EC402 classes

November 29, 2010

Antoine Goujard

PS6 Question 1

The model is:

$y_i = \beta_1.x_{1i} + \beta_2.x_{2i} + \varepsilon_i$

By PS5, we know:

$\hat{\beta} = \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}$ and $s^2 = \frac{RSS}{n-k} = \frac{2}{27}$

And $V(\hat{\beta}) = \sigma_\varepsilon^2.(X'X)^{-1} = \sigma_\varepsilon^2.\frac{1}{3}.\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

Also, $\hat{V}(\hat{\beta}) = s^2.(X'X)^{-1} = \underbrace{\frac{2}{27}.\frac{1}{3}}_{2/81}.\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

a) Now we want to test: $H_0 : \ \beta_1 = 0$ **vs** $H_1 : \ \beta_1 \neq 0$

To perform this test we need to know the distribution of the test statistic under H_0 so in finite samples we need to make some assumptions on the distribution of the disturbance ε_i. We can make two sets of assumptions:

-S1: Under A1, A2, A3F, A5N, we have: $\hat{\beta} \sim \mathcal{N}_k(\beta, \sigma_\varepsilon^2.(X'X)^{-1})$.

-S2: Under A1, A2, A3RFI, A5N, we have: $\hat{\beta}|X \sim \mathcal{N}_k(\beta, \sigma_\varepsilon^2.(X'X)^{-1})$.

With A1, A2, A3F, A3FI, A5N as in the lecture notes p24.

A1. $\rho(X) = k \leq n$.

A2. $y = X\beta + \varepsilon$ and $E(\varepsilon) = 0$.

A3F. X is fixed in repeated samples .

A3FI. x_{ik} is independent of ε_j for all observations i, j and all variable k.

A5N. $\varepsilon \sim \mathcal{N}_n(0, \sigma_\varepsilon^2.I_n)$

Rk: we can slightly change the set S2 of assumptions saying that: A1, A2 hold and that $\varepsilon|X \sim \mathcal{N}_n(0, \sigma_\varepsilon^2.I_n)$. The conclusion will be the same: $\hat{\beta}|X \sim \mathcal{N}_k(\beta, \sigma_\varepsilon^2.(X'X)^{-1})$. Indeed, A3FI+A5N $\Rightarrow \varepsilon|X \sim \mathcal{N}_n(0, \sigma_\varepsilon^2.I_n)$

Let's assume that we are in the case S2. Under H_0 if we know σ_ε^2 we can test H_0 using the fact that $\hat{\beta}_1|X \sim_{H_0} \mathcal{N}(0, \sigma_\varepsilon^2.(X'X)^{-1}_{11})$.

Here we need to estimate σ_ε^2 by s^2. We will use 3 facts from the lecture notes:

1/ $(n-k).\frac{s^2}{\sigma_\varepsilon^2}|X \sim \chi^2_{n-k}$.

2/ $\hat{\beta}|X \sim \mathcal{N}_k(\beta, \sigma_\varepsilon^2.(X'X)^{-1})$

3/ $\hat{\beta}$ and s^2 are independent (conditional on X).(*)

(*)**[Technical part]** Conditional on X, the only random part of $\hat{\beta}-\beta$ is $(X'X)^{-1}X'\varepsilon|X$. And s^2 is a deterministic function of $z := M_x\varepsilon$ because $s^2 = \frac{z'z}{n-k}$. We can prove that $\hat{\beta}$ is independent of z because:

$$\begin{pmatrix}\hat{\beta}-\beta \\ z\end{pmatrix} = \begin{pmatrix}(X'X)^{-1}X' \\ M_x\end{pmatrix}_{(k+n)\times n} .\varepsilon$$

And $(X'X)^{-1}X'M_x = 0_{k\times n}$. Moreover $\varepsilon|X \sim \mathcal{N}_n(0, \sigma_\varepsilon^2.I_n)$.

So we find: $\begin{pmatrix}\hat{\beta}-\beta \\ z\end{pmatrix}|X \sim \mathcal{N}_{n+k}(\begin{pmatrix}0 \\ 0\end{pmatrix}, \begin{pmatrix}\sigma_\varepsilon^2.(X'X)^{-1} & 0 \\ 0 & \sigma_\varepsilon^2.M_x\end{pmatrix})$

[Back to PS6Q1a]

We define the t statistic under $H_0;\ \beta_1 = 0$ as:

$t_0 = \frac{\hat{\beta}_1}{se(\hat{\beta}_1)}|X \sim_{H_0} t_{n-k}$

We have also $t_0 = \frac{\hat{\beta}_1}{se(\hat{\beta}_1)} \sim_{H_0} t_{n-k}$

As the distribution t_{n-k} does not depend on X.

We found a t_{n-k} distribution because:

$$t_0 = \frac{\hat{\beta}_1}{\sqrt{V(\hat{\beta}_1)}}.\frac{\sqrt{V(\hat{\beta}_1)}}{\sqrt{\hat{V}(\hat{\beta}_1)}} = \underbrace{\frac{\hat{\beta}_1}{\sqrt{V(\hat{\beta}_1)}}}_{|X\sim\mathcal{N}(0,1)} . \frac{1}{\sqrt{\underbrace{(n-k).\frac{s^2}{\sigma_\varepsilon^2}}_{|X\sim\chi^2_{n-k}}/(n-k)}}$$

In our case, $se(\hat{\beta}_1) = \sqrt{4/81} = 2/9$ so $t_0 = 9/2 * 1/3 = 3/2 = 1.5$.

And $|t_9^*(2.5\%)| = 2.262$. So we can not reject H_0 at the 5% significance level.

b) We test: $H_0:\ \beta_2 = 0$ **vs** $H_1:\ \beta_2 \neq 0$

Now, $t_0 = 1/3 * \sqrt{81/4} = 1.5$. So we have the same conclusion.

c) We test: $H_0:\ \beta_1 = \beta_2 = 0$ **vs** $H_1:\ \beta_1 \neq 0$ **or** $\beta_2 \neq 0$

In matrix notation, $H_0 : R.\beta = q$ with $q = 0_{2\times 1}$ and $R = I_2$.

(Here clearly $\rho(R) = r = 2$, but it may be important to check before performing your computations that your constraints are linearly independent).

The idea of the Wald test is to see if $\hat{\beta}$ satisfies approximately the restrictions imposed under H_0. That is if $R.\hat{\beta} \simeq q$.

Under the set of assumptions S2, we have $\hat{\beta}|X \sim \mathcal{N}_k(\beta, \sigma_\varepsilon^2.(X'X)^{-1})$.

So it comes (under H_0) $R\hat{\beta} - q|X \sim \mathcal{N}_r(0, \sigma_\varepsilon^2.R(X'X)^{-1}R')$

Thus by PS2 Q5b, we have: $(R\hat{\beta}-q)'(\sigma_\varepsilon^2.R(X'X)^{-1}R')^{-1}(R\hat{\beta}-q)|X \sim_{H_0} \chi^2_r$

Unfortunately we can not use this statistic because we do not know σ_ε^2. So we work with:

$F = \frac{(R\hat{\beta}-q)'(s^2.R(X'X)^{-1}R')^{-1}(R\hat{\beta}-q)}{r}$

$F = \frac{(R\hat{\beta}-q)'(\sigma_\varepsilon^2.R(X'X)^{-1}R')^{-1}(R\hat{\beta}-q)/r}{\frac{(n-k)*s^2}{\sigma_\varepsilon^2}/(n-k)}|X \sim_{H_0} F(r, n-k)$

rk: the numerator depends only on $\hat{\beta}$ and the denominator is a function of s^2 so they are independent conditional on X, following two χ^2 distributions with the right degrees of freedom divided by r and $n-k$.

In our case, $R = I_2$ and $q = 0_{2\times 1}$ so this simplifies to:
$F = \hat{\beta}'(s^2.(X'X)^{-1})^{-1}\hat{\beta}/2 = s^{-2}\hat{y}'\hat{y}/2 = 27/2 * 2/3 * 1/2 = 4.5$ (see PS5 Q1 for the value of $\hat{y}'\hat{y}$).
Here $F^*_{2,9}(5\%) = 4.26$ so we can reject H_0 at the 5% significance level.

d) The results are consistent because the first two tests are testing the marginal explanatory power of x_1 controlling for x_2 and the marginal explanatory power of x_2 controlling for x_1. The last one tests the joint explanatory power of x_1 and x_2. This suggests that the two explanatory variables have some degree of collinearity so that once we control for one, adding the other does not add much information to the model. However, the two explanatory variables have a significant joint explanatory power on the variations of y.
rk. This case is possible but the other one is not. If we do not reject H_0 for the last joint hypothesis, we should not reject the simple hypothesis (at the same significance level).

PS6 Question 2

(*) $y_t = \beta_1 + \beta_2.x_{2t} + \beta_3.x_{3t} + \beta_4.x_{4t} + \varepsilon_t$
with $\rho(X'X) = 4$ and $\varepsilon|X \sim \mathcal{N}_n(0, \sigma^2_\varepsilon.I_n)$.
This gives us: $\hat{\beta}|X \sim \mathcal{N}_k(\beta, \sigma^2_\varepsilon.(X'X)^{-1})$.
a) The explanation to test $H_0 : R.\beta = q$ is detailed in Q3. Under H_0 the F statistic has a $F_{r,T-k}$ distributions, where r is the number of independent linear constraints, T the number of observations and k is the number of parameters of the unconstrained model (here $k = 4$).

b)RSS_U comes from fitting the model (*).
i) $H_0 :\ \beta_2 - 3\beta_3 = 4$ and $\beta_1 = 2\beta_4$. Let's impose these constraints in model (*). This can be done by substitution because we know that under H_0, $\beta_1 = 2\beta_4$ and $\beta_2 = 4 + 3\beta_3$. We get:
$y_t = 2\beta_4 + (4 + 3\beta_3).x_{2t} + \beta_3.x_{3t} + \beta_4.x_{4t} + \varepsilon_t$
Or $\underbrace{y_t - 4x_{2t}}_{u_t} = \beta_3.\underbrace{(3x_{2t} + x_{3t})}_{v_t} + \beta_4.\underbrace{(2 + x_{4t})}_{w_t} + \varepsilon_t$
So to compute RSS_R we need to regress u_t on v_t and w_t without a constant.
Moreover we have $r = 2$, so this fully characterize the F-statistic and its distribution under H_0.