Online Appendix for "Sentiment and speculation in a market with heterogeneous beliefs"

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1 Miscellaneous results

First of all, let us write down a version of the de Moivre–Laplace theorem; this theorem is essentially a special case of the Central limit theorem that first appeared in 1716 in de Moivre's "The Doctrine of Chances." For a proof, see the textbook of Chung (2012).

Theorem 1. *Suppose* $0 < p_n < 1$, $p_n + q_n = 1$, $p_n \to p$ *and*

$$x_k = \frac{k - np_n}{\sqrt{np_nq_n}},\ 0 \le k \le n$$

Let A *be an arbitrary, fixed positive number. Then in the range of* k *such that* $|x_k| \le A$ *we get*

$$\binom{n}{k} p_n^k q_n^{n-k} \sim \frac{1}{\sqrt{2\pi np_nq_n}} e^{-\frac{x_k^2}{2}}$$

where the convergence is uniform and the notation $\sim$ *means that the ratio of the right hand side to the left hand side tends to 1 as* $n \to \infty$.[1] *Moreover if* S_n *has the Binomial*(n, p_n) *distribution then, for any 2 constants* $a < b$ *we have:*

$$\lim_{n\to\infty} P\left(a \le \frac{S_n - np_n}{\sqrt{np_nq_n}} \le b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}dx$$

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[1]The notation $\sim$ for asymptotic equivalence, is first defined by De Bruijn, in his book *Asymptotic Methods in Analysis*. In fact $f(x) \sim g(x)$, as $x \to \infty$ is equivalent to $f(x) = g(x)(1 + o(1))$.

We also use a stronger version of the de Moivre–Laplace theorem presented in Osius (1989) which implies that there is convergence of the moment generating function of a standardized binomial to that of a standardized Normal (there is *convergence of infinite exponential order*).

We use a central limit theorem for beta-binomial random variables that appears[2] in Paul and Plackett (1978) in a slightly generalized form that allows α and β to have a term proportional to $\sqrt{N}$.

Theorem 2. *If* $Y \sim BB(\overline{\lambda}N, \alpha, \beta)$*, where* $\overline{\lambda} > 0$*,* $\alpha = \theta N + \eta\sqrt{N}, \beta = \theta N - \eta\sqrt{N}$*,*[3] *and we let* $N \to \infty$*, then:*

$$\frac{Y - \frac{1}{2}\overline{\lambda}N - \frac{\eta}{2\theta}\overline{\lambda}\sqrt{N}}{\sqrt{\frac{(\overline{\lambda}+2\theta)}{8\theta}\overline{\lambda}N}} \to N(0,1)$$

Note that the convergence of Beta-Binomial distribution to Normal holds not only in distribution but also in moment generating functions. Indeed, by the Moment Continuity Theorem, convergence in distribution of subgaussian random variables implies convergence of moment generating functions.

1.1 Proof of Lemma 2

This section provides a proof of Lemma 2, which we restate in its general form:

Lemma. *If* $Y_1 \sim BB(T, \overline{\alpha}, \lambda\overline{\alpha})$ *and* $Y_2 \sim BB(T, \underline{\alpha}, \lambda\underline{\alpha})$*, for* $\overline{\alpha} > \underline{\alpha}$ *and* $\lambda > 0$ *then* Y_1 *second order stochastically dominates* Y_2*.*

Proof. First note that $\mathbb{E}[Y_1] = \mathbb{E}[Y_2] = \frac{T}{1+\lambda}$. Write $f_{\overline{\alpha}}(\cdot)$ and $f_{\underline{\alpha}}(\cdot)$ for the probability mass functions of Y_1 and Y_2, respectively. It is enough to show that the likelihood ratio $f_{\overline{\alpha}}(k)/f_{\underline{\alpha}}(k)$ is increasing for integers $k \in [0, T/(1+\lambda))$ and decreasing for integers $k \in (T/(1+\lambda), T]$. This implies that Y_1 second order stochastically dominates Y_2, by Theorem 2.2 of Ramos et al. (2000).[4]

We start by showing that

$$\frac{B(k+\overline{\alpha}, T-k+\lambda\overline{\alpha})}{B(k+\underline{\alpha}, T-k+\lambda\underline{\alpha})}$$

(that is, the likelihood ratio, up to a positive constant of proportionality) is increasing for integers $k \in [0, T/(1+\lambda)]$. Pick k_1 between 1 and $T/(1+\lambda)$ and let $k_2 = k_1 - 1$. We

[2]We caution the reader that there is a typo in the theorem as stated by Paul and Plackett (1978): the random variable is not correctly standardized.

[3]In fact the theorem holds in the more general case where $\alpha = \theta_1 N + \eta\sqrt{N}, \beta = \theta_2 N - \eta\sqrt{N}$.

[4]This result applies for continuous random variables, but it is straightforward to adapt the result to the discrete case which is relevant here.

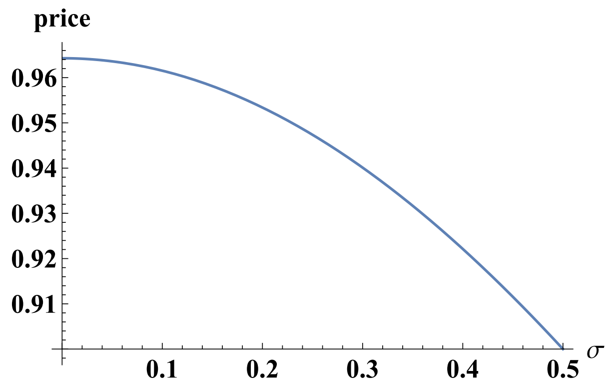


Figure 1: The time 0 price of the risky asset in the example of Figure 2, if agents learn over time, where $\sigma = \frac{1}{\sqrt{4(1+\zeta)}}$ is the standard deviation of the median agent's prior belief.

must show that

$$\frac{\Gamma(k_1+\overline{\alpha})\Gamma(T-k_1+\lambda\overline{\alpha})}{\Gamma(k_1+\underline{\alpha})\Gamma(T-k_1+\lambda\underline{\alpha})} > \frac{\Gamma(k_2+\overline{\alpha})\Gamma(T-k_2+\lambda\overline{\alpha})}{\Gamma(k_2+\underline{\alpha})\Gamma(T-k_2+\lambda\underline{\alpha})}\,. \tag{1}$$

As $\Gamma(z+1) = z\Gamma(z)$ for any positive real z, this reduces to

$$\frac{k_2+\overline{\alpha}}{T-k_1+\lambda\overline{\alpha}} > \frac{k_2+\underline{\alpha}}{T-k_1+\lambda\underline{\alpha}}\,,$$

which is equivalent to $k_1 + \lambda k_2 < T$. This holds because $k_1 \leq T/(1+\lambda)$ and $k_2 < k_1$. Conversely, if $k_2 = k_1 - 1 \geq \frac{T}{1+\lambda}$ then the inequality (1) reverses as then $k_1 + \lambda k_2 > T$. □

2 Learning

Figure 1 shows how the time 0 price of the risky asset in the illustrative example provided in Figure 2 of the main paper varies if agents learn over time. The variable on the x-axis is σ, the standard deviation of the median agent's prior belief; in the example in the main paper, $\sigma = 0$. The maximum possible standard deviation is $\sigma = 1/2$. As $1/p_{m,t}$ is convex in this example (which could be checked directly, but in this case can be seen immediately using the sufficient condition that $\log p_{m,t}$ is weakly concave), the price declines as investors' prior uncertainty increases (i.e., as ζ decreases), as shown more generally in Result 4.

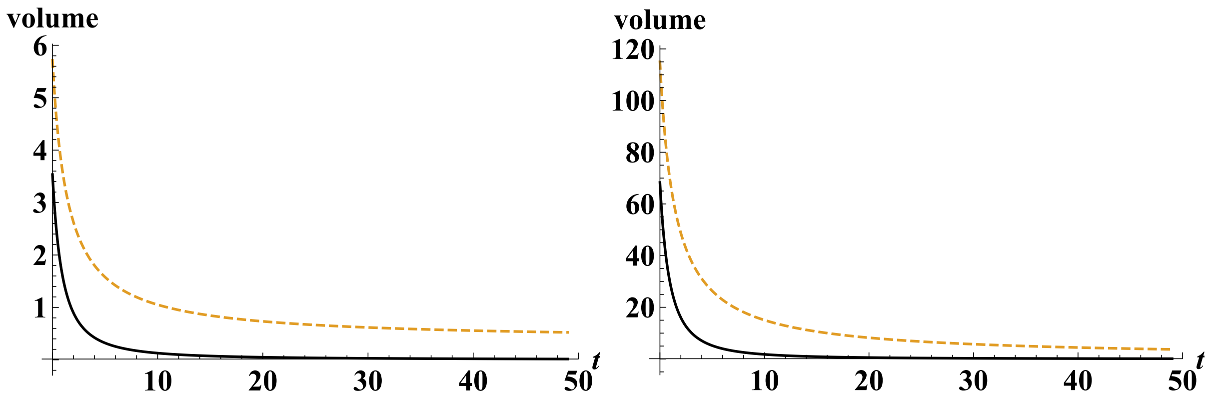


Figure 2: Volume (solid) and gross leverage (dashed) over time, with $\varepsilon = 0.3$ (left) or $\varepsilon = 0.9$ (right).

3 Volume and leverage

Recall that the *leverage ratio* of investor h, which we define as the ratio of funds borrowed, $x_h p - w_h$, to wealth, w_h, is

$$\frac{x_h p - w_h}{w_h} = \frac{h - H_{m,t}}{H_{m,t} - p^*}.$$

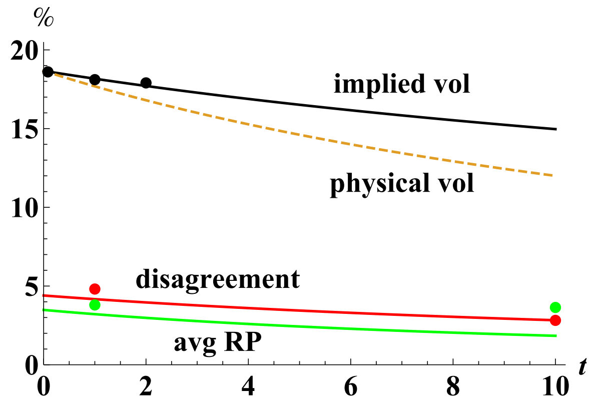
If $p_u > p_d$ then $p^* < H_{m,t}$, by equation (6) of the paper; in this case the above equation shows that people who are optimistic relative to the representative investor borrow from pessimists. We can define *gross leverage* as the total dollar amount these optimists borrow, scaled by aggregate wealth:

$$\begin{aligned} \frac{\int_{H_{m,t}}^{1} (x_h p - w_h)\, f(h)\, dh}{p} &= \int_{H_{m,t}}^{1} \frac{w_h f(h)}{p} \frac{h - H_{m,t}}{H_{m,t} - p^*}\, dh \\ &= \frac{H_{m,t}^{m+\alpha}(1 - H_{m,t})^{n+\beta}}{(m + \alpha + n + \beta) B(\alpha + m, \beta + n)(H_{m,t} - p^*)}. \end{aligned}$$

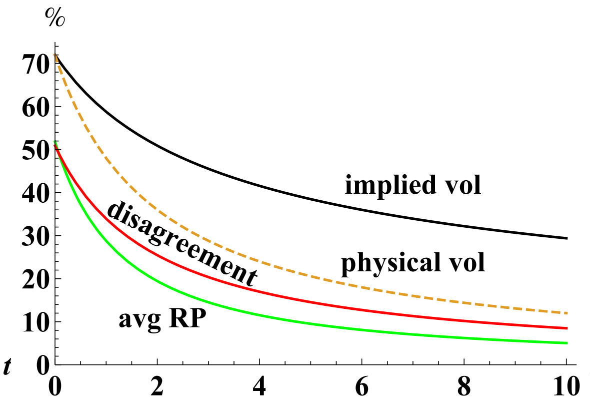
Conversely, if $p_u < p_d$ then optimists are lenders and pessimists borrowers. In either case, we can define gross leverage as the absolute value of the above expression,

$$\frac{H_{m,t}^{m+\alpha}(1 - H_{m,t})^{n+\beta}}{(m + \alpha + n + \beta) B(\alpha + m, \beta + n)\, |H_{m,t} - p^*|}. \tag{2}$$

The left panel of Figure 2 shows the time series of volume and gross leverage in the risky bond example, assuming bad news arrives each period. (If good news arrives at any stage, volume drops permanently to zero.) There is a burst of trade at first: volume substantially exceeds the total supply of the asset initially, as agents with extreme views



(a) $\theta = 1.8$.



(b) $\theta = 0.2$.

Figure 3: Term structures of implied and physical volatility, mean expected returns and disagreement in the baseline (left) and crisis (right) calibrations.

undertake highly leveraged trades, but declines rapidly over time as wealth becomes concentrated in the hands of investors with similar beliefs. The right panel shows the corresponding series if $\varepsilon = 0.9$.

Assuming bad news arrives in the transition from time t to time $t+1$, the volume of trade (in terms of the number of units of the risky asset transacted) is

$$\frac{1}{2}\int_0^1 \left| \frac{(1-h)^t}{\frac{1}{1+t}} \frac{h-p_t^*}{H_{0,t}-p_t^*} - \frac{(1-h)^{t+1}}{\frac{1}{2+t}} \frac{h-p_{t+1}^*}{H_{0,t+1}-p_{t+1}^*} \right| dh = \frac{4(1+t)^{1+t}}{(3+t)^{3+t}} \left(1+t+\frac{1+\varepsilon T}{1-\varepsilon}\right),$$

where we include the factor of $1/2$ to avoid double-counting.

Gross leverage, in the same transition, calculated from (2), is

$$\left(\frac{1+t}{2+t}\right)^{2+t} \left(1+\frac{1+T}{1+t}\frac{\varepsilon}{1-\varepsilon}\right).$$

Volume and gross leverage are each increasing in ε and in T. The safer the bond is, the more aggressively agents trade on their disagreement without risking ruin, as the relative safety of the asset permits agents to take on more leverage: extremists on both sides of the market are trying to "pick up nickels in front of a steamroller."

4 Two calibrations in the Brownian limit

Figure 3 shows the two calibrations discussed in the main paper.

5 Option prices in the Poisson limit

To state our results in an economical way, we write

$$A = e^{-J}\left[1 - \frac{\sigma^2\omega(T-t)(e^J-1)}{1+\sigma^2\omega t}\right] < 1 \quad \text{and} \quad B = \left[1 - \frac{\sigma^2\omega(T-t)(e^J-1)}{1+\sigma^2\omega t}\right]^{1/\sigma^2}, \tag{3}$$

so that the price of the risky asset, at time t, if q jumps have taken place, is BA^q.

The option pricing result is most cleanly stated when the strike $K = BA^k$, where $k \geq 0$ is an integer. For strikes not of this form, options are priced by interpolating linearly in strike: that is, if $\underline{K} = BA^{k+1}$ and $\overline{K} = BA^k$ and $K = \chi\underline{K} + (1-\chi)\overline{K}$ for $\chi \in (0,1)$, then the price of an option with strike K is the convex combination of the prices of options with strikes $\underline{K}$ and $\overline{K}$, with weights χ and $1-\chi$, respectively. (To see this, note that the price of a butterfly spread constructed using options of all three strikes is zero, because the probability of the underlying asset's price lying strictly between $\underline{K}$ and $\overline{K}$ at expiry is zero by definition of $\underline{K}$ and $\overline{K}$.)

Result 1. *The time 0 price of a put option, expiring at time t, with strike BA^k (where $k \geq 0$ is an integer, and where B and A are defined as in (3)) is*[5]

$$p_0\left(1-\lambda\right)^{1/\sigma^2}\lambda^{1+k}\binom{k+1/\sigma^2}{k+1}\left[\frac{1}{A}F\left(1, 1+k+1/\sigma^2, 2+k, \lambda/A\right) - F\left(1, 1+k+1/\sigma^2, 2+k, \lambda\right)\right],$$

where the price of the underlying asset is $p_0 = \left[1 - \sigma^2\omega T\left(e^J-1\right)\right]^{1/\sigma^2}$, $\lambda = \sigma^2\omega t/(1+\sigma^2\omega t)$, and $F(\cdot,\cdot,\cdot,\cdot)$ is Gauss's hypergeometric function.

In the case $\sigma = 1$, this simplifies to $\left(\frac{\omega t}{1+\omega t}\right)^{k+1}(1+\omega T)(e^J-1)$, whereas if beliefs are homogeneous ($\sigma = 0$) then the price of the put option is $e^{-\omega t - \omega T\left(e^J-1\right)}\sum_{q>k}\frac{(\omega t)^q}{q!}\left(e^{J(q-k)}-1\right)$.

Proof of Result 1. Following the same logic as in the proof of Result 7 of the paper, the put price is $p_0\,\mathbb{E}\left[\left(A^{k-q}-1\right)^+\right]$. As $n \to \infty$, a beta binomial distribution with parameters n, α, ρn approaches a negative binomial distribution with parameters α and $1/(1+\rho)$, so q is asymptotically distributed NegativeBinomial$\left(1/\sigma^2, \lambda\right)$. Thus the put price is

$$p_0\sum_{q>k}\binom{q+1/\sigma^2-1}{q}\left(1-\lambda\right)^{1/\sigma^2}\lambda^q\left(A^{k-q}-1\right)$$

which reduces to the given formulas in the cases $\sigma > 0$ and $\sigma = 1$.

[5]We write $\binom{n}{k}$ for the generalized binomial coefficient $\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$, which is defined for $k \in \mathbb{N}$ and arbitrary n.

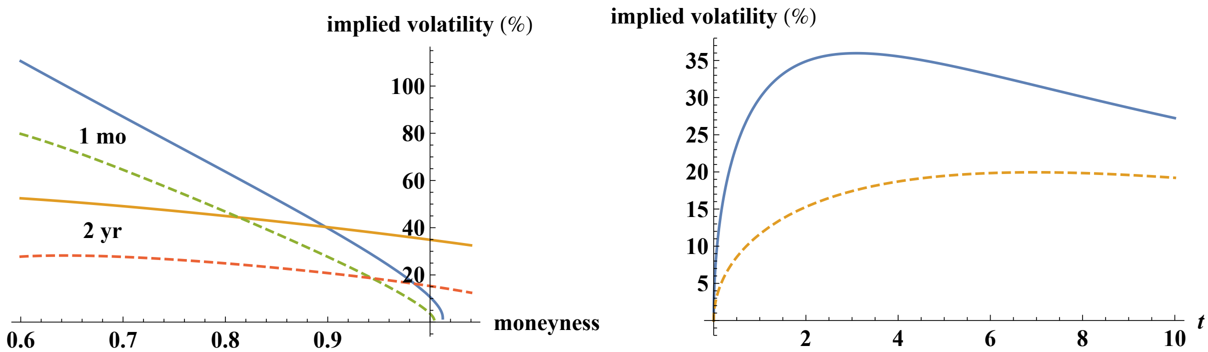


Figure 4: Left: The volatility smile for options of maturity 1 month and 2 years, in heterogeneous (solid) and homogeneous (dashed) belief economies. Right: The term structure of at-the-money implied volatility plotted against time-to-expiry, t, in heterogeneous (solid) and homogeneous (dashed) belief economies. Both panels use the baseline calibration.

As $\sigma \to 0$, the negative binomial random variable q converges to a Poisson random variable with mean ωt, and A approaches e^{-J}. Thus the put price is

$$p_0 \sum_{q=0}^{\infty} e^{-\omega t} \frac{(\omega t)^q}{q!} \left(e^{J(q-k)} - 1\right)^+ = e^{-\omega t - \omega T\left(e^J - 1\right)} \sum_{q>k} \frac{(\omega t)^q}{q!} \left(e^{J(q-k)} - 1\right) . \qquad \square$$

Figure 4 illustrates. The left panel plots the Black–Scholes implied volatility for short-dated ($t = 1/12$) and long-dated ($t = 2$) options across a range of strikes, both with and without heterogeneity. Short-dated options exhibit a steeper smirk than long-dated options. Heterogeneity increases the level of volatility and further steepens the smirk relative to the homogeneous economy. The right panel plots the term structure of implied volatility for at-the-money options, which exhibits a hump shape in the presence of heterogeneous beliefs.

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