$y_t = \beta_1 + \beta_2 x_t + \varepsilon_t \quad t = 1, \dots T$

We are told to assume

$A1: \ r(X) = k \geq T$

$A2: \ y = X\beta + \varepsilon, \ E(\varepsilon) = 0_{T\times 1}$

$A3F: \ X$ is fixed or non-stochastic.

$ASN: \ \varepsilon_t$ iid $\mathcal{N}(0, \sigma^2)$

rk1 we can also state ASN as: $\varepsilon_{T\times 1} \sim \mathcal{N}(0_{T\times 1}, \sigma^2 I_T)$

rk2 ASN incorporates A4GM as: $E(\varepsilon\varepsilon') = \sigma^2 I_T$

We have a) No autocorrelation: $Cov(\varepsilon_t, \varepsilon_{t'}) = 0$ if $t \neq t'$.

b) homoskedasticity: $V(\varepsilon_t) = \sigma^2$ constant over t.

If σ^2 is unknown the usual test-statistic with known critical values is $t = \dfrac{\hat{\beta}_2 - \beta_2}{\sqrt{\hat{V}(\hat{\beta}_2)}}$ and $\boxed{t \sim t_{T-k}}$

t is a pivotal statistic as the distribution t_{T-k} does not depend on the unknown parameters: β_1, β_2 and σ^2.

Proof We need to rewrite t as: $t = \dfrac{U}{\sqrt{V/T-k}}$ where:

a) $U \sim \mathcal{N}(0,1)$

b) $V \sim \chi^2_{T-k}$

c) U is independent of V.

Then a, b, c imply $t \sim t_{T-k}$.

Step 1: $\hat{\beta} - \beta = (X'X)^{-1}X'\varepsilon \sim \mathcal{N}_k(0_{k\times 1}, \sigma^2 (X'X)^{-1})$ By A3F, ASN.

Step 2: $\hat{\beta}_2 - \beta_2 = a'(\hat{\beta} - \beta)$ where $a = \binom{0}{1}_{2\times 1}$ vector that selects the second element of β.

$\hat{\beta}_2 - \beta_2 \sim \mathcal{N}(a'.0 = 0, \sigma^2 a'(X'X)^{-1}a)$

Thus, $\dfrac{\hat{\beta}_2 - \beta_2}{\sqrt{\sigma^2 a'(X'X)^{-1}a}} \sim \mathcal{N}(0,1)$

Step3 Note that

$$t = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{\hat{V}(\hat{\beta}_2)}} = \frac{\hat{\beta}_2 - \beta_2}{\sqrt{s^2 a'(X'X)^{-1} a}}$$

$$= \underbrace{\boxed{\frac{\hat{\beta}_2 - \beta_2}{\sqrt{\sigma^2 a'(X'X)^{-1} a}}}}_{\sim \mathcal{N}(0,1) \text{ By step 2}} \times \frac{1}{\sqrt{\underbrace{\left(\frac{s^2(T-k)}{\sigma^2}\right)}_{\text{has a } \chi^2_{T-k} \text{ distribution. (proof in step4).}} / T-k}}$$

Step4

$$\frac{s^2(T-k)}{\sigma^2} = \frac{1}{T-k} \times (T-k) \frac{\hat{\varepsilon}'\hat{\varepsilon}}{\sigma^2} \quad \text{using } \hat{\varepsilon} = \Pi_x \varepsilon$$

$$= \frac{\varepsilon'}{\sigma} \Pi_x \frac{\varepsilon}{\sigma}$$

- But by ASN $\frac{\varepsilon}{\sigma} \sim \mathcal{N}(0, I_T)$
- Π_x is symmetric, idempotent, $r(\Pi_x) = tr(\Pi_x) = T-k$.
- Hence by $\boxed{\text{PS2, Q7}}$ we have, $\frac{s^2(T-k)}{\sigma^2} \sim \chi^2_{T-k}$.

Step 5 s^2 and $\hat{\beta}_2 - \beta_2$ are independent so using step2, step 3 and step4, $t \sim \chi^2_{T-k}$

s^2 and $\hat{\beta}_2 - \beta_2$ are independent because, we assume ASN, $\varepsilon \sim \mathcal{N}(0, \sigma^2 I_T)$ and s^2 depends only of $\Pi_x \varepsilon$ that is orthogonal to the random part of $\hat{\beta}_2 = (X'X)^{-1}X'\varepsilon$ (recall that A3F, X fixed holds).

Full proof of independence (not required)

- $\hat{\beta}_2 - \beta_2 = (X'X)^{-1}X'\varepsilon$
- S^2 depends only on $M_X \varepsilon$.

But $M_X \varepsilon$ and $(X'X)^{-1}X'\varepsilon$ are independent.

Step 1 $\begin{pmatrix} M_X\varepsilon \\ (X'X)^{-1}X'\varepsilon \end{pmatrix}$ is jointly normal.

$$\begin{pmatrix} M_X \varepsilon \\ (X'X)^{-1}X'\varepsilon \end{pmatrix} = \underbrace{\begin{pmatrix} M_X \\ (X'X)^{-1}X' \end{pmatrix}}_{(T+k)\times T} \underbrace{\varepsilon}_{T\times 1} \sim \mathcal{N}\Bigg(\underbrace{0}_{(T+k)\times 1}, \begin{pmatrix} A & B \\ C & D \end{pmatrix}\Bigg)$$

Step 2

$$A = M_X \sigma^2 I_T M_X' = \sigma^2 M_X$$

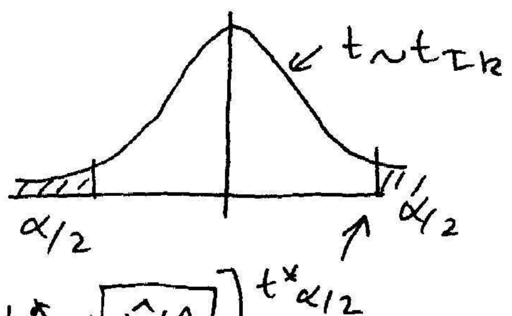
$$D = (X'X)^{-1}X'\sigma^2 I_T X (X'X)^{-1} = \sigma^2 (X'X)^{-1}$$

$$B = M_X \sigma^2 I_T X (X'X)^{-1} = \sigma^2 \underbrace{M_X X}(X'X)^{-1} = \underset{T\times k}{0}$$

$$C = (X'X)^{-1}X'\sigma^2 I_T M_X' = \sigma^2 (X'X)^{-1}\underbrace{X'M_X} = 0_{k\times T}$$

So $M_X\varepsilon$ and $(X'X)^{-1}X'\varepsilon$ are jointly normal with 0 var covar matrix. Hence they are independent.

As we have $\dfrac{\hat{\beta} - \beta_2}{\sqrt{\hat{V}(\hat{\beta}_2)}} \sim t_{T-k}$



$$\beta_2 \in \left[\hat{\beta}_2 - t^*_{\alpha/2}\sqrt{\hat{V}(\hat{\beta}_2)},\ \hat{\beta}_2 + t^*_{\alpha/2}\sqrt{\hat{V}(\hat{\beta}_2)}\right]$$

with probability $1-\alpha$ (two sided, symmetric CI).