (\ln c) q\,dt$ subject to

$\int c\,dt \le K_0 = 1$ a.s., $c \ge 0$ } defines $c \in \mathcal{C}$ cf (3.3–3.4) above

where $0 < q(\omega,t) \le$ const a.e.(m), $E\int q\,dt \le 1$ and $E\int_{q<1} (\ln q)\, q\,dt > -\infty$

Consider $\ln c$ separately for $c \ge 1$, $c \le 1$

Take $0 < \epsilon < 1$, ϵ small

$\frac{d}{dc}\left[(\ln c)^{1+\epsilon}\right] = (1+\epsilon)(\ln c)^{\epsilon}/c$; for $c > 1$, this is $< (1+\epsilon)(c-1)^{\epsilon}/c < (1+\epsilon)c^{\epsilon-1}$.

$\therefore (\ln c)^{1+\epsilon} - \underbrace{(\ln 1)^{1+\epsilon}}_{\text{zero}} < (1+\epsilon)\int_{c=1}^{c} c^{\epsilon-1}\,dc = \frac{1+\epsilon}{\epsilon} c^{\epsilon}\Big|_1^{c} < \frac{1+\epsilon}{\epsilon} c$

$\sup_c \int_{c\ge1} (\ln c)^{1+\epsilon}\,d\mu \le \frac{1+\epsilon}{\epsilon} \sup_c \int_{c\ge1} c\cdot q \cdot dm \le \frac{1+\epsilon}{\epsilon} \sup_{\text{a.e.}(m)} q \cdot \underbrace{E\int_{c\ge1} c\,dt}_{\le K_0 = 1}$

$\le \frac{1+\epsilon}{\epsilon} \sup q.$

$\therefore$ the family $\{c\, I_{c\ge1} : c \in \mathcal{C}\}$ is weakly sequentially compact

cf Existence Lemma, p.51
and Assumption (iv) p.54
of RES 1978.

On the other hand
$E\int_{\underline{c}<1} (\ln \underline{c})\, q\,dt = E\int_{q<1} (\ln q)\, q\,dt$ if we set $\underline{c} = q$,
which is $> -\infty$ by assumption.

Consider $h\underline{c}$, $0 < h \le 1$ to apply Assumption (iii), p.53 of RES 1978 (here $\underline{c}$ replaces g_+ in 1978 reference)

$E\int_{\underline{c}<1} \ln(h\underline{c})\, q\,dt = E\int_{q<1} (\ln h + \ln q)\, q\,dt$
$= (\ln h)\, E\int_{q<1} q\,dt + E\int_{q<1} \ln q \cdot q\,dt > -\infty.$

So both A(iii) and A(iv) are satisfied,
$\therefore$ an optimum exists, see Theorem 3, page 56.