EC402 classes

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Comments on PS1 and PS2

[T] means technical and can be omitted in 1^{st} read and 2^{nd} read, just here if you have already seen these technicalities somewhere else. **[O]** means optional or indirectly related to the problem sets.

PS1.Q3 [T]

Let $f(x, y)$ be the joint pdf of (X, Y). The question is when can we interchange the order of integration in:
$A := \int \int g(x, y)f(x, y)dxdy$
And $A_1 := \int \int g(x, y)f(x, y)dydx$
So that: $A = A_1$
Sufficient conditions are:
- Tonelli's theorem: eg. $g()$ is continuous (or measurable if you know this notion) and non negative.
- Fubini's theorem: eg. $g()$ is continuous (or measurable) and $\int \int |g(x, y)|f(x)f(y)dxdy$ is finite.
When $\int \int |g(x, y)|f(x, y)dxdy < \infty$ we say that $g()$ is integrable and $A = A_1$. We always make this assumption, so that we can always interchange the order of the integrals.

PS1.Q4 [O]

Counterexample to prove that: $E(Y|X) = E(Y)$ does not imply X independent of Y.
Take a bivariate discrete case:

p	X=Y^2	Y
1/3	1	-1
1/3	0	0
1/3	1	1

Then $E(Y|X) = E(Y) = 0$ but $E(X|Y) \neq E(X)$. And we showed in class that the statement "X is independent of Y" implies conditional mean independence (ie. that: $E(Y|X) = E(Y)$ and $E(X|Y) = E(X)$).
Counterexample to prove that: $Cov(X, Y) = 0$ does not imply $E(X|Y) = E(X)$.
Take the same example:

p	X=Y^2	Y	X × Y
1/3	1	-1	-1
1/3	0	0	0
1/3	1	1	1

Then $E(XY) = E(Y) = 0$ so $Cov(X, Y) = 0$ but $E(X|Y) \neq E(X)$.

PS2. Useful matrix algebra definitions and results

1. $A_{n\times n}$ **symmetric** matrix is such that $A = A'$.
2. $A_{n\times n}$ **idempotent** matrix is such that $A^2 = A$.
3. $A_{n\times n}$ **orthogonal** matrix is such that $AA' = A'A = I_n$.
4. $A_{n\times n}$ **invertible** matrix is such that $\exists A^{-1},\ A^{-1}A = I_n$.
5. $A_{m\times n}$ **full column rank** matrix is such that for $x_{n\times 1} \neq 0$ then $A.x \neq 0$.
6. $A_{n\times n}$ **positive definite/semidefinite** matrix is such that for $x \neq 0$ then $x'Ax > / \geq 0$.

1. **Diagonalization of symmetric matrices**. If $A_{n\times n}$ is symmetric then there exists $S_{n\times n}$ such that $S'S = I_n$ and $\Lambda = diag(\lambda_1, ..., \lambda_n)$ with the λ_is the eigenvalues of A such that $S'AS = \Lambda$ or $A = S\Lambda S'$.
2. $A_{n\times n}$ **symmetric** matrix has **real eigenvalues**.
3. $A_{n\times n}$ **idempotent** matrix has **eigenvalues** $= 0$ or $= 1$.
4. $A_{m\times n}$ and $G_{p\times m}$ is full column rank (ie. $r(G) = m$) then $r(GA) = r(A)$.
5. $A_{m\times n}$ and $G_{n\times p}$ is full row rank (ie. $r(G) = n$) then $r(AG) = r(A)$.

PS2.Q4 Quadratric form and Chi-squarred distribution

If M is $n\times n$ symmetric, idempotent and rank M is J and if $x \sim \mathcal{N}(0, I_n)$ then $z := x'.M.x \sim \chi^2_J$.

We can do the proof in 4 steps:
Step 1: By the **diagonalization theorem** for symmetric matrices we have:
$z := x'.M.x = x'.S\Lambda S'.x$ where $\Lambda = diag(\lambda_1, ..., \lambda_n)$ and $S_{n\times n}$ such that $S'S = I_n$.
So that $z := x'.M.x = u'.\Lambda.u$ with $u := S'.x$
But $u := S'.x \sim \mathcal{N}(0, S'S) = \mathcal{N}(0, I_n)$
Step 2: $\forall i, \quad \lambda_i \in \{0, 1\}$ (we proved this in class using that M is **idempotent**).
Step 3: $tr(\Lambda) = \sum_i \lambda_i = r(M) = J$
Step 4: $z = \sum_i \lambda_i u_i^2$ with the u_i's iid $\mathcal{N}(0, 1)$ so $z \sim \chi^2_J$.

Rk: We can prove step 3, using step 1 and step 2. We have:
$tr(M) = tr(S\Lambda S') = tr(\Lambda S'S) = tr(\Lambda) = \sum_i \lambda_i = r(\Lambda)$
But $r(\Lambda) = r(M)$ because S and S' are square invertible matrices.