The Forward Premium Puzzle in a Two-Country World

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The forward premium puzzle

Uncovered interest parity relates next year's spot exchange rate, e_{t+1}, to today's spot exchange rate, e_t, and 1-yr interest rates in each country, i_{1,t} and i_{2,t}:

$$\mathbb{E} \log e_{t+1} \stackrel{?}{=} \log e_t + i_{1,t} - i_{2,t}$$

- "If country 2 has a lower interest rate than country 1 then (surely??) this should be compensated by the expected appreciation of its currency"
- Using covered interest parity, this is equivalent to

$$\mathbb{E} \log e_{t+1} \stackrel{?}{=} \log f_t$$

The forward premium puzzle

- UIP fails badly in the data
- Regress exchange rate changes on interest rate differentials

$$\Delta \log e_{t+1} = a_0 + a_1(i_{1,t} - i_{2,t}) + \varepsilon_{t+1}$$

- UIP is said to hold if $a_0 = 0$ and $a_1 = 1$
- Empirically, a1 is around zero or even negative

The forward premium puzzle

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- UIP is said to hold if $a_0 = 0$ and $a_1 = 1$
- Empirically, a_1 is around zero or even negative
- Carry trade generates positive excess returns: borrow in low interest-rate currencies, invest in high interest-rate currencies. Currency movements do not undo the interest differential, and may even help the trade

Related papers

- Vast empirical literature: Frankel (1980), Hansen and Hodrick (1980), Fama (1984), Brunnermeier, Nagel and Pedersen (2008), Jurek (2009), Jorda and Taylor (2009)
- Responses
 - Segmented markets: Alvarez, Atkeson and Kehoe (2009)
 - Disasters: Farhi-Fraiberger-Gabaix-Ranciere-Verdelhan (2009)
 - Country size, nontradables: Hassan (2009)
 - ▶ Long-run risk: Colacito and Croce (2010), Bansal-Shaliastovich (2010)
 - Habits: Verdelhan (2010)
- Cole and Obstfeld (1991), Zapatero (1995), Pavlova and Rigobon (2007), Stathopoulos (2009)
 - ► All rely on log utility and unit elasticity of substitution between goods



- The structure of this paper is comparatively simple
- Infinite horizon, continuous time
- Two countries, producing outputs D_{1t} and D_{2t} at time t
- Assets are priced by a representative agent
- Two important caveats: I have nothing to say about
 - nominal issues—model is fully real
 - Spatial issues—countries are distinguished only in that the rep agent views their outputs as imperfect substitutes

• Assets priced by a representative agent with power utility

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$$C_t \equiv \left[w^{1/\eta} D_{1t}^{rac{\eta-1}{\eta}} + (1-w)^{1/\eta} D_{2t}^{rac{\eta-1}{\eta}}
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• So units matter: there is an intratemporal relative price (exchange rate between the two goods) and each good has its own interest rate

Setup Preferences

$$C_t \equiv \left[w^{1/\eta} D_{1t}^{rac{\eta-1}{\eta}} + (1-w)^{1/\eta} D_{2t}^{rac{\eta-1}{\eta}}
ight]^{rac{\eta}{\eta-1}}$$

- $\eta = 1$: Cobb-Douglas case, $C_t \propto D_{1t}^w D_{2t}^{1-w}$
- $\eta = \infty$: Perfect substitutes case, $C_t = D_{1t} + D_{2t}$

- Output growth is i.i.d. over time, but may be correlated across assets; formally, (log D_{1,t}, log D_{2,t}) is a Lévy process
- Allows for dividends to follow geometric Brownian motions, or compound Poisson processes, or a combination of both, or many other possibilities
- (Will mean that the log exchange rate follows a random walk)

Technology

Output growth is summarized by the cumulant-generating function, $\mathbf{c}(\theta_1, \theta_2)$, defined by

$$\mathbf{c}(\theta_1, \theta_2) = \log \mathbb{E}\left[\left(\frac{D_{1,t+1}}{D_{1,t}}\right)^{\theta_1} \left(\frac{D_{2,t+1}}{D_{2,t}}\right)^{\theta_2}\right]$$

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• Example: if the two output streams are independent GBMs,

$$\mathbf{c}(\theta_1, \theta_2) = \mu_1 \theta_1 + \mu_2 \theta_2 + \frac{1}{2} \sigma_1^2 \theta_1^2 + \frac{1}{2} \sigma_2^2 \theta_2^2$$

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• Example: if the two output streams are correlated GBMs,

$$\mathbf{c}(\theta_1,\theta_2) = \mu_1\theta_1 + \mu_2\theta_2 + \frac{1}{2}\sigma_1^2\theta_1^2 + \frac{\kappa\sigma_1\sigma_2\theta_1\theta_2}{\kappa\sigma_1\sigma_2\theta_1\theta_2} + \frac{1}{2}\sigma_2^2\theta_2^2$$

• Notice: positively correlated fundamentals iff $\frac{\partial^2 \mathbf{c}}{\partial \theta_1 \partial \theta_2} > 0$

Technology

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• Example: if the two output streams are also subject to jumps,

$$\mathbf{c}(\theta_1, \theta_2) = \mu_1 \theta_1 + \mu_2 \theta_2 + \frac{1}{2} \sigma_1^2 \theta_1^2 + \kappa \sigma_1 \sigma_2 \theta_1 \theta_2 + \frac{1}{2} \sigma_2^2 \theta_2^2 + \omega_1 (e^{\mu_{J,1}\theta_1 + \frac{1}{2} \sigma_{J,1}^2 \theta_1^2} - 1) + \omega_2 (e^{\mu_{J,2}\theta_2 + \frac{1}{2} \sigma_{J,2}^2 \theta_2^2} - 1)$$

Technology

Output growth is summarized by the cumulant-generating function, $\mathbf{c}(\theta_1, \theta_2)$, defined by

$$\mathbf{c}(\theta_1, \theta_2) = \log \mathbb{E}\left[\left(\frac{D_{1,t+1}}{D_{1,t}}\right)^{\theta_1} \left(\frac{D_{2,t+1}}{D_{2,t}}\right)^{\theta_2}\right]$$

• Example: if the two output streams are also subject to jumps,

$$\mathbf{c}(\theta_1, \theta_2) = 0.02\theta_1 + 0.02\theta_2 + \frac{1}{2}0.1^2\theta_1^2 + 0 \cdot 0.1^2\theta_1\theta_2 + \frac{1}{2}0.1^2\theta_2^2 + \\ + 0.02(e^{-0.2\theta_1 + \frac{1}{2}0.1^2\theta_1^2} - 1) + 0.02(e^{-0.2\theta_2 + \frac{1}{2}0.1^2\theta_2^2} - 1)$$

Some identities

• Intratemporal price of good 2 in units of good 1

$$e_t \equiv \frac{u_2(D_{1t}, D_{2t})}{u_1(D_{1t}, D_{2t})}$$

• Stochastic discount factor depends on units

$$M_{1,t+1} \equiv e^{-\rho} \frac{u_1(D_{1,t+1}, D_{2,t+1})}{u_1(D_{1t}, D_{2t})} \quad M_{2,t+1} \equiv e^{-\rho} \frac{u_2(D_{1,t+1}, D_{2,t+1})}{u_2(D_{1t}, D_{2t})}$$

 Backus, Foresi and Telmer (2001), Brandt, Cochrane and Santa-Clara (2006) start from

$$\frac{e_{t+1}}{e_t} = \frac{M_{2,t+1}}{M_{1,t+1}}$$

• I follow these papers and refer to e_t as the exchange rate

Some identities

• Start from

$$\frac{e_{t+1}}{e_t} = \frac{M_{2,t+1}}{M_{1,t+1}}$$

• Take logs, then expectations:

$$\mathbb{E}_t \Delta \log e_{t+1} = \mathbb{E}_t \log M_{2,t+1} - \mathbb{E}_t \log M_{1,t+1}$$

• "Ignore risk":

$$\mathbb{E}_t \Delta \log e_{t+1} \quad \approx \quad \log \mathbb{E}_t M_{2,t+1} - \log \mathbb{E}_t M_{1,t+1} = i_{1,t} - i_{2,t}$$

• Allowing for risk,

$$\mathbb{E}_{t}\Delta \log e_{t+1} = i_{1,t} - i_{2,t} + \underbrace{\log \mathbb{E}_{t}M_{1,t+1} - \mathbb{E}_{t}\log M_{1,t+1}}_{L_{t}(M_{1,t+1})} - \underbrace{\left(\log \mathbb{E}_{t}M_{2,t+1} - \mathbb{E}_{t}\log M_{2,t+1}\right)}_{L_{t}(M_{2,t+1})}$$

Two assets, Cobb-Douglas case, is easy $u(D_{1t}, D_{2t}) \propto (D_{1t}^w D_{2t}^{1-w})^{1-\gamma}/(1-\gamma)$

$$P_{10} = \int_{t=0}^{\infty} \mathbb{E} \left[e^{-\rho t} \frac{u_1(D_{1t}, D_{2t})}{u_1(D_{10}, D_{20})} \cdot D_{1t} \right] dt$$

$$= \int_{t=0}^{\infty} \mathbb{E} \left[e^{-\rho t} \left(\frac{D_{1t}}{D_{10}} \right)^{w(1-\gamma)-1} \left(\frac{D_{2t}}{D_{20}} \right)^{(1-w)(1-\gamma)} \cdot D_{1t} \right] dt$$

$$= \frac{D_{10}}{\rho - \mathbf{c}(w(1-\gamma), (1-w)(1-\gamma))}$$

- Details aren't important
- Cobb-Douglas is tractable because everything is multiplicative (consumption aggregator, log dividend growth i.i.d.)
- Price-dividend ratio is constant-intratemporal price adjusts

Two assets are hard if $\eta > 1$

Consider the log utility, perfect substitution case, $C_t = D_{1t} + D_{2t}$:

$$P_{1,0} = \mathbb{E} \int_0^\infty e^{-\rho t} \left(\frac{C_t}{C_0}\right)^{-\gamma} \cdot D_{1,t} dt$$

= $(D_{10} + D_{20}) \int_0^\infty e^{-\rho t} \mathbb{E} \left[\frac{D_{1t}}{D_{1t} + D_{2t}}\right] dt$
= $(D_{10} + D_{20}) \int_0^\infty e^{-\rho t} \mathbb{E} \left[\frac{1}{1 + D_{2t}/D_{1t}}\right] dt$

- Cochrane, Longstaff, Santa-Clara (2008), Martin (2009)
- Intratemporal price is constant—valuation ratios adjust

Perpetuity prices, $\eta = 1$



Figure: Outputs

log D_i follow independent B.M. with drift 0.02, volatility 0.1. On top of this, N(-0.2, 0.1) disasters arrive independently at rate 0.02.

$$\rho = 0.04, \gamma = 4.$$



Figure: Large-country units



Figure: Small-country units

Perpetuity prices, $\eta = \infty$



Figure: Outputs

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Figure: Large-country units



Figure: Small-country units

Perpetuity prices, $\eta = 2$



Figure: Outputs

log D_i follow independent B.M. with drift 0.02, volatility 0.1. On top of this, N(-0.2, 0.1) disasters arrive independently at rate 0.02. $\rho = 0.04, \gamma = 4.$



Figure: Large-country units



Figure: Small-country units

Perpetuity prices



Figure: Large units, $\eta = 1$ Figure: Large units, $\eta = 2$ Figure: Large units, $\eta = \infty$



Figure: Small units, $\eta = 1$ Figure: Small units, $\eta = 2$ Figure: Small units, $\eta = \infty$

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The Forward Premium Puzzle

A suggestive special case

Suppose $D_{1t} \equiv 1$ and D_{2t} is always smaller than 1 (eg asset 2 is subject to random downward jumps at random times). Then,

$$\mathbb{E}\left[\frac{1}{1+D_{2t}}\right] = \mathbb{E}\left[1-D_{2t}+D_{2t}^2-\ldots\right]$$
$$= \sum_{n=0}^{\infty} (-1)^n D_{20}^n \mathbb{E}\left[\left(\frac{D_{2t}}{D_{20}}\right)^n\right]$$
$$= \sum_{n=0}^{\infty} (-1)^n D_{20}^n e^{\mathbf{c}(0,n)t}$$

A suggestive special case

Substituting back, we find that

$$P_{1,0}/D_{1,0} = \frac{1}{\sqrt{s(1-s)}} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1-s}{s}\right)^{n+1/2}}{\rho - \mathbf{c}(0,n)},$$

where the state variable s is the *output share* of asset 1, which by assumption starts out greater than 0.5 and increases towards 1 over time

• In the general case, the analogous trick is to express the term inside the expectation as a Fourier integral, rather than as a geometric series

General case

Prices: The price-dividend ratio of asset 1 is

$$\frac{1}{\sqrt{s^{\widehat{\gamma}}(1-s)^{\widehat{\gamma}}}}\int_{-\infty}^{\infty}\frac{\mathscr{F}_{\widehat{\gamma}}(v)\left(\frac{1-s}{s}\right)^{iv}}{\rho-\mathbf{c}[\chi(1-\widehat{\gamma}/2-iv),\chi(-\widehat{\gamma}/2+iv)]}\,dv$$

where

$$s \equiv \frac{D_1}{D_1 + e \cdot D_2}$$

$$\widehat{\gamma} \equiv (\gamma \eta - 1)/(\eta - 1)$$

$$\chi \equiv 1 - 1/\eta$$

$$\mathscr{F}_{\widehat{\gamma}}(v) \equiv \frac{1}{2\pi} \cdot B(\widehat{\gamma}/2 + iv, \widehat{\gamma}/2 - iv) \quad (B \text{ is the beta function})$$

- Similar "integral formulas" for interest rates and expected returns
- Can be evaluated numerically—effectively instantly—or analytically in special cases

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Figure: Price-dividend ratio on asset 1, plotted against *s*, in the imperfect substitution case $\eta = 2$ (black) and the perfect substitution case (dashed red).

- As the country's share of global output declines, P/D increases
- This effect is muted in the imperfect substitution case

General case

Expected exchange rate appreciation:

$$\begin{array}{ll} \displaystyle \frac{\mathbb{E}de_t}{e_t} & = & \mathbf{c}(1-\chi,\chi-1)\,dt \\ \displaystyle \frac{\mathbb{E}d(1/e_t)}{1/e_t} & = & \mathbf{c}(\chi-1,1-\chi)\,dt \end{array}$$

- Average of these two is positive—Siegel (1972)
- In "own units", expected returns look like $\mathbb{E}dP/P + D/P dt$
- 1-unit return on an asset paying 2-goods: $\mathbb{E}d(eP)/eP + D/P dt$



Figure: Siegel's paradox.

General case

Interest rates: The zero-coupon yield to time T in country 1, $\mathscr{Y}_{T,1}(s)$, is

$$-\frac{1}{T}\log\left\{\frac{1}{\sqrt{s^{\widehat{\gamma}}(1-s)^{\widehat{\gamma}}}}\int_{-\infty}^{\infty}\mathscr{F}_{\widehat{\gamma}}(\nu)\left(\frac{1-s}{s}\right)^{i\nu}e^{-\left[\rho-\mathsf{c}\left[\chi(1-1/\chi-\widehat{\gamma}/2-i\nu),\chi(-\widehat{\gamma}/2+i\nu)\right]\right]T}\,d\nu\right\}$$

The instantaneous riskless rate in country 1 is

$$\frac{1}{\sqrt{s^{\widehat{\gamma}}(1-s)^{\widehat{\gamma}}}}\int_{-\infty}^{\infty}\mathscr{F}_{\widehat{\gamma}}(\mathsf{v})\left(\frac{1-s}{s}\right)^{i\mathsf{v}}\left[\rho-\mathsf{c}[\chi(1-1/\chi-\widehat{\gamma}/2-i\mathsf{v}),\chi(-\widehat{\gamma}/2+i\mathsf{v})]\right]\,d\mathsf{v}$$

The long rate is a constant, independent of the current state *s*:

$$\mathscr{Y}_{\infty,1} = \max_{\theta \in [0, \gamma - 1/\eta]} \rho - \mathbf{c}(\theta - \gamma, -\theta)$$

Each good has its own interest rate "Own-rates of interest" (Sraffa 1932)

[W]e need not stretch our imagination and think of an organised loan market amongst savages bartering deer for beavers. Loans are currently made in the present world in terms of every commodity for which there is a forward market. When a cotton spinner borrows a sum of money for three months and uses the proceeds to purchase spot, a quantity of raw cotton which he simultaneously sells three months forward, he is actually 'borrowing cotton' for that period. The rate of interest which he pays, per hundred bales of cotton, is the number of bales that can be purchased with the following sum of money: the interest on the money required to buy spot 100 bales, plus the excess (or minus the deficiency) of the spot over the forward prices of the 100 bales

Interest rates



Figure: The riskless rate (black solid), perpetuity yield (red dashed), and long rate (blue dotted) in 1-units plotted against *s*. $\rho = 0.04$, $\gamma = 4$, $\eta = 2$.

- Hump shape: precautionary savings effect—more technological diversification when *s* close to 0.5
- Tilt: Rates higher when country 1 is small than when it is large

Violation of UIP

- Small country has higher interest rate
- What about its currency? Uncovered interest parity?
- With symmetric fundamentals, as here, e_t and $1/e_t$ are each expected to appreciate (Siegel's paradox)
- Therefore uncovered interest parity fails: and not only does expected exchange rate movement not undo interest rate differentials, it actually adds to the risk premium

The fundamental asymmetry

Why the risk premium on small-country bonds?

- What you fear: bad news for big country
- When this happens, small good is in higher relative supply
- Relative price of small good declines
- Bond does badly in big-country units
- And hence is risky

 In the presence of jumps, the carry trade experiences occasional disastrous losses when big country has bad news, leading to small country currency devaluation

Excess returns



Figure: Left: Good-1 bond risk premia in 1-units (black solid) and in 2-units (blue dotted) and, for comparison, in the perfect substitutes case (red dashed). Right: Asset 1 risk premia in 1-units (black solid) and in 2-units (blue dotted) and in the perfect substitutes case (red dashed).

Violation of UIP

Result (Failure of UIP)

In the regression

$$\log e_{t+1} - \log e_t = a_0 + a_1(i_{1,t} - i_{2,t}) + \varepsilon_{t+1}$$

the model generates $plim(a_1) = 0$.

Proof

•
$$e_t \propto (D_{1t}/D_{2t})^{1/\eta}$$

- $\log e_t = \text{constant} + (\log D_{1t} \log D_{2t})/\eta$
- $(\log D_{1t}, \log D_{2t})$ is a Lévy process $\implies \log e_t$ is too
- So $\log e_{t+1} \log e_t$ is independent of time-t information
- As a result, $\operatorname{cov}(\log e_{t+1} \log e_t, i_{1,t} i_{2,t}) = 0$

• But var
$$(i_{1,t} - i_{2,t}) \neq 0$$

Violation of UIP



Figure: Forward price to time t of good 2 in 1-units ($F_{0\to t}$, black solid), expected future spot prices ($\mathbb{E} e_t = \mathbb{E} 1/e_t$, red dashed), and forward price of good 1 in 2-units ($1/F_{0\to t}$, blue dotted), plotted against t.

- Forward price of small- (large-) country output lies below (above) its expected future spot price
- In logs, this diagram would be symmetric

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The Forward Premium Puzzle

The small-country limit

- How general is this? Do results depend on parameter tuning?
- Look at the small-country limit in which country 1 is very small and country 2 is very large: D_{1t}/D_{2t} → 0
- Can get everything in closed form: P/D, interest rates, risk premia
- Lots of things to look at: risk premium on large country in own units and in foreign units; risk premium in small country in own units and in foreign units; risk premium on foreign bonds from each perspective
- Solution technique: analyze the integral by looking at residues in the complex plane. In asymmetric limit, only one residue matters

Solving the model in the small-country limit



Useful to think of the integral formulas as limits of path integrals

Solving the model in the small-country limit



By residue theorem, integral is $2\pi i \times \text{sum}$ of residues of integrand

- Residue of f(·) at a: coefficient on (z − a)⁻¹ in a series expansion of a function f(z) at a point a where f(a) = ∞
- For a very small country, only nearest residue to real axis matters

Result (Risk premia in the small-country limit.)

In foreign units, the good-i perpetuity earns the risk premium $XS^*_{B,i}$, where

$$\begin{array}{lll} XS^*_{B,1} & = & \mathbf{c}(\chi-1,1-\chi) + \mathbf{c}(0,-\gamma) - \mathbf{c}(\chi-1,1-\chi-\gamma) \\ XS^*_{B,2} & = & \mathbf{c}(1-\chi,\chi-1) + \mathbf{c}(\chi-1,1-\chi-\gamma) - \mathbf{c}(0,-\gamma). \end{array}$$

Excess returns on "equity", denominated in own units, XS_i, are given by

$$XS_1 = \mathbf{c}(1,0) + \mathbf{c}(\chi - 1, 1 - \chi - \gamma) - \mathbf{c}(\chi, 1 - \chi - \gamma)$$

$$XS_2 = \mathbf{c}(0,1) + \mathbf{c}(0,-\gamma) - \mathbf{c}(0,1-\gamma).$$

Excess returns on "equity", denominated in foreign units, XS_i^* , are given by

$$\begin{aligned} XS_1^* &= \mathbf{c}(\chi, 1-\chi) + \mathbf{c}(0, -\gamma) - \mathbf{c}(\chi, 1-\chi-\gamma) \\ XS_2^* &= \mathbf{c}(1-\chi, \chi) + \mathbf{c}(\chi-1, 1-\chi-\gamma) - \mathbf{c}(0, 1-\gamma). \end{aligned}$$

Result $(\mathbf{c}(\theta_1, \theta_2) = \mu \theta_1 + \mu \theta_2 + \frac{1}{2}\sigma^2 \theta_1^2 + \kappa \sigma^2 \theta_1 \theta_2 + \frac{1}{2}\sigma^2 \theta_2^2)$ In foreign units, the good-*i* perpetuity earns the risk premium $XS_{B,i}^*$, where

$$\begin{aligned} XS_{B,1}^* &= \gamma \sigma^2 (1-\kappa)(1-\chi) \\ XS_{B,2}^* &= -(\gamma+2\chi-2)\sigma^2 (1-\kappa)(1-\chi) \end{aligned}$$

Excess returns on "equity", denominated in own units, XS_i , are given by

$$XS_1 = \gamma \kappa \sigma^2 + \sigma^2 (1 - \kappa)(1 - \chi)$$

$$XS_2 = \gamma \sigma^2$$

Excess returns on "equity", denominated in foreign units, XS_i^* , are given by

$$\begin{split} &XS_1^* = \gamma \kappa \sigma^2 + \gamma \sigma^2 (1-\kappa)(1-\chi) \\ &XS_2^* = \gamma \sigma^2 - (\gamma + 2\chi - 1)\sigma^2 (1-\kappa)(1-\chi). \end{split}$$

Results in the limit

Result

Asset pricing in the big country in own units looks like closed-economy asset pricing: can pretend there is only one tree, $C_t = D_{2t}$, to find interest rates, P/D, risk premia

Nonparametric assumptions on dividend processes

Three properties the CGF may or may not possess

- Symmetry: Countries have same distribution of output growth $\rightarrow \mathbf{c}(\theta_1, \theta_2) = \mathbf{c}(\theta_2, \theta_1)$
- Convex difference: Restricts higher cumulants of output growth (holds in lognormal case, or with disasters) → c(θ₁, θ₂) c(θ₁ + t, θ₂ + t) is convex in (θ₁, θ₂) for all t > 0, θ₁, and θ₂
- Second Se

Why bother with this nonparametric approach? Because disasters are a double-edged sword: tweaking the tails has a strong effect, so important to check that results are not sensitive to particular calibrations

Linked fundamentals property

• The two countries have linked fundamentals if the CGF is *supermodular*

 $\begin{aligned} \mathbf{c}(\theta_1, \theta_2) + \mathbf{c}(\phi_1, \phi_2) &\leq \mathbf{c} \left(\max \left\{ \theta_1, \phi_1 \right\}, \max \left\{ \theta_2, \phi_2 \right\} \right) + \\ &+ \mathbf{c} \left(\min \left\{ \theta_1, \phi_1 \right\}, \min \left\{ \theta_2, \phi_2 \right\} \right) \end{aligned}$

• Topkis (1978): Sufficient condition is $\frac{\partial^2 \mathbf{c}(\theta_1, \theta_2)}{\partial \theta_1 \partial \theta_2} \ge 0$

$$\begin{array}{c|c} -0.5 \bigoplus & -0.2 \\ & -0.4 \\ & -0.6 \\ \oplus & -0.8 \\ & -1.0 \\ & -1.2 \end{array} \qquad \bigcirc \qquad 1.0$$

Results in the limit

Result (Failure of UIP)

Suppose Property 1 (Symmetry) holds.

- The small country has a higher interest rate than the large country
- The excess return on the small country's perpetuity is positive in large-country units
- UIP fails in the strong sense that the small country has a higher interest rate and its exchange rate is expected to appreciate

Proof: $R_{f,small} = \rho - \mathbf{c}(\chi - 1, 1 - \chi - \gamma)$, and $R_{f,big} = \rho - \mathbf{c}(0, -\gamma)$. So, for the first part, must show that

$$\mathbf{c}(0,-\gamma)-\mathbf{c}(\chi-1,1-\chi-\gamma)\stackrel{?}{>}0$$

$R_{f,\text{small}} > R_{f,\text{big}}$



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$R_{f,\text{small}} > R_{f,\text{big}}$



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$R_{f,\text{small}} > R_{f,\text{big}}$



Result (An exorbitant privilege)

Suppose Properties 1 (Symmetry) and 2 (Convex Difference) hold.

- Then UIP also fails for the large country, which has the "exorbitant privilege" of paying a negative risk premium on its bonds in small-country units
- Stronger than the previous result, because (Siegel again) expected exchange rate movements are positive

Results in the limit

Result ("Equity carry trades")

Suppose Properties 1, 2, and 3 hold. Then there is a critical value $\eta^* \in (1, \infty)$ —where $\eta^* = 2$ in the lognormal case—such that

$$\begin{aligned} 0 < XS_1 < XS_1^* < XS_2^* < XS_2 & \text{ if } \eta > \eta^* \\ 0 < XS_1 < XS_2^* < XS_1^* < XS_2 & \text{ if } \eta < \eta^*. \end{aligned}$$

We also have $XS_{B,1}^* \le XS_1^*$. If η is sufficiently large then we have a total ordering of risk premia: $XS_{B,2}^* < 0 < XS_{B,1}^* < XS_1 < XS_1^* < XS_2^* < XS_2$.

I don't know what the empirical evidence is here

Sample proof: $XS_1 > 0$...holds iff $\mathbf{c}(1,0) + \mathbf{c}(\chi - 1, 1 - \chi - \gamma) - \mathbf{c}(\chi, 1 - \chi - \gamma) > 0$







Conclusions

- Analyzed a model that lets both intertemporal (asset) prices and intratemporal (goods) prices move around
- UIP is violated even in this simple model
- If countries have same output growth distribution then the small country's bonds earn positive excess returns in large country units
- Jumps not needed for the mechanism
- Jumps are not what makes the model hard to solve
- Methodological contribution: nonparametric approach, without loglinearizing, or assuming log utility or Cobb-Douglas aggregator