Forecasting Crashes with a Smile

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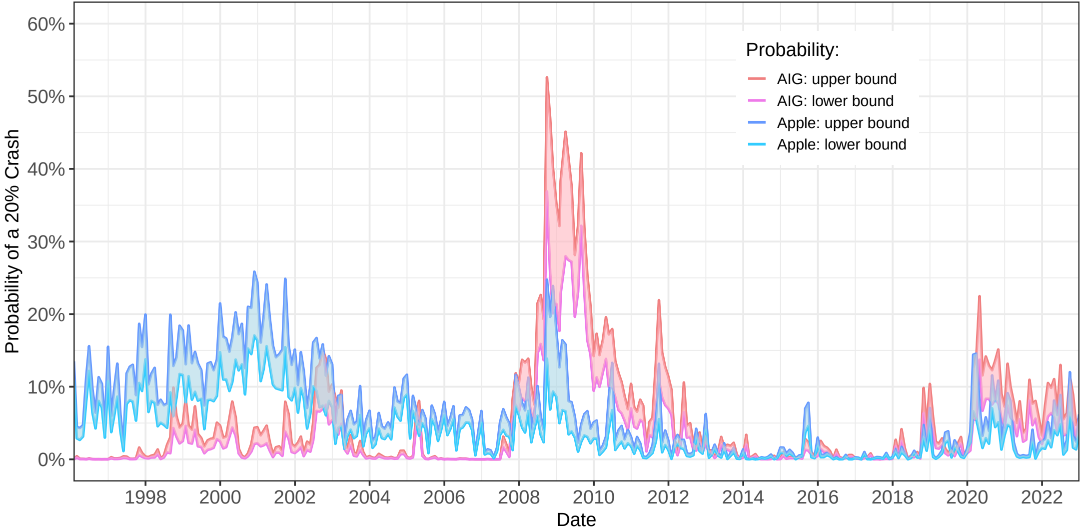
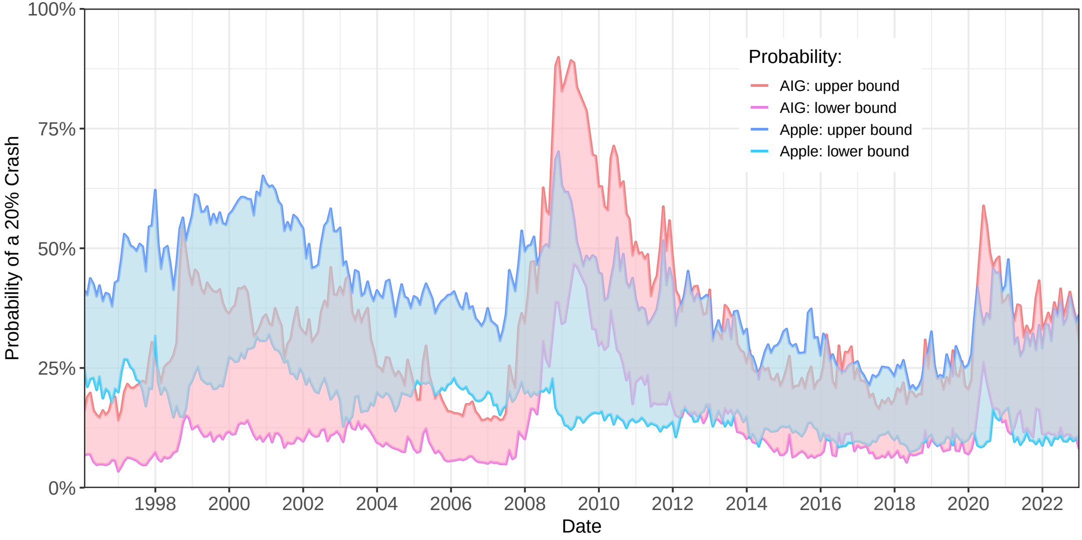
What is the chance that Apple stock drops 20% over the next month?

- We derive bounds on this quantity using index options and individual stock options
- No distributional assumptions
- The bounds are observable in real time
- We argue that the lower bound should be expected to be closer to the truth
- And show that it forecasts well in and out of sample

Probabilities of a 20% decline over the next month



Probabilities of a 20% decline over the next year

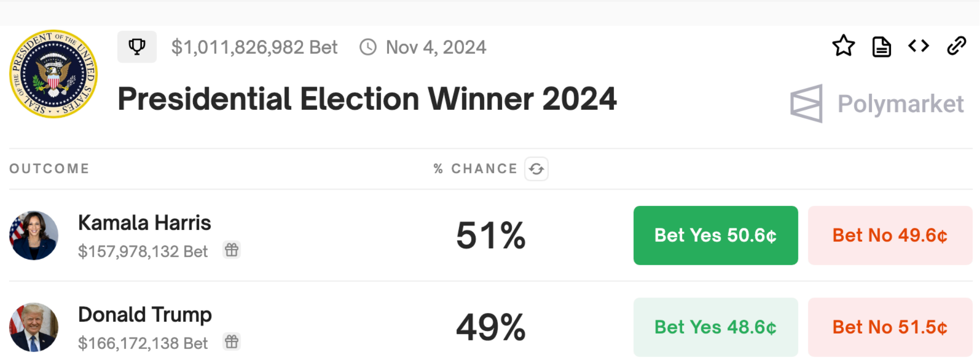


Today

1. Theory
2. Data
3. In-sample tests
4. Out-of-sample tests
5. Industry crash risk series
6. Explaining crash probabilities

Theory

We can infer risk-neutral probabilities directly from asset prices



We can infer risk-neutral probabilities directly from asset prices

- The risk-neutral probability that the market declines by 20% over the next month can be calculated from index options expiring in a month

$$\mathbb{P}^*[R \leq 0.8] = R_f \times \underbrace{\frac{1}{R_f}\mathbb{E}^*[\boldsymbol{I}(R \leq 0.8)]}_{\text{price of a binary option}} = R_f \times \underbrace{\text{put}'(0.8)}_{\text{slope of put prices}}$$

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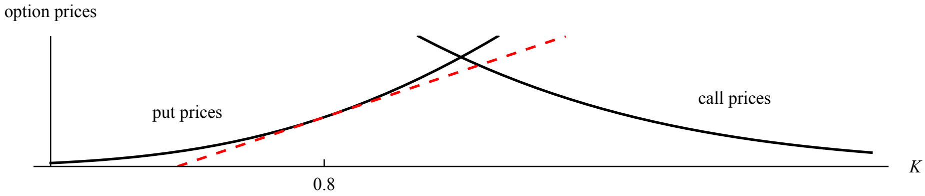

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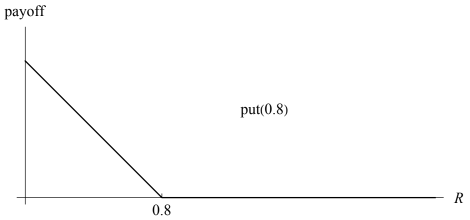

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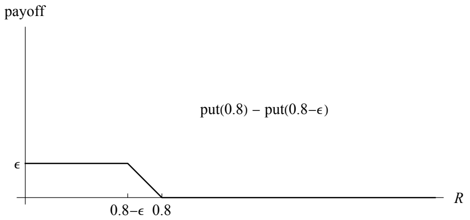

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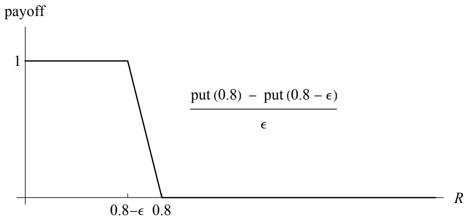

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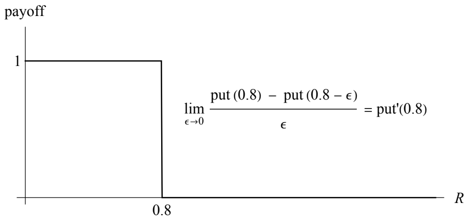


Strengths and weaknesses of risk-neutral probabilities

- Risk-neutral probabilities perform quite well in forecasting crashes
- But they overstate the probability of a crash
- And the extent to which they overstate varies
- They overstate most in scary times and for scary ($\approx$ high beta) stocks
- This is unfortunate! These are the situations, and stocks, for which a crash indicator is most useful

So we want true, not risk-neutral, probabilities

- We require an assumption (implicit or explicit) to link the true and risk-neutral probabilities—that is, about the stochastic discount factor
- We take the perspective of a one-period marginal investor with power utility who chooses to hold the market. So the SDF must be $M = R_m^{-\gamma}/\lambda$ for some constant λ
- The true expectation of a random payoff X then satisfies

$$\mathbb{E}[X] = \mathbb{E}[\underbrace{\lambda M R_m^{\gamma}}_{\equiv 1} X] = \lambda\, \mathbb{E}[M \times (R_m^{\gamma} X)] = \lambda \frac{\mathbb{E}^*[R_m^{\gamma} X]}{R_f}$$

- Eliminate λ by considering the case $X = 1$:

$$\mathbb{E}[X] = \frac{\mathbb{E}^*[R_m^{\gamma} X]}{\mathbb{E}^*[R_m^{\gamma}]}$$

So we want true, not risk-neutral, probabilities

- In the case $\gamma = 0$, our approach simply forecasts using risk-neutral probabilities
- Bad news: Our hypothetical investor understands market risk, but does not "know" about various anomalies demonstrated in the empirical finance literature
  - …, momentum, value, profitability, …
- Good news: We don't need to make the standard, undesirable, assumption that historical measures are good proxies for the forward-looking risk measures that come out of theory

Theory (1)

- Setting $X = \boldsymbol{I}(R_i \leq q)$, this implies that the crash probability of stock i is

$$\mathbb{P}[R_i \leq q] = \frac{\mathbb{E}^*\left[R_m^\gamma \boldsymbol{I}(R_i \leq q)\right]}{\mathbb{E}^*\left[R_m^\gamma\right]}$$

- To calculate $\mathbb{E}^*\left[R_m^\gamma\right]$, we need marginal risk-neutral distribution of R_m
  - Easy, using index option prices (Breeden and Litzenberger, 1978)
- To calculate $\mathbb{E}^*\left[R_m^\gamma \boldsymbol{I}(R_i \leq q)\right]$, we need the joint distribution of (R_m, R_i)
  - Problem: Joint risk-neutral distribution is not observable given assets that are traded in practice (Martin, 2018, "Options and the Gamma Knife")
  - This is a general theme: we are often interested in covariances and other features of the joint distribution in asset pricing
  - The case $i = m$ is easy. But *testing* the theory is hard because crashes are rare

A 2×2 example

- Suppose the risk-neutral probability of a crash in Apple is 5%
- Suppose the risk-neutral probability of a crash in the market is also 5%
- These numbers can be calculated from options on Apple and options on the market
- But they are consistent with different joint distributions, eg,

		Apple	
		Crash	No crash
S&P 500	Crash	5%	0%
	No crash	0%	95%

		Apple	
		Crash	No crash
S&P 500	Crash	0%	5%
	No crash	5%	90%

A 2×2 example

		Apple: Crash	Apple: No crash
S&P 500	Crash	5%	0%
	No crash	0%	95%

		Apple: Crash	Apple: No crash
S&P 500	Crash	0%	5%
	No crash	5%	90%

- In the left-hand world, AAPL is risky
  - Risk-neutral probability of a crash will overstate the true probability of a crash
- In the right-hand world, AAPL is a hedge
  - Risk-neutral probability will understate the true probability of a crash
- Moral: Even if we can't observe the joint distribution, we may be able to derive bounds on the true crash probability

Theory (2)

$$\mathbb{P}[R_i \leq q] = \frac{\mathbb{E}^*\left[R_m^\gamma \boldsymbol{I}(R_i \leq q)\right]}{\mathbb{E}^*\left[R_m^\gamma\right]}$$

- We do not observe the joint risk-neutral distribution, so cannot calculate the right-hand side
- But we do observe the individual (marginal) risk-neutral distributions of R_m and R_i, from options on the market and on stock i
- The Fréchet–Hoeffding theorem provides upper and lower bounds on the right-hand side, as in the 2×2 example

Theory (3)

Result (Bounds on the probability of a crash)

We have

$$\frac{\mathbb{E}^*\left[R_m^\gamma \boldsymbol{I}(R_m \leq q_l)\right]}{\mathbb{E}^*\left[R_m^\gamma\right]} \leq \mathbb{P}[R_i \leq q] \leq \frac{\mathbb{E}^*\left[R_m^\gamma \boldsymbol{I}(R_m \geq q_u)\right]}{\mathbb{E}^*\left[R_m^\gamma\right]}$$

- The three elements are

$$\mathbb{E}^*\left[R_m^\gamma\right] = R_f^\gamma + \gamma(\gamma-1)R_f\left[\int_0^{R_f} K^{\gamma-2}\text{put}_m(K)\,\mathrm{d}K + \int_{R_f}^{\infty} K^{\gamma-2}\text{call}_m(K)\,\mathrm{d}K\right]$$

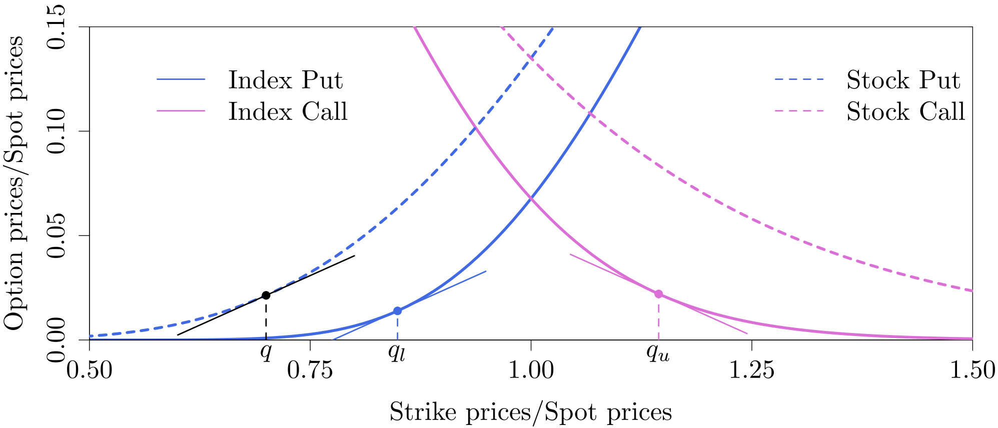
$$\mathbb{E}^*\left[R_m^\gamma \boldsymbol{I}\left(R_m \leq q_l\right)\right] = R_f q_l^\gamma\left[\text{put}_m'(q_l) - \gamma\frac{\text{put}_m\left(q_l\right)}{q_l}\right] + \gamma(\gamma-1)R_f\int_0^{q_l} K^{\gamma-2}\text{put}_m(K)\,\mathrm{d}K$$

$$\mathbb{E}^*\left[R_m^\gamma \boldsymbol{I}\left(R_m \geq q_u\right)\right] = R_f q_u^\gamma\left[\gamma\frac{\text{call}_m(q_u)}{q_u} - \text{call}'(q_u)\right] + \gamma(\gamma-1)R_f\int_{q_u}^{\infty} K^{\gamma-2}\text{call}_m(K)\,\mathrm{d}K$$

Theory (4)

- The stock-i-specific quantiles q_l and q_u are such that

$$\mathbb{P}^*[R_m \leq q_l] = \mathbb{P}^*[R_i \leq q] = \mathbb{P}^*[R_m \geq q_u]$$



Theory (5)

- Bounds from the Fréchet–Hoeffding theorem are attainable in principle
  - Lower bound achieved for a stock that is comonotonic with the market—i.e., whose return is a (potentially nonlinear) increasing function of the market return
  - Upper bound achieved for a stock that is countermonotonic with the market—i.e., whose return is a (potentially nonlinear) decreasing function of the market return
- Intuitively, asset prices will tend to overstate crash probabilities if crashes are scary; or understate crash probabilities if crashes occur in good times
- A priori, we expect that the scary case is the relevant one, and hence that the lower bound should be closer to the truth in practice

Theory (6)

Result (Bounds on the probability of a crash)

We have

$$\frac{\mathbb{E}^*\left[R_m^\gamma \boldsymbol{I}(R_m \leq q_l)\right]}{\mathbb{E}^*\left[R_m^\gamma\right]} \leq \mathbb{P}[R_i \leq q] \leq \frac{\mathbb{E}^*\left[R_m^\gamma \boldsymbol{I}(R_m \geq q_u)\right]}{\mathbb{E}^*\left[R_m^\gamma\right]}$$

Further theoretical results

- Both $\mathbb{P}[R_i \leq q]$ and $\mathbb{P}^*[R_i \leq q]$ lie in between the bounds
- When $\gamma = 0$, the lower and upper bounds both equal $\mathbb{P}^*[R_i \leq q]$, and $\mathbb{P}^*$ and $\mathbb{P}$ coincide
- As γ increases, the bounds widen monotonically, so higher γ is more conservative
- As $\gamma \to \infty$, the bounds become trivial: the lower bound tends to zero and the upper bound tends to one

Data

Data

- S&P 500 index and stock constituents from **Compustat**
- Risk-free rates and implied volatilities from **OptionMetrics**
  - Monthly from 1996/01 to 2022/12
  - On average around 492 firms each month
  - Options maturing in $1, 3, 6$ and 12 months
  - Over $155,000$ firm-month observations per maturity
- Firm characteristics from **Compustat**
- Price, return, and volume data from **CRSP**
- Focus on “crashes” of $10\%, 20\%$ and 30% at horizons $\tau = 1, 3, 6$ and 12 months
- I'll often focus on the case of a 20% decline over one month
- We set risk aversion, γ, equal to 2

Summary statistics

		averaged across firms (number of obs. $T = 324$)				averaged across time (number of obs. $N = 1044$)			
	maturity	1	3	6	12	1	3	6	12
$q = 0.7$, down by over 30%									
realized	mean	0.006	0.029	0.057	0.093	0.009	0.038	0.073	0.115
	s.d.	0.019	0.064	0.100	0.120	0.025	0.067	0.103	0.147
lower bound	mean	0.004	0.025	0.051	0.076	0.006	0.030	0.056	0.082
	s.d.	0.007	0.019	0.023	0.023	0.013	0.032	0.042	0.049
risk-neutral	mean	0.007	0.044	0.098	0.167	0.009	0.050	0.104	0.173
	s.d.	0.012	0.037	0.050	0.056	0.017	0.045	0.061	0.071
upper bound	mean	0.009	0.060	0.139	0.253	0.011	0.066	0.146	0.259
	s.d.	0.016	0.053	0.077	0.094	0.020	0.056	0.078	0.093

Summary statistics

		averaged across firms (number of obs. $T = 324$)				averaged across time (number of obs. $N = 1044$)			
	maturity	1	3	6	12	1	3	6	12
$q = 0.8$, down by over 20%									
realized	mean	0.021	0.069	0.110	0.152	0.029	0.084	0.130	0.173
	s.d.	0.048	0.107	0.140	0.158	0.059	0.092	0.129	0.165
lower bound	mean	0.022	0.073	0.102	0.123	0.027	0.079	0.110	0.133
	s.d.	0.020	0.028	0.027	0.027	0.029	0.046	0.052	0.056
risk-neutral	mean	0.031	0.113	0.174	0.236	0.037	0.120	0.182	0.246
	s.d.	0.031	0.050	0.053	0.058	0.036	0.058	0.065	0.072
upper bound	mean	0.038	0.144	0.234	0.340	0.044	0.152	0.243	0.352
	s.d.	0.040	0.071	0.082	0.097	0.042	0.069	0.079	0.089

Summary statistics

		averaged across firms (number of obs. $T = 324$)				averaged across time (number of obs. $N = 1044$)			
	maturity	1	3	6	12	1	3	6	12
$q = 0.9$, down by over 10%									
realized	mean	0.096	0.172	0.211	0.238	0.110	0.190	0.231	0.254
	s.d.	0.123	0.170	0.184	0.193	0.089	0.119	0.152	0.182
lower bound	mean	0.109	0.168	0.195	0.209	0.118	0.179	0.206	0.218
	s.d.	0.036	0.031	0.027	0.023	0.050	0.055	0.056	0.056
risk-neutral	mean	0.136	0.228	0.286	0.341	0.145	0.239	0.297	0.350
	s.d.	0.050	0.051	0.051	0.049	0.056	0.061	0.063	0.063
upper bound	mean	0.156	0.277	0.367	0.466	0.166	0.290	0.378	0.476
	s.d.	0.064	0.074	0.080	0.085	0.062	0.070	0.073	0.073

In-sample tests

Empirical tests

- $\boldsymbol{I}(R_i \leq q) = 0 + 1 \times \underbrace{\mathbb{E}[\, \boldsymbol{I}(R_i \leq q) \,]}_{\mathbb{P}[R_i \leq q]} + \varepsilon$
- So a regression of the realized crash indicator $\boldsymbol{I}(R_i \leq q)$ onto an ideal crash probability measure $\mathbb{P}[R_i \leq q]$ would yield zero constant term and a unit regression coefficient
- If the lower bound is close to the truth, then in a regression

$$\boldsymbol{I}[R_{i,t \to t+\tau} \leq q] = \alpha^L + \beta^L \, \mathbb{P}^L_{i,t}(\tau, q) + \varepsilon_{i,t+\tau},$$

we should find $\alpha^L \approx 0$ and $\beta^L \approx 1$ at any horizon τ and for any crash size q

In-sample tests (1)

Down by 30% ($q = 0.7$)

	lower bound				risk-neutral				upper bound			
maturity	1	3	6	12	1	3	6	12	1	3	6	12
α	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01
	(0.00)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)	(0.01)	(0.01)
	[0.00]	[0.00]	[0.01]	[0.01]	[0.00]	[0.00]	[0.01]	[0.01]	[0.00]	[0.00]	[0.01]	[0.01]
β	0.95	1.03	1.09	1.05	0.66	0.60	0.59	0.56	0.51	0.43	0.39	0.35
	(0.15)	(0.12)	(0.11)	(0.10)	(0.11)	(0.08)	(0.07)	(0.07)	(0.09)	(0.06)	(0.05)	(0.05)
	[0.16]	[0.14]	[0.18]	[0.15]	[0.11]	[0.11]	[0.11]	[0.11]	[0.10]	[0.09]	[0.08]	[0.07]
R^2	3.90%	5.37%	5.17%	3.91%	3.77%	4.56%	4.01%	3.06%	3.63%	4.16%	3.41%	2.47%

In-sample tests (1)

with time fixed effects

Down by 30% ($q = 0.7$)

	lower bound				risk-neutral				upper bound			
maturity	1	3	6	12	1	3	6	12	1	3	6	12
β	0.93	1.05	1.11	1.14	0.68	0.70	0.74	0.78	0.55	0.55	0.58	0.60
	(0.14)	(0.10)	(0.08)	(0.08)	(0.10)	(0.07)	(0.05)	(0.05)	(0.09)	(0.05)	(0.04)	(0.04)
	[0.16]	[0.13]	[0.12]	[0.11]	[0.13]	[0.09]	[0.10]	[0.07]	[0.09]	[0.07]	[0.06]	[0.06]
R^2-proj	3.27%	4.81%	5.06%	4.54%	3.21%	4.52%	4.87%	4.50%	3.16%	4.39%	4.74%	4.43%

In-sample tests (2)

Down by 20% ($q = 0.8$)

	lower bound				risk-neutral				upper bound			
maturity	1	3	6	12	1	3	6	12	1	3	6	12
α	0.00	−0.01	−0.01	0.02	0.00	−0.01	−0.02	0.00	0.00	−0.01	−0.01	0.01
	(0.00)	(0.01)	(0.01)	(0.01)	(0.00)	(0.01)	(0.01)	(0.01)	(0.00)	(0.01)	(0.01)	(0.02)
	[0.00]	[0.01]	[0.01]	[0.02]	[0.00]	[0.01]	[0.01]	[0.02]	[0.00]	[0.01]	[0.02]	[0.03]
β	0.92	1.03	1.15	1.07	0.68	0.69	0.73	0.66	0.56	0.51	0.49	0.41
	(0.11)	(0.09)	(0.09)	(0.08)	(0.09)	(0.07)	(0.07)	(0.07)	(0.08)	(0.06)	(0.06)	(0.06)
	[0.11]	[0.13]	[0.15]	[0.13]	[0.09]	[0.10]	[0.11]	[0.12]	[0.07]	[0.08]	[0.10]	[0.10]
R^2	5.65%	5.15%	4.76%	3.69%	5.48%	4.50%	3.89%	2.96%	5.32%	4.11%	3.22%	2.30%

In-sample tests (2)

with time fixed effects

Down by 20% ($q = 0.8$)

	lower bound				risk-neutral				upper bound			
maturity	1	3	6	12	1	3	6	12	1	3	6	12
β	0.93	1.03	1.13	1.10	0.73	0.80	0.89	0.87	0.62	0.67	0.74	0.71
	(0.09)	(0.07)	(0.06)	(0.06)	(0.07)	(0.05)	(0.05)	(0.05)	(0.06)	(0.04)	(0.04)	(0.04)
	[0.10]	[0.10]	[0.09]	[0.09]	[0.07]	[0.07]	[0.07]	[0.06]	[0.07]	[0.07]	[0.07]	[0.06]
R^2-proj	4.49%	4.65%	4.55%	4.01%	4.39%	4.53%	4.48%	4.00%	4.33%	4.45%	4.40%	3.98%

Intermission: Probability of a rise of at least 20%

	lower bound				risk-neutral				upper bound			
maturity	1	3	6	12	1	3	6	12	1	3	6	12
α	0.00	0.01	0.09	0.34	0.00	0.00	0.04	0.24	0.00	−0.01	0.03	0.21
	(0.00)	(0.00)	(0.01)	(0.02)	(0.00)	(0.01)	(0.01)	(0.03)	(0.00)	(0.01)	(0.01)	(0.03)
	[0.00]	[0.01]	[0.01]	[0.03]	[0.00]	[0.01]	[0.02]	[0.04]	[0.00]	[0.01]	[0.02]	[0.04]
β	1.35	1.58	1.32	0.12	1.03	1.17	1.08	0.46	0.85	0.91	0.82	0.42
	(0.13)	(0.11)	(0.11)	(0.15)	(0.10)	(0.09)	(0.09)	(0.12)	(0.09)	(0.08)	(0.07)	(0.09)
	[0.13]	[0.14]	[0.15]	[0.21]	[0.11]	[0.13]	[0.15]	[0.17]	[0.09]	[0.10]	[0.11]	[0.13]
R^2	6.95%	5.78%	2.51%	0.01%	7.28%	6.66%	3.79%	0.38%	7.36%	6.81%	4.21%	0.72%

- For rises, the upper bound would be tight in the comonotonic case
- At the one year horizon, it is harder to predict booms than crashes (perhaps because booms are more idiosyncratic so comonotonicity is further from the truth)

In-sample tests (3)

Down by 10% ($q = 0.9$)

	lower bound				risk-neutral				upper bound			
maturity	1	3	6	12	1	3	6	12	1	3	6	12
α	−0.02	−0.01	−0.01	0.03	−0.02	−0.02	−0.02	0.00	−0.02	0.00	0.01	0.05
	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.02)	(0.02)	(0.03)	(0.01)	(0.02)	(0.02)	(0.03)
	[0.01]	[0.02]	[0.02]	[0.03]	[0.01]	[0.02]	[0.03]	[0.04]	[0.01]	[0.03]	[0.04]	[0.05]
β	1.05	1.07	1.12	1.01	0.88	0.83	0.80	0.68	0.75	0.63	0.54	0.41
	(0.08)	(0.07)	(0.07)	(0.08)	(0.08)	(0.08)	(0.08)	(0.09)	(0.07)	(0.07)	(0.07)	(0.08)
	[0.08]	[0.11]	[0.12]	[0.12]	[0.07]	[0.11]	[0.12]	[0.13]	[0.08]	[0.12]	[0.12]	[0.11]
R^2	5.46%	3.71%	3.38%	2.41%	5.46%	3.39%	2.80%	1.83%	5.35%	3.03%	2.16%	1.23%

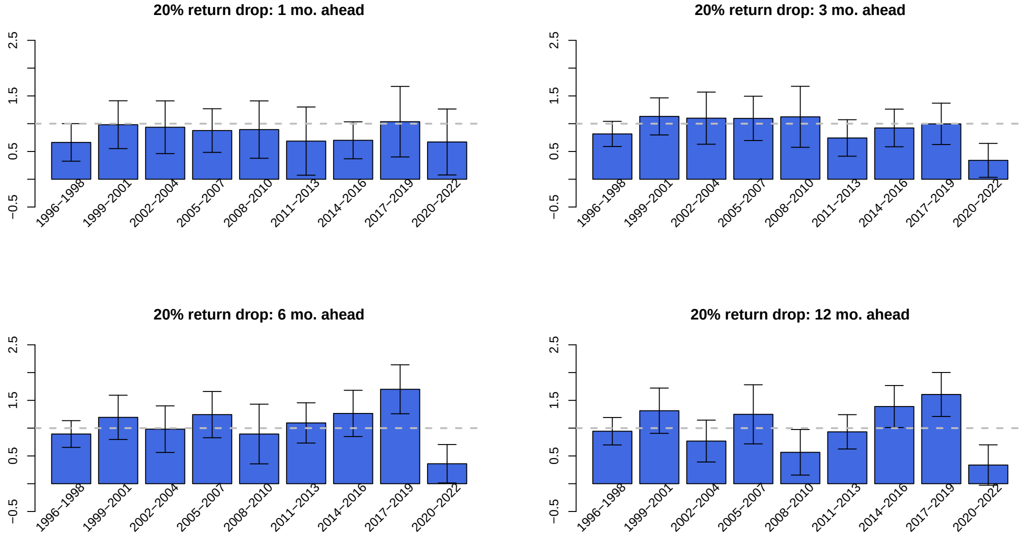
In-sample tests (3)

with time fixed effects

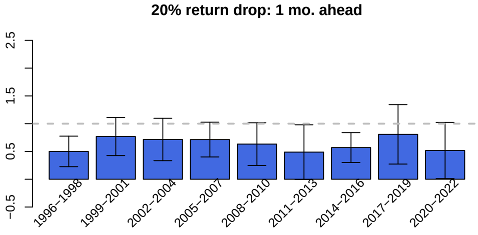
Down by 10% ($q = 0.9$)

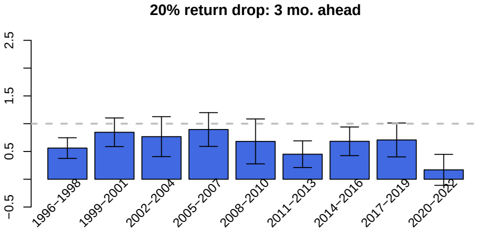
	lower bound				risk-neutral				upper bound			
maturity	1	3	6	12	1	3	6	12	1	3	6	12
β	0.99	0.99	1.05	1.05	0.88	0.89	0.94	0.93	0.80	0.79	0.83	0.82
	(0.06)	(0.05)	(0.06)	(0.06)	(0.05)	(0.05)	(0.05)	(0.05)	(0.05)	(0.04)	(0.04)	(0.05)
	[0.06]	[0.07]	[0.08]	[0.08]	[0.05]	[0.07]	[0.07]	[0.08]	[0.05]	[0.06]	[0.06]	[0.06]
R^2-proj	4.02%	3.15%	3.14%	2.85%	3.99%	3.12%	3.12%	2.83%	3.96%	3.08%	3.09%	2.82%

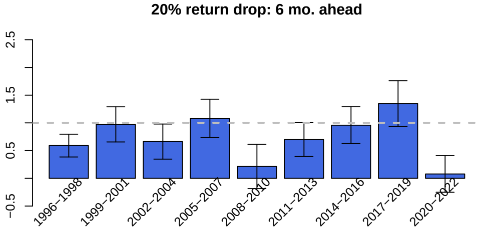
Estimated β, by year: lower bound

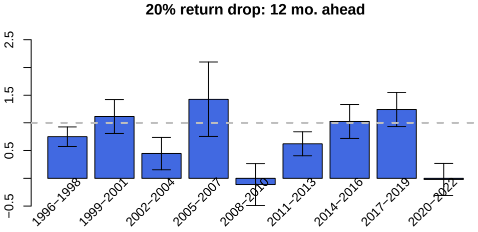


Estimated β, by year: risk-neutral probabilities









Lower bound vs. risk-neutral probabilities

- Risk-neutral probabilities overstate true crash probabilities
- The extent to which they overstate varies over time and across stocks
- We should expect risk-neutral probabilities to overstate most—hence estimated β coefficients to be lowest—in scary times or for scary ($\approx$ high beta) stocks
- The lower bound adjusts for scariness, so estimated β coefficients are more stable
- This gives the lower bound an advantage when we look at OOS performance

Fréchet–Hoeffding vs. Cauchy–Schwarz

- Here's another approach that does not work as well. Write

$$\mathbb{P}\left[R_i \leq q\right] = \mathbb{P}^*\left[R_i \leq q\right] + \frac{\text{cov}^*\left[R_m^\gamma, \boldsymbol{I}(R_i \leq q)\right]}{\mathbb{E}^*\left[R_m^\gamma\right]}$$

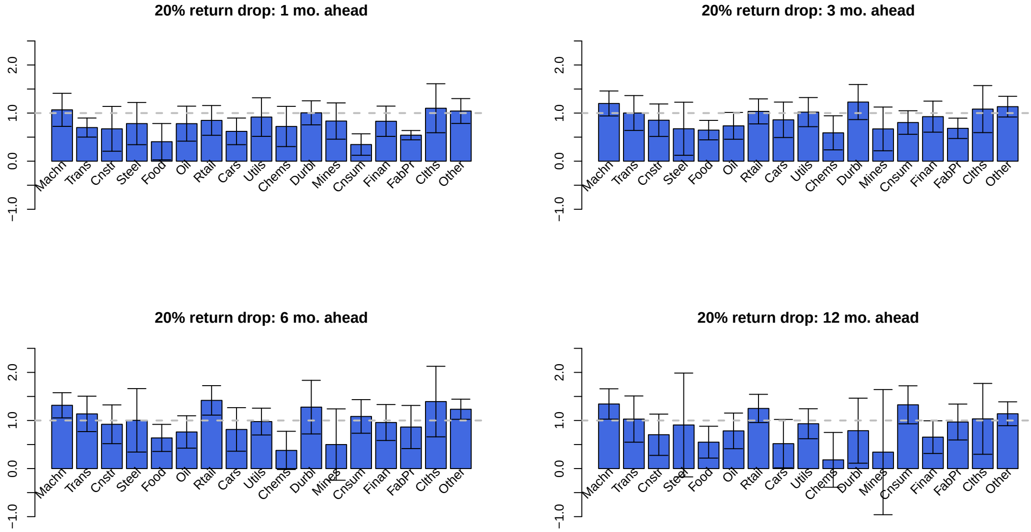
- Risk-neutral correlation, $\rho^*\left[R_m^\gamma, \boldsymbol{I}(R_i \leq q)\right]$, must lie between plus and minus one, so

$$\mathbb{P}^*\left[R_i \leq q\right] - \frac{\sigma^*\left[R_m^\gamma\right]\sigma^*\left[\boldsymbol{I}(R_i \leq q)\right]}{\mathbb{E}^*\left[R_m^\gamma\right]} \leq \mathbb{P}\left[R_i \leq q\right] \leq \mathbb{P}^*\left[R_i \leq q\right] + \frac{\sigma^*\left[R_m^\gamma\right]\sigma^*\left[\boldsymbol{I}(R_i \leq q)\right]}{\mathbb{E}^*\left[R_m^\gamma\right]}$$

  where $\sigma^*\left[\cdot\right]$ denotes risk-neutral volatility

- But $\rho^*\left[R_m^\gamma, \boldsymbol{I}(R_i \leq q)\right]$ cannot reach plus or minus one: one variable is continuous, the other discrete!
- So these bounds are very weak: typically around three times wider for the 1 month/20% pair and around 10 times wider for the 1 month/30% pair

Estimated β, by industry: lower bound



Competitor variables from the literature

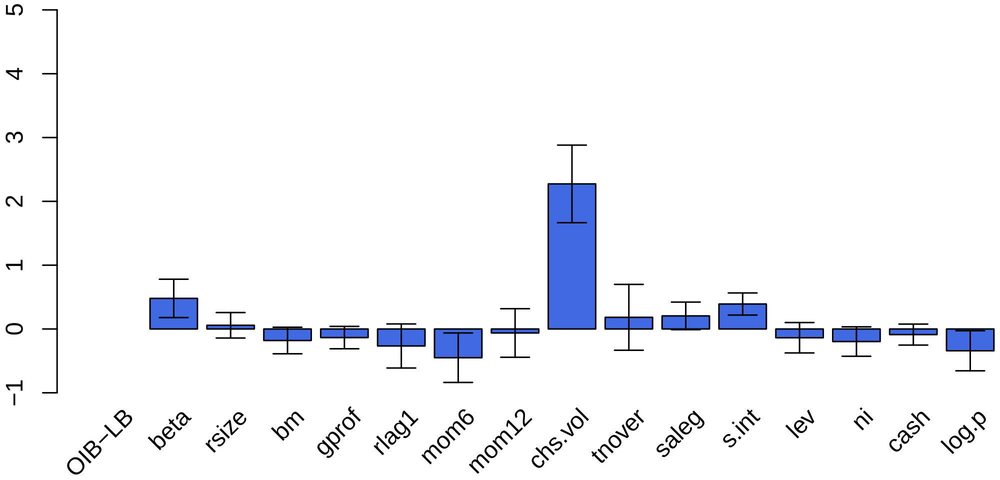
- We compare against 15 variables drawn from the literature
  - Stock characteristics: CAPM beta, (log) relative size, book-to-market, profitability (gross profit/assets), momentum (prior 2-6 and 2-12 month returns), lagged return
  - Chen–Hong–Stein, 2001: realized volatilities (standard deviations of daily market-adjusted returns over the last six months) and monthly turnover (shares traded scaled by shares outstanding)
  - Greenwood–Shleifer–You, 2019: sales growth
  - Asquith–Pathak–Ritter, 2005; Nagel, 2005: short interest (shares shorted/shares held by institutions)
  - Campbell–Hilscher–Szilagyi, 2008: leverage (debt/total assets), net income/total assets, cash/total assets, log price per share (winsorized from above at $15)
- All variables are standardized to unit standard deviation for comparability

In-sample tests (4)

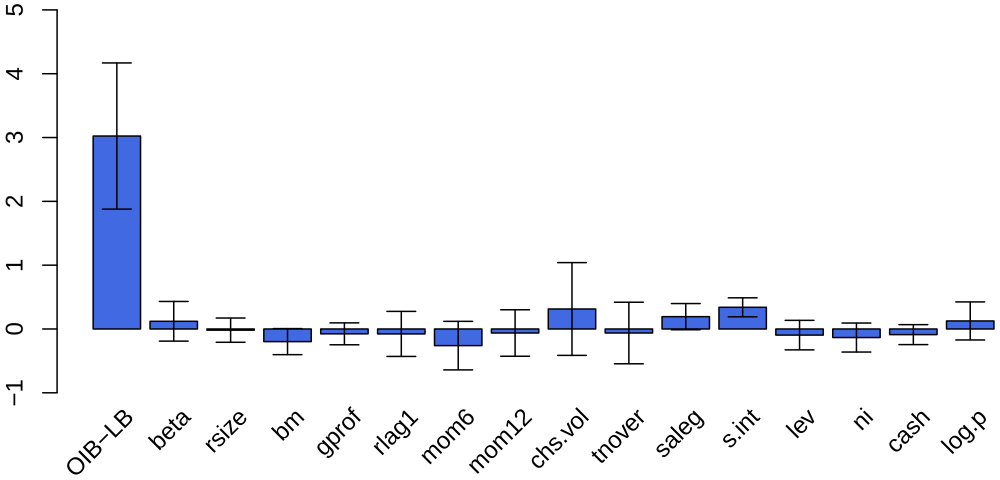
Asterisks indicate t-statistics above 4

	$\boldsymbol{I}(R_{t\to t+1} \leq 0.8)$					
$\mathbb{P}^L[R_{t\to t+1} \leq 0.8]$		3.40^*	3.02^*		4.41	2.72^*
		(0.41)	(0.58)		(3.08)	(0.33)
$\mathbb{P}^*[R_{t\to t+1} \leq 0.8]$				2.81^*	-1.39	
				(0.66)	(3.36)	
CHS-volatility	2.27^*		0.31	0.44	0.32	0.50
	(0.31)		(0.37)	(0.44)	(0.39)	(0.18)
short int.	0.39^*		0.34^*	0.37^*	0.33^*	0.27^*
	(0.09)		(0.08)	(0.08)	(0.08)	(0.06)
	⋮		⋮	⋮	⋮	⋮
R^2/R^2-proj.	4.49%	5.65%	5.82%	5.69%	5.83%	4.72%

In-sample tests (4)



In-sample tests (4)



Out-of-sample tests

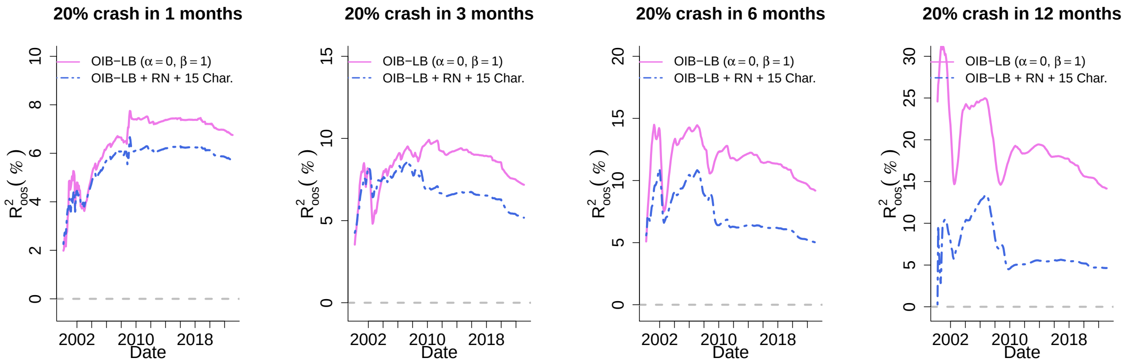
We compare OOS forecast performance of two models

1. Competitor model uses 15 characteristics, lower bound, and risk-neutral probabilities
   - We train predictive models using expanding or rolling windows
     - variable selection using elastic net
     - tuning parameters for sparsity: 5-fold cross validation based on the training sample
   - Then make out-of-sample forecasts for the rest of the sample
2. Our lower bound, directly used to forecast with fixed $\alpha = 0$ and $\beta = 1$
   - Nothing is estimated

- Performance measure: out-of-sample R^2
- Diebold–Mariano tests reject the null of equal forecasting accuracy
- Very similar results for a "kitchen sink" competitor that also uses interactions and squares of the 15 original characteristics (for a total of 137 variables)

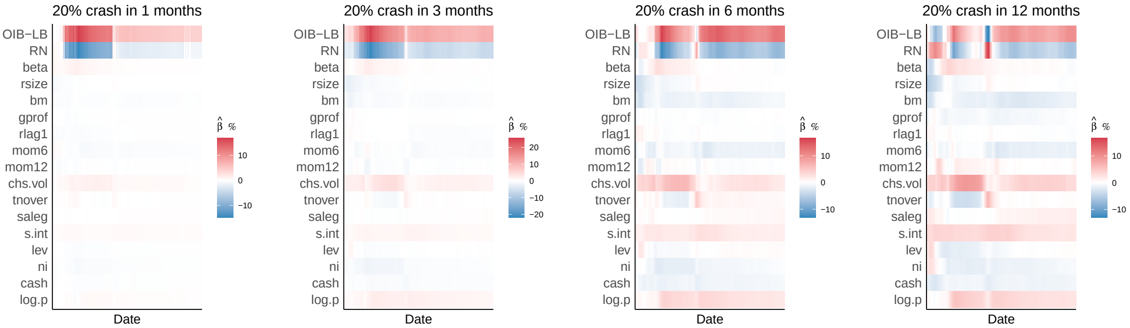
Out-of-sample forecasts

R^2, expanding window, competing against in-sample mean crash probabilities (firm-specific)



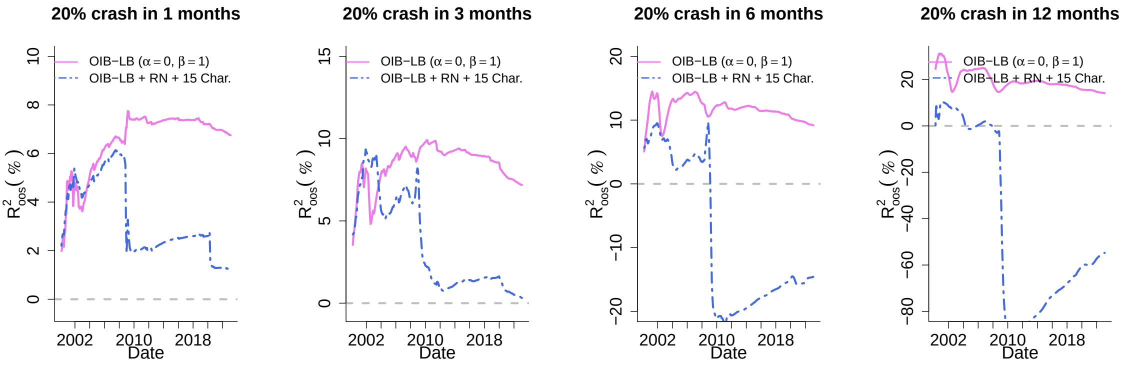
Out-of-sample forecasts

$\hat{\beta}$, expanding window, competing against in-sample mean crash probabilities (firm-specific)



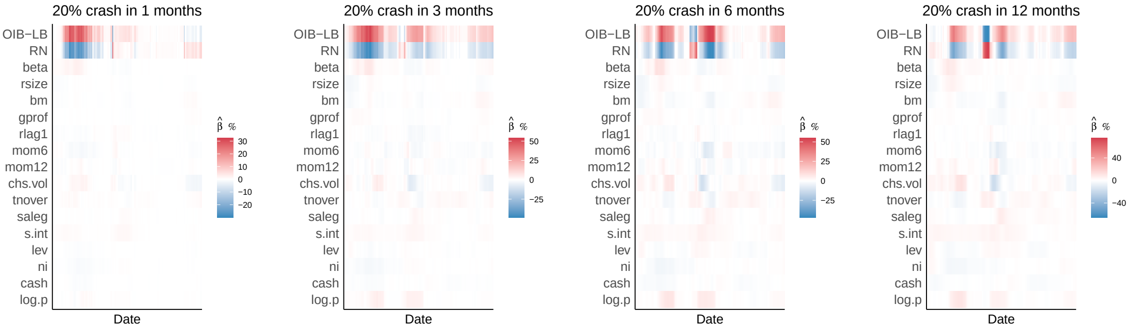
Out-of-sample forecasts

R^2, 3yr rolling window, competing against in-sample mean crash probabilities (firm-specific)



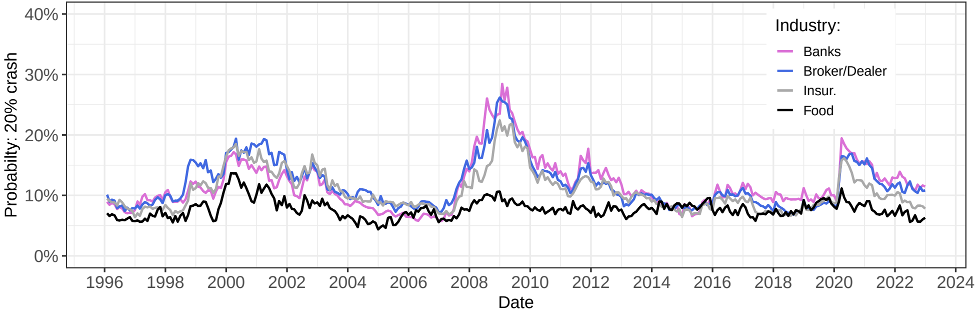
Out-of-sample forecasts

$\hat{\beta}$, 3yr rolling window, competing against in-sample mean crash probabilities (firm-specific)



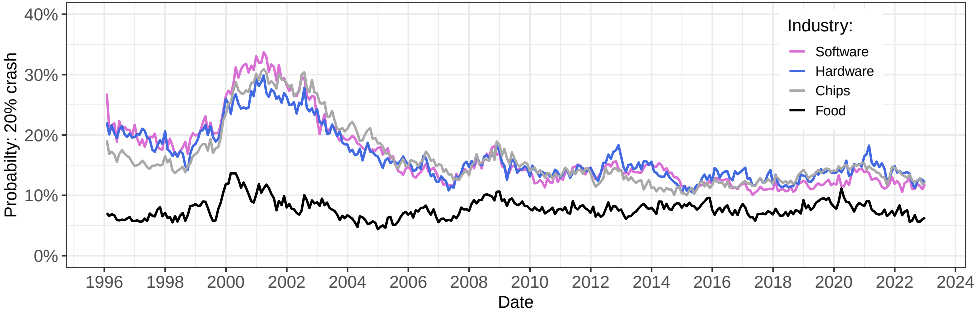
Industry crash risk

Industry average crash probabilities



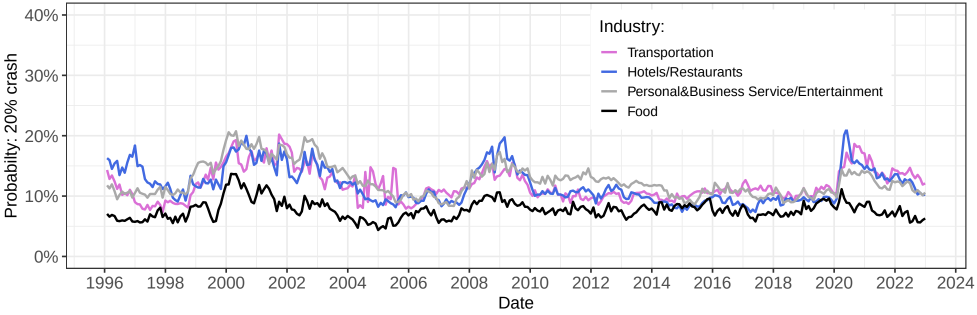
- Substantial variation in crash probability over time and across industries
- News about crash risk is not just idiosyncratic: related industries' probabilities comove

Industry average crash probabilities



- Substantial variation in crash probability over time and across industries
- News about crash risk is not just idiosyncratic: related industries' probabilities comove

Industry average crash probabilities



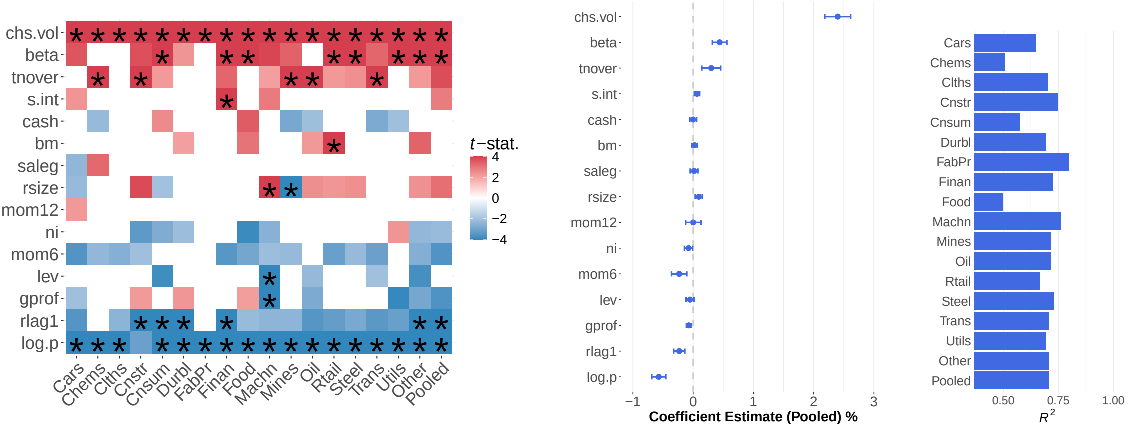
- Substantial variation in crash probability over time and across industries
- News about crash risk is not just idiosyncratic: related industries' probabilities comove

Explaining crash probabilities

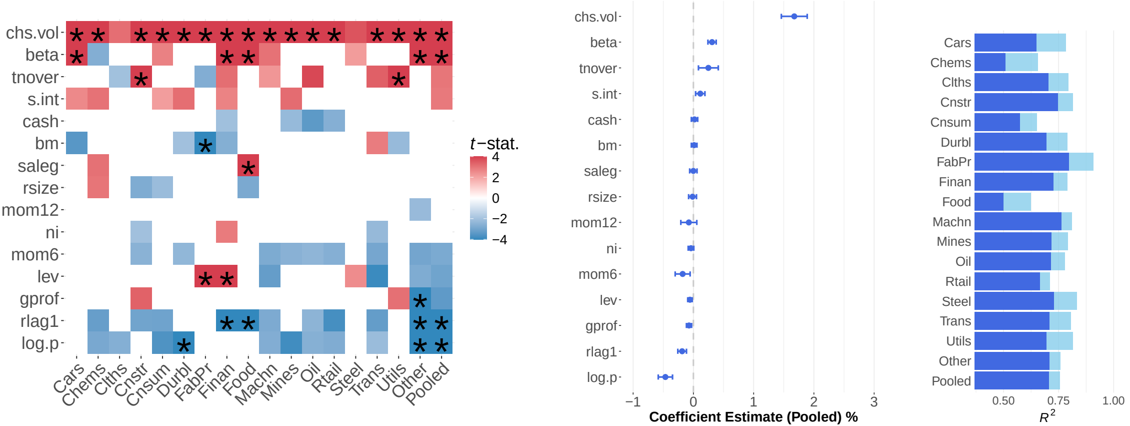
Explaining crash probabilities

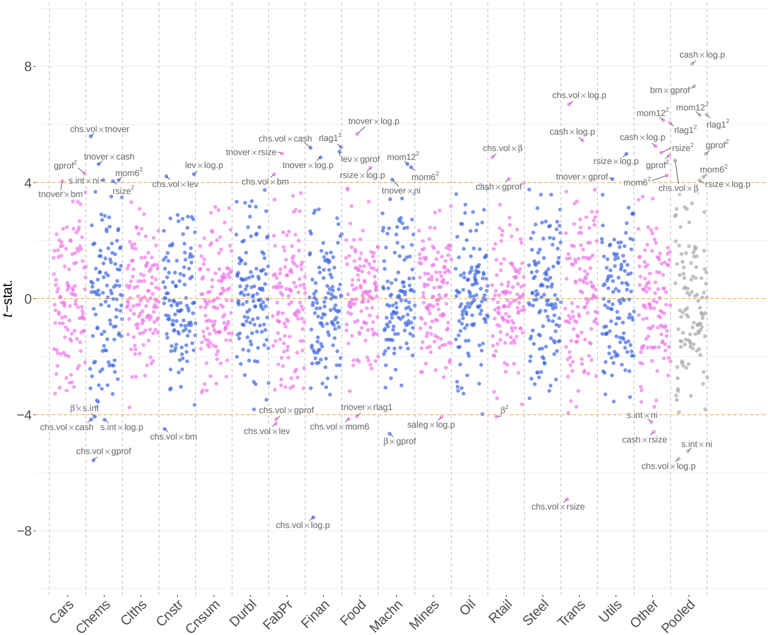
- If you accept the lower bound as a tolerable measure of crash risk, then we can use it to “de-noise” the realized crash event indicator
- This boosts power to detect variables that influence a stock’s likelihood of crashing: we find R^2 on the order of 70–75%
- Crash risk is higher for
  - stocks with high beta, CHS volatility, share turnover, and short interest (Chen, Hong and Stein, 2001; Hong and Stein, 2003)
  - penny stocks, with low log share price (Campbell, Hilscher and Szilagyi, 2008)
  - stocks with poor recent returns, either over the past month or from month -6 to -1
  - unprofitable stocks or stocks with low net income

Panel A: Regressions of the lower bound onto 15 characteristics



Panel B: Regressions onto characteristics, squared characteristics, and interaction terms




t−stat.
8
4
0
−4
−8
Cars
Chems
Clths
Cnstr
Cnsum
Durbl
FabPr
Finan
Food
Machn
Mines
Oil
Rtail
Steel
Trans
Utils
Other
Pooled
cash × log.p
bm × gprof
chs.vol × log.p
chs.vol × tnover
tnover × log.p
chs.vol × cash
tnover × cash
gprof²
s.int × ni
tnover × bm
mom6²
rsize²
lev × log.p
chs.vol × lev
tnover × rsize
chs.vol × bm
lev × gprof
rsize × log.p
mom12²
tnover × ni
chs.vol × β
cash × gprof
tnover × gprof
rlag1²
gprof²
rsize × log.p
β × s.int
chs.vol × cash
s.int × log.p
chs.vol × gprof
chs.vol × bm
chs.vol × lev
tnover × rlag1
chs.vol × mom6
saleg × log.p
β × gprof
β²
s.int × ni
cash × rsize
chs.vol × log.p
chs.vol × rsize

Interactions

(c) Interaction terms

	est.	s.e.	t-stat.
cash×log.p	0.113	0.014	8.10
bm×gprof	0.108	0.015	7.32
chs.vol×log.p	−0.154	0.028	−5.52
s.int×ni	−0.056	0.011	−5.26
chs.vol×beta	0.173	0.036	4.76
rsize×log.p	0.088	0.022	4.07
lev×rlag1	−0.056	0.015	−3.81
beta×s.int	0.053	0.015	3.60

⋮

- Crash risk is higher for
  - penny stocks with low cash, or that are small or volatile
  - unprofitable growth stocks
  - stocks with high short interest and low net income
  - high beta stocks with high CHS volatility or short interest
  - highly levered stocks with low lagged returns

Summary

- We derive bounds on crash probabilities and show that the lower bound successfully forecasts crashes in and out of sample
- For one month forecasts of 20% crashes, we find
  - t-stats in the range 5 to 13
  - estimated coefficient 10 times larger than the next most important competitor variable
- Risk-neutral probabilities also perform well in sample, but overstate crash probabilities—and time variation in overstatement hurts out-of-sample performance
- Our results depend on one key assumption: the form of the SDF
- This is a strong assumption, but it allows us to avoid the undesirable (and commonly made) assumption that backward-looking historical measures are good proxies for the forward-looking measures that come out of theory
- It seems the price of our assumption is worth paying