PS7

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Binary response model

Random effects probit (xtprobit)

Probit case: Assume $y_{it}^* = x_{it}'\beta + \varepsilon_{it}$

Assuming $\varepsilon_{it} \sim \mathcal{N}(0,1)$ i.i.d we get the usual probit model that can be estimated by MLE.

If we have a one factor error, ε_{it}, we have: $\varepsilon_{it} = \alpha_i + \nu_{it}$

To find a MLE, we need to make some extra assumptions on the disturbances, that will give us the **random effects probit model**:

1. $y_{it}^* = x_{it}'\beta + \varepsilon_{it} = x_{it}'\beta + \alpha_i + \nu_{it}$ with $y_{it} = 1_{y_{it}^*>0}$.
2. $\alpha_i | X \sim \mathcal{N}(0, \sigma_\alpha^2)$ i.i.d over i.
3. $\nu_{it} | X, \alpha_i \sim \mathcal{N}(0,1)$ i.i.d over i and t.

Rk. 1 Assumptions 2 and 3 imply that the correlation between two disturbances of the same individual is constant over time. This is sometimes called an equi-correlation assumption. Assumption 2 is the **random effects** assumption. It states that the distribution of α_i conditional on X does not depend on X.

Rk. 2 Here the panel data is balanced. So we can write: $E(\varepsilon\varepsilon'|X) = I_n \otimes \Omega$ where $\Omega_{T\times T} = (1+\sigma_\alpha^2).\begin{pmatrix} 1 & \rho & ... \\ & \ddots & \\ ... & \rho & 1 \end{pmatrix}$, with $\rho = \frac{\sigma_\alpha^2}{1+\sigma_\alpha^2}$.

Then, $P(y_{it} = 1|X, \alpha_i) = \Phi(x_{it}'\beta + \alpha_i)$ by A1 and A3.

Thus, $P(y_{it} = 1|x_{it}) = \int \Phi(x_{it}'\beta + \alpha) f_{\alpha|x_{it}}(\alpha) d\alpha$.

Or, $P(y_{it} = 1|x_{it}) = \int \Phi(x_{it}'\beta + \alpha) f_\alpha(\alpha) d\alpha$ by A2.

Thus $P(y_{it} = 1|x_{it}) = \int \Phi(x_{it}'\beta + \alpha)\frac{1}{\sigma_\alpha}\varphi(\frac{\alpha}{\sigma_\alpha}) d\alpha$ by A2.

However, in general, $P(y_{i1}, ..., y_{iT}|X_i) \neq \prod_t P(y_{it}|x_{it})$

As conditional on X_i, the random part of the y_{it}s are the $\varepsilon_{it} = \alpha_i + \nu_{it}$ which share the same α_i.

Hence we have to apply the same argument to the likelihood of all the observations of an individual, i:

$P(y_{i1}, ..., y_{iT}|x_i, \alpha_i) = P(y_i|x_i, \alpha_i) = \prod_t P(y_{it}|x_i, \alpha_i) = \prod_t P(y_{it}|x_{it}, \alpha_i)$ by A3.

So, $P(y_i|x_i) = \int \prod_t P(y_{it}|x_i, \alpha) f_{\alpha|x_i}(\alpha) d\alpha = \int \prod_t P(y_{it}|x_i, \alpha) f_\alpha(\alpha) d\alpha$ by

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A2.
And, $P(y_i|x_i) = \int \prod_t P(y_{it}|x_{it}, \alpha)\frac{1}{\sigma_\alpha}\varphi(\frac{\alpha}{\sigma_\alpha})d\alpha$ by A2.
Note that by A3, we have: $P(y_{it}|x_{it}, \alpha) = \Phi(x'_{it}\beta+\alpha)^{y_{it}}(1-\Phi(x'_{it}\beta+\alpha))^{1-y_{it}}$
Thus, we obtain the likelihood for the full sample:
$L(y_1, ..., y_n|x_1, ..., x_n) = \prod_i P(y_i|x_i)$ by A2 and A3.
$L(y_1, ..., y_n|x_1, ..., x_n) = \prod_i \int \prod_t P(y_{it}|x_{it}, \alpha)\frac{1}{\sigma_\alpha}\varphi(\frac{\alpha}{\sigma_\alpha})d\alpha$

Under the assumptions of the random effects probit (one factor model), this is a parametric and fully efficient estimator. We obtain estimators of β and σ_α. The estimator $\hat{\sigma}_\alpha$ can be used to test the presence of individual specific effects, α_i, under the random effects assumption.

Extension to ARMA errors
Now we assume:

1. $y^*_{it} = x'_{it}\beta + \varepsilon_{it} = x'_{it}\beta + \alpha_i + \nu_{it}$ with $y_{it} = 1_{y^*_{it}>0}$.
2. $\alpha_i|X \sim \mathcal{N}(0, \sigma^2_\alpha)$ i.i.d over i.
3. $\nu_{it}|X, \alpha_i \sim ARMA(p, q)$.

$\nu_{it}|X, \alpha_i \sim ARMA(p, q)$ means that:
$\nu_{it} = \sum_p \alpha_p.\nu_{it-p} + \sum_q \gamma_q.\zeta_{it-q}$ where ζ_{it} are iid over i and t.
We get a general likelihood:

$$L = \prod_{i=1}^n(\underbrace{\int ... \int_{a_{it}}^{b_{it}} ... \int}_{T_i \text{ integrals}} f_{\varepsilon_{T_i}}(\varepsilon_{i1}, ..., \varepsilon_{iT_i}).d\varepsilon_{T_i})$$

where if $y_{it} = 1$ then $a_{it} = -x'_{it}\beta$ and $b_{it} = +\infty$.
while if $y_{it} = 0$ then $b_{it} = -x'_{it}\beta$ and $a_{it} = -\infty$.
The integral does not simplify in this case because of the temporal relationship between the transitory shocks ν_{it}. We have to use simulation based methods.

We can used SML, simulated maximum likelihood:

1. The parameters of interest are $\theta_{(K+p+q+1)\times 1} = (\beta', \nu', \gamma', \sigma_\alpha)'$.
2. We maximize $\tilde{L}(\theta, R) = 1/n \sum_i^n ln(\tilde{l}(y_i, \theta, R))$ where R is the number of simulations and $\tilde{l}$ the simulated likelihood.
3. We first draw R, $T \times 1$ uniform random vectors $\tilde{u}_r$. We will keep fixed these vectors in all the following process.
4. Let $\theta^{(k)}$ be the value of the parameters at iteration k. To obtain $\tilde{L}$ we proceed as follows:

- From our assumptions about the pdf of ε and our parameters' values, $\theta^{(k)}$, we get the simulated disturbances: $\tilde{\varepsilon}_{i,r}^{(n)} = F_{\varepsilon_i}^{-1}(\tilde{u}_r, \theta^{(n)})$. This gives us: $\tilde{y}_{i,r}^{*(n)}$ and $\tilde{y}_{i,r}^{(n)}$.
- Using the R simulations we get: $\tilde{l}(y_i, \theta^{(n)}, R)$ for each i and we compute $\tilde{L}(\theta^{(n)}, R)$.

5. We iterate step 4 among the possible values of θ to maximize the simulated log-likelihood.

Random effects tobit

We assume:
$y_{it}^* = x_{it}'\beta + \varepsilon_{it} = x_{it}'\beta + \alpha_i + \nu_{it}$
$y_{it} = 1_{y_{it}^*>0} * y_{it}^*$

By the same arguments as before we have:
$f(y_i|X_i) = \int f(y_i|X_i, \alpha).f_\alpha(\alpha|X_i)d\alpha$
Assuming $\alpha_i|X \sim \mathcal{N}(0, \sigma_\alpha^2)$ i.i.d over i, this becomes:
$f(y_i|X_i) = \int f(y_i|X_i, \alpha).\frac{1}{\sigma_\alpha}\varphi(\frac{\alpha}{\sigma_\alpha})d\alpha$
Assuming $\nu_{it}|X, \alpha_i \sim \mathcal{N}(0, \sigma_\nu^2)$ i.i.d over i and t, we get:
$f(y_i|X_i) = \int \prod_t f(y_{it}|x_{it}, \alpha).\frac{1}{\sigma_\alpha}\varphi(\frac{\alpha}{\sigma_\alpha})d\alpha$

- If $y_{it} = 0$ then $f(y_{it}|x_{it}, \alpha) = \Phi(\frac{-x_{it}'\beta - \alpha}{\sigma_\nu})$
- If $y_{it} > 0$ then $f(y_{it}|x_{it}, \alpha) = \frac{1}{\sigma_\nu}\varphi(\frac{y_{it} - x_{it}'\beta - \alpha}{\sigma_\nu})$

By independence of the observations over the individuals i we get the likelihood as:
$L = \prod_{i=1}^n f(y_i|X_i) = \prod_{i=1}^n [\int \prod_t f(y_{it}|x_{it}, \alpha).\frac{1}{\sigma_\alpha}\varphi(\frac{\alpha}{\sigma_\alpha})d\alpha]$.