Stabilizing Weighted Graphs

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Matchings and w-vertex covers

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- By LP duality,

$$\nu(G) \leq \nu_f(G) = \tau_f(G).$$

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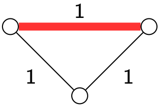

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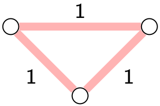

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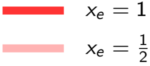

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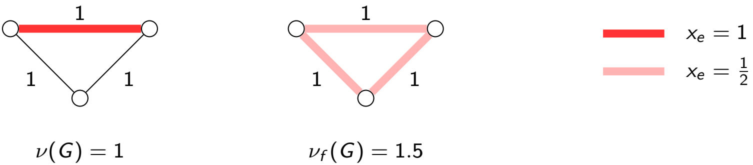

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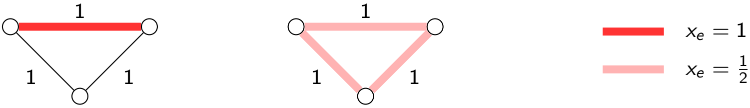

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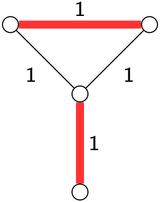

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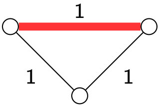

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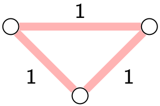

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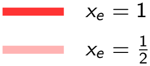

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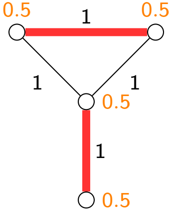

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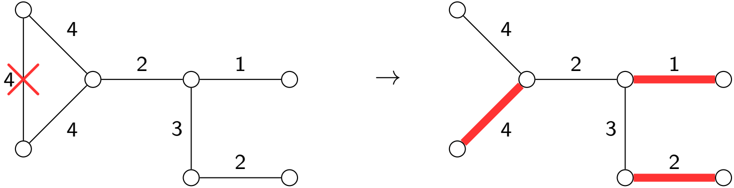

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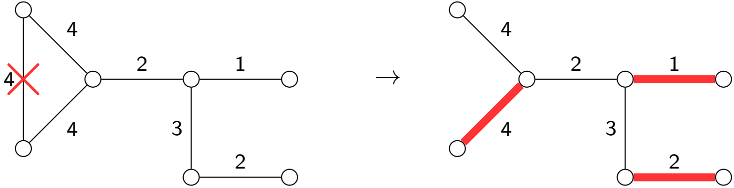

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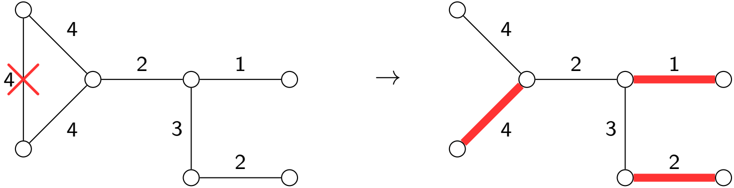

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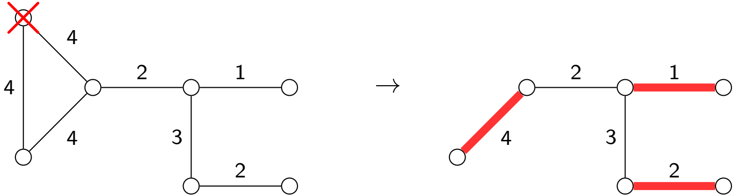

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- Why are stable graphs interesting?
  - Motivated by network bargaining games and cooperative matching games.

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A stable outcome exists $\Leftrightarrow$ A balanced outcome exists $\Leftrightarrow$ G is stable

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e.g. by blocking some players $\boxed{\text{Vertex-stabilizer}}$

by blocking some deals $\boxed{\text{Edge-stabilizer}}$

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- Other variants [Mishra et al. '11, Biró et al. '12, Könemann et al. '15].

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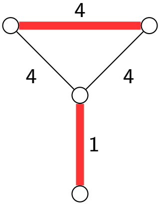
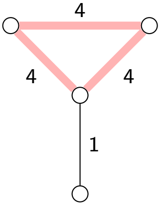


$\nu(G) = 5$



$\nu_f(G) = 6$

Main results

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Thm 1: There exists a polynomial time algorithm that computes a minimum vertex-stabilizer S for a weighted graph G. Moreover,

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Thm 2: Deciding whether a graph G has a vertex-stabilizer S where $\nu(G \setminus S) = \nu(G)$ is **NP**-complete.

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Thm 3: There is no constant factor approximation for the minimum edge-stabilizer problem unless $\mathbf{P} = \mathbf{NP}$.

Thm 4: There exists an efficient $O(\Delta)$-approximation algorithm for the minimum edge-stabilizer problem.

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Thm [Balinski '70]: A fractional matching $\hat{x}$ in G is basic if and only if

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where $\mathcal{X}$ is the set of basic maximum-weight fractional matchings in G.

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  - An edge uv is tight if $y_u + y_v = w_{uv}$.

Preliminaries

Thm [Balinski '70]: A fractional matching $\hat{x}$ in G is basic if and only if

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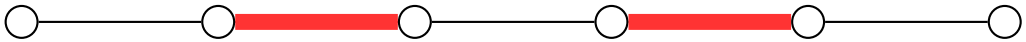

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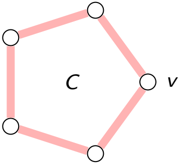

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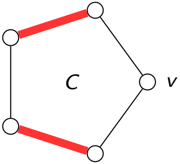

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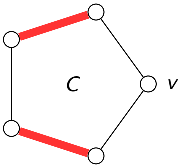

Def. An alternating path is valid if it

- starts with an exposed vertex or a matched edge
- ends with an exposed vertex or a matched edge

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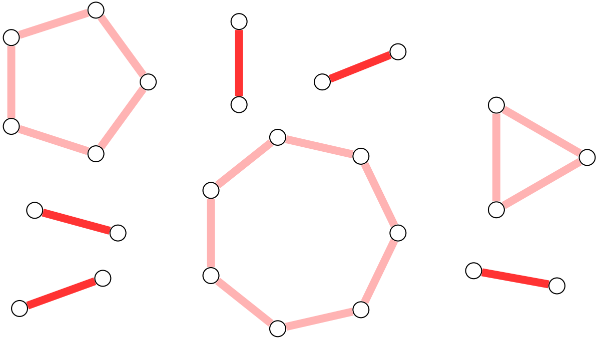

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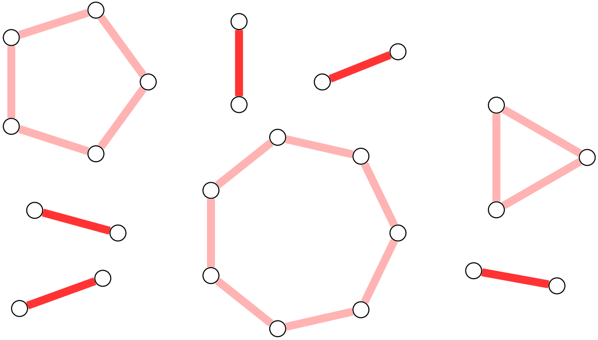

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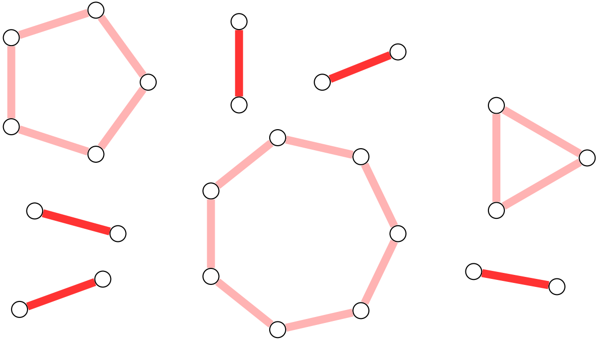

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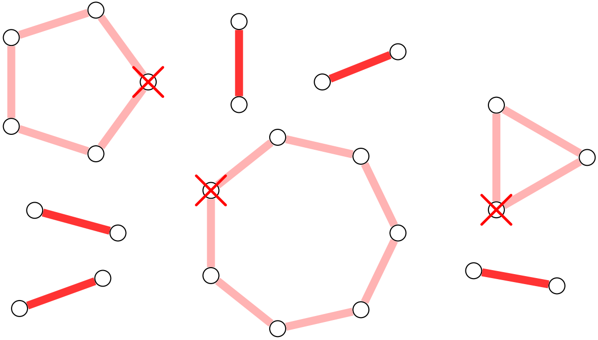

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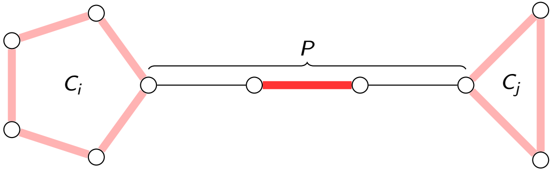

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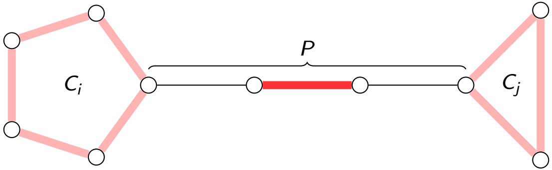

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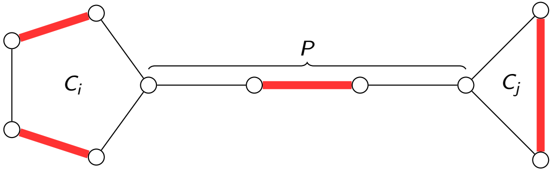

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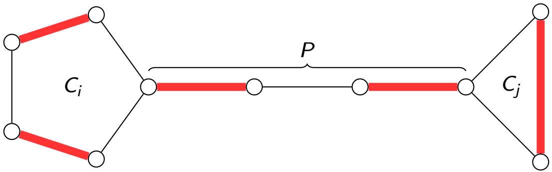

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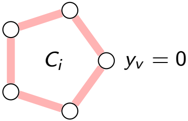

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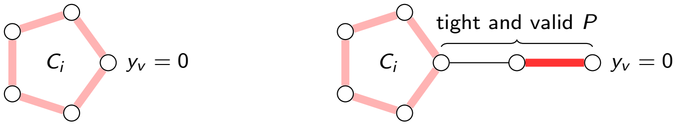

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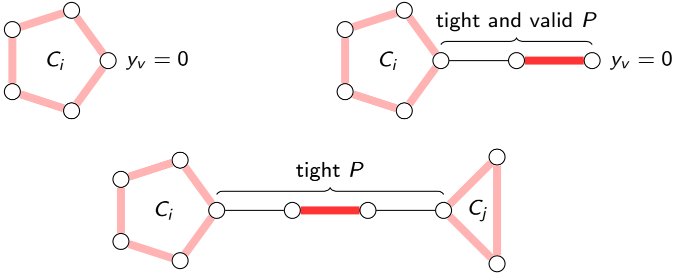

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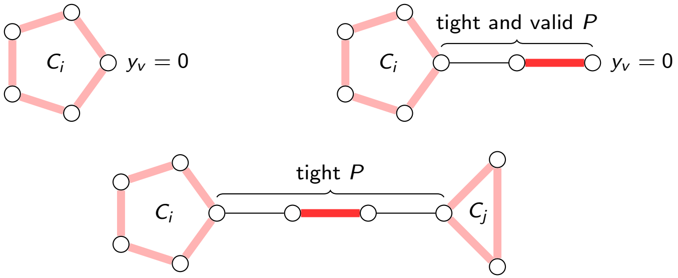

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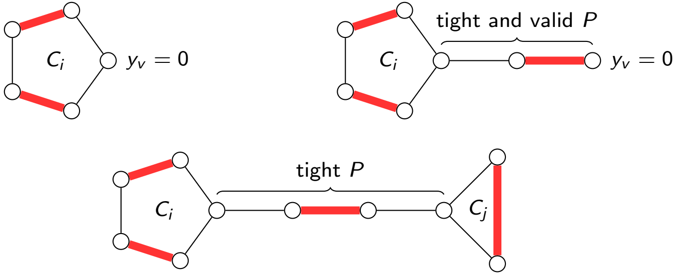

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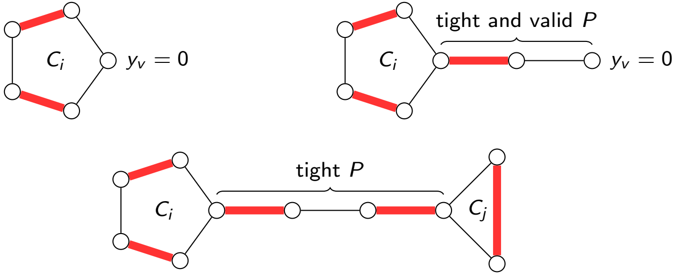

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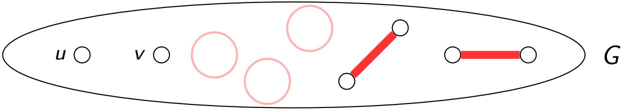

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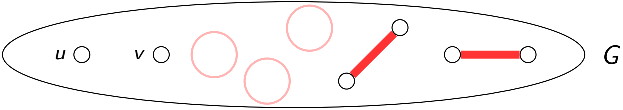

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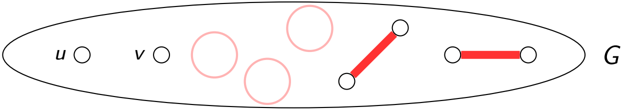

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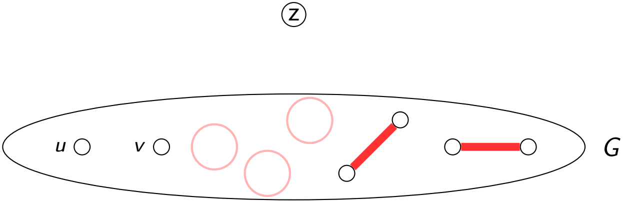

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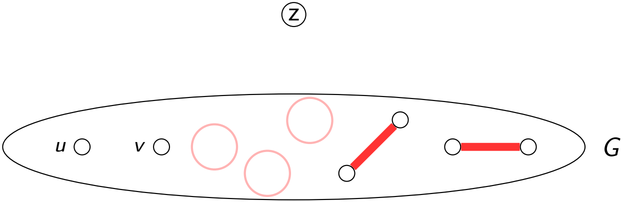

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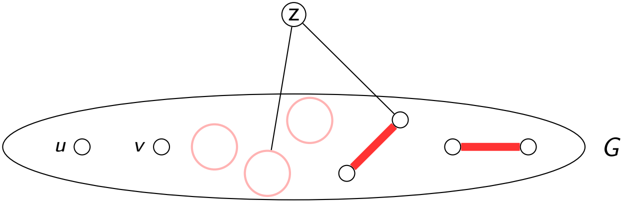

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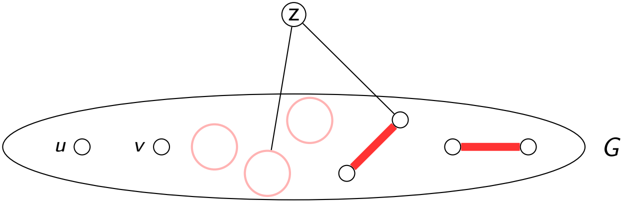

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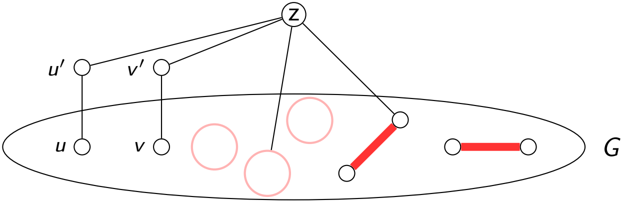

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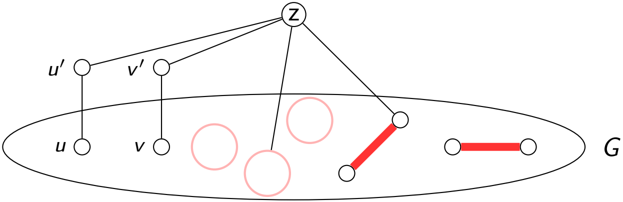

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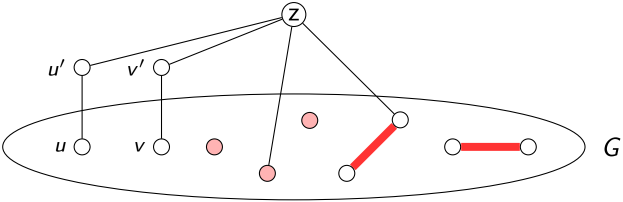

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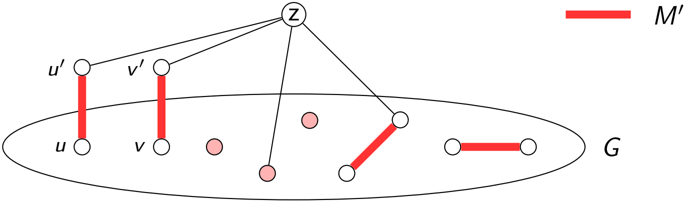

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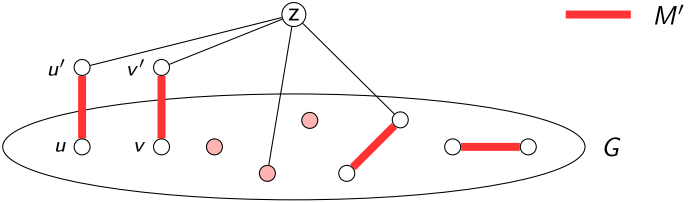

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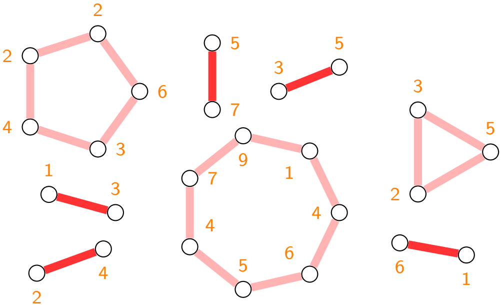

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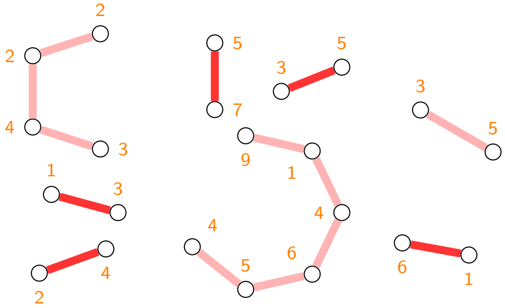

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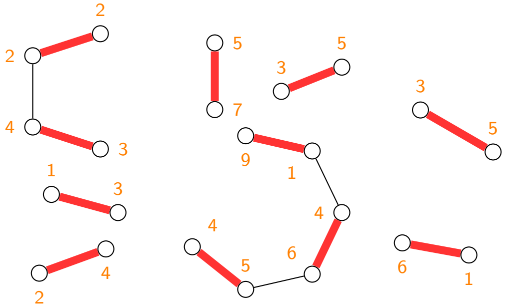

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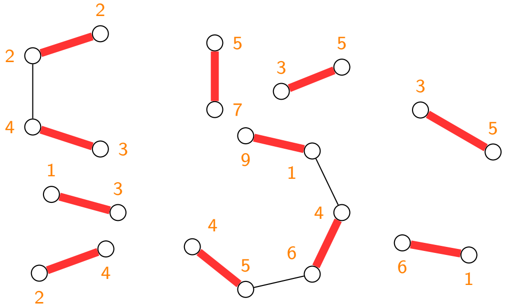

Optimality - $\gamma(G)$ is a lower bound on the size of a vertex-stabilizer.

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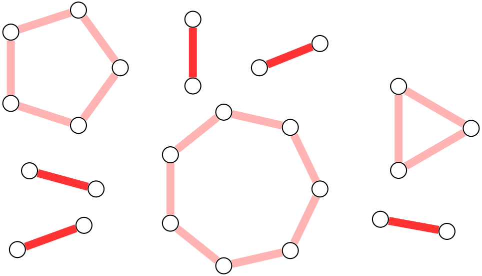

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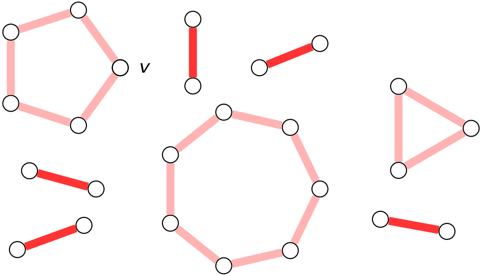

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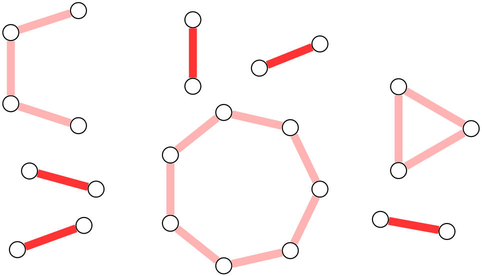

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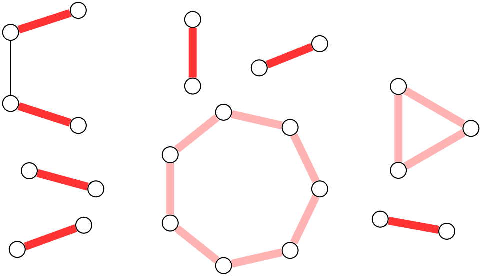

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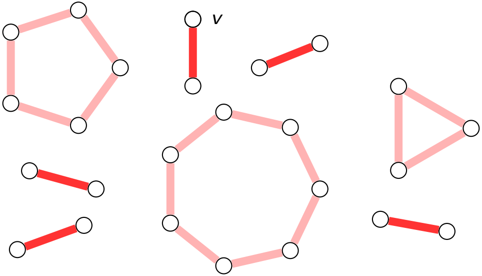

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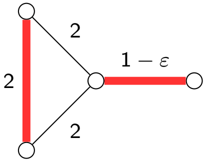

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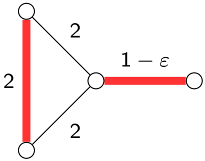

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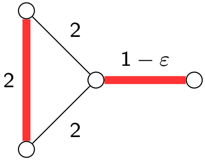

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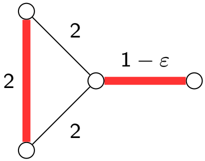

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NP-complete!

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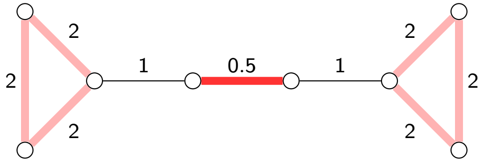

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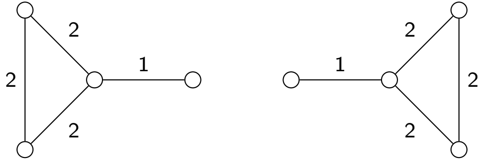

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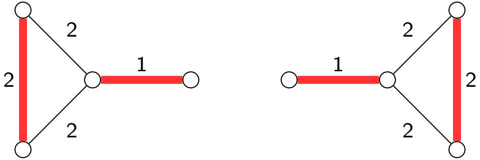

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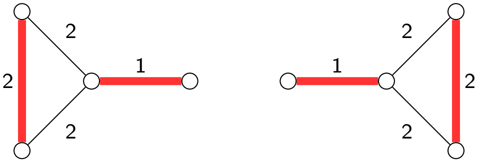

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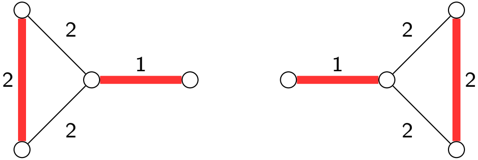

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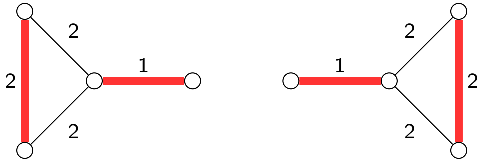

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Thm 4: There exists an $O(\Delta)$-approximation algorithm for the minimum edge-stabilizer problem.

Additional results

- Given a set of deals M, remove as few players as possible such that M is realizable as a stable outcome.

$\rightarrow$ Find a minimum vertex-stabilizer S such that
M is a maximum-weight matching in $G \setminus S$.

- A solution to this problem is called an M-vertex-stabilizer.

Thm [Ahmadian et al. '16]: If M is a maximum matching in an unweighted graph, then it is polytime solvable.

Thm 6: The problem is **NP**-hard on unweighted graphs. Moreover, no $(2-\varepsilon)$-approximation algorithm exists for any $\varepsilon > 0$ assuming UGC.

Thm 7: The problem admits a 2-approximation algorithm on weighted graphs. Furthermore, if M is a maximum-weight matching, then it is polytime solvable.

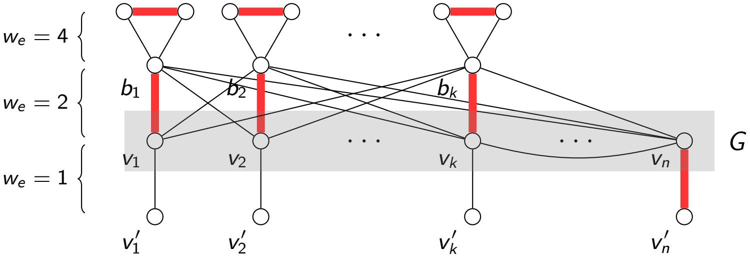
Thank you!

Appendix 1

Thm 2: Deciding whether a graph has a weight-preserving vertex-stabilizer is **NP**-complete.

Proof: Reduction from the independent set problem.

Construct the gadget graph G^* as follows:



G has an independent set of size k

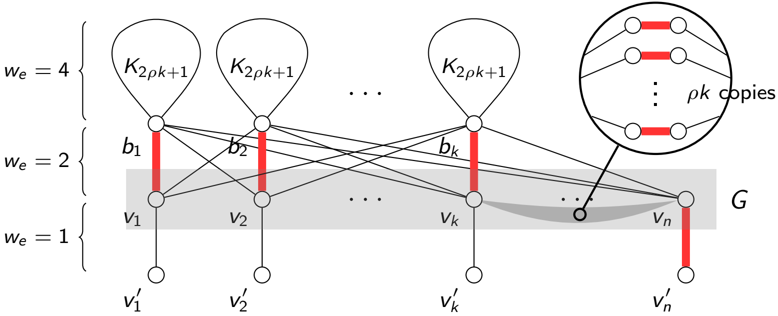
$\Leftrightarrow$

G^* has a weight-preserving vertex-stabilizer. $\square$

Appendix 2

Thm 3: There is no constant factor approximation for the minimum edge-stabilizer problem unless **P** = **NP**.

Proof: Suppose we have an α-approximation algorithm. Set $\rho = \lceil \alpha \rceil$.



- If G has an independent set of size k, then OPT $\leq k$. Else, OPT $\geq (\rho + 1)k$. $\square$