Forecasting Crashes with a Smile

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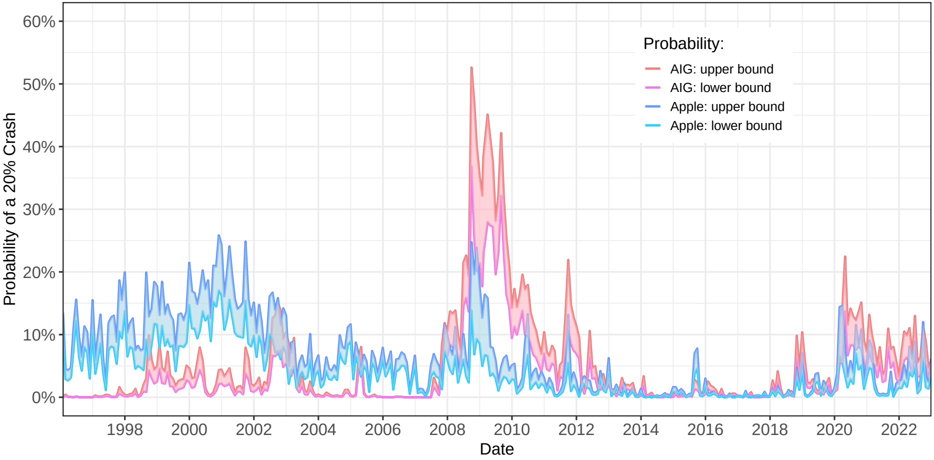
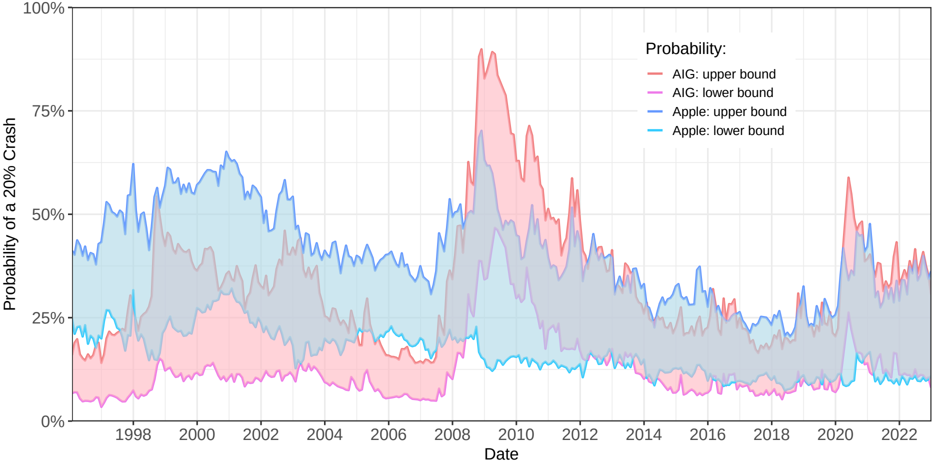
Introduction

- What is the probability that a given stock drops by 20% over the next month?
- We derive bounds on this quantity using index options and individual stock options
- The bounds are observable in real time
- They perform well in and out of sample
- No distributional assumptions

Probabilities of a 20% decline over the next month



Probabilities of a 20% decline over the next year



Background (1)

- We can infer risk-neutral probabilities directly from asset prices
- For example, the risk-neutral probability that the market declines by 10% over the next month can be calculated from index options expiring in a month

$$\mathbb{P}^*[R \leq 0.9] = R_f \times \underbrace{\frac{1}{R_f}\mathbb{E}^*[\boldsymbol{I}(R \leq 0.9)]}_{\text{price of a binary option}} = R_f \times \underbrace{\text{put}'(0.9)}_{\text{slope of put prices}}$$

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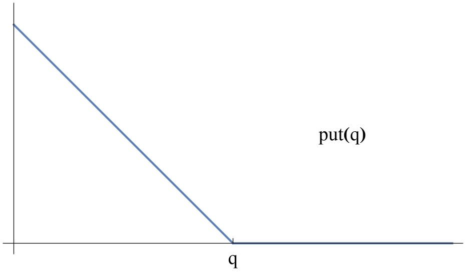

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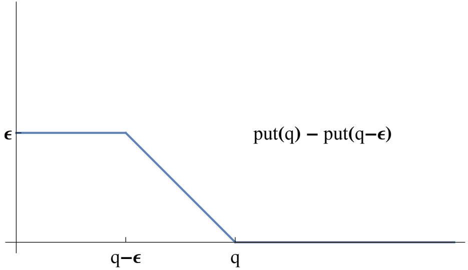

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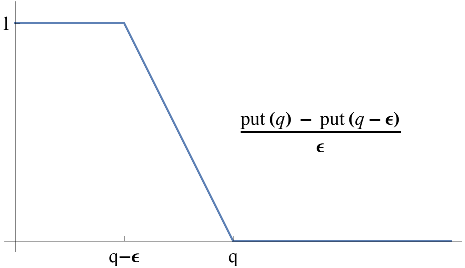

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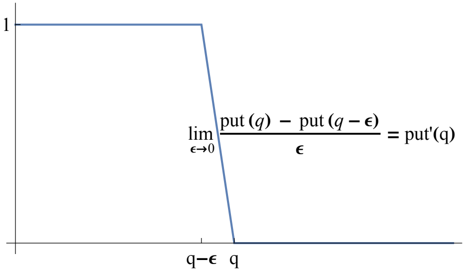

Background (2)

- For forecasting, we would like to measure true, not risk-neutral, probabilities
- We require an assumption (implicit or explicit) to link the true and risk-neutral probabilities—that is, about the stochastic discount factor
- Simple example: think from the perspective of a marginal investor with log utility who chooses to invest fully in the market
  - From this investor's perspective, $1/R_m$ must be a stochastic discount factor
  - This implies that

$$\mathbb{E}X = \mathbb{E}\left[\left(\frac{1}{R_m}\right) XR_m\right] = \frac{1}{R_f}\,\mathbb{E}^*\left(XR_m\right)$$

  - We can infer beliefs about X from the *price* of XR_m

Theory (1)

- We take the perspective of a one-period marginal investor with power utility who chooses to hold the market
- It follows that the SDF is of the form $M = R_m^{-\gamma}/\lambda$ for some constant λ
- γ is the coefficient of relative risk aversion

Theory (2)

- The true expectation of a random payoff X then satisfies

$$\mathbb{E}[X] = \mathbb{E}[\underbrace{\lambda M R_m^{\gamma}}_{\equiv 1} X] = \lambda\, \mathbb{E}[M \times (R_m^{\gamma} X)] = \lambda \frac{\mathbb{E}^*[R_m^{\gamma} X]}{R_f}$$

- Applied in the case $X = 1$, we must have $\lambda = \frac{R_f}{\mathbb{E}^*[R_m^{\gamma}]}$
- So,

$$\mathbb{E}[X] = \frac{\mathbb{E}^*[R_m^{\gamma} X]}{\mathbb{E}^*[R_m^{\gamma}]}$$

Theory (3)

- Consider $X = \boldsymbol{I}(R_i \leq q)$, the crash probability of a stock

$$\mathbb{P}[R_i \leq q] = \frac{\mathbb{E}^* \left[R_m^\gamma \boldsymbol{I}(R_i \leq q)\right]}{\mathbb{E}^* \left[R_m^\gamma\right]}$$

- To calculate $\mathbb{E}^* \left[R_m^\gamma\right]$, we need marginal distribution of R_m
  - Easy, using index option prices (Breeden and Litzenberger, 1978)
- To calculate $\mathbb{E}^* \left[R_m^\gamma \boldsymbol{I}(R_i \leq q)\right]$, we need the joint distribution of (R_m, R_i)
  - Problem: Joint risk-neutral distribution is not observable given assets that are traded in practice (Martin, 2018)
  - This is a general theme: we are often interested in covariances and other features of the joint distribution

A 2×2 example

- Suppose the risk-neutral probability of a crash in Apple is 5%
- Suppose the risk-neutral probability of a crash in the market is also 5%
- These numbers can be calculated from options on Apple and options on the market
- But they are consistent with different joint distributions, eg,

		Apple	
		Crash	No crash
S&P 500	Crash	5%	0%
	No crash	0%	95%

		Apple	
		Crash	No crash
S&P 500	Crash	0%	5%
	No crash	5%	90%

A 2 × 2 example

		Apple	
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	No crash	0%	95%

		Apple	
		Crash	No crash
S&P 500	Crash	0%	5%
	No crash	5%	90%

- In the left-hand world, AAPL is risky
  - Risk-neutral probability of a crash will overstate the true probability of a crash
- In the right-hand world, AAPL is a hedge
  - Risk-neutral probability will understate the true probability of a crash
- Moral: Even if we can't observe the joint distribution, we may be able to derive bounds on the true crash probability

Theory (4)

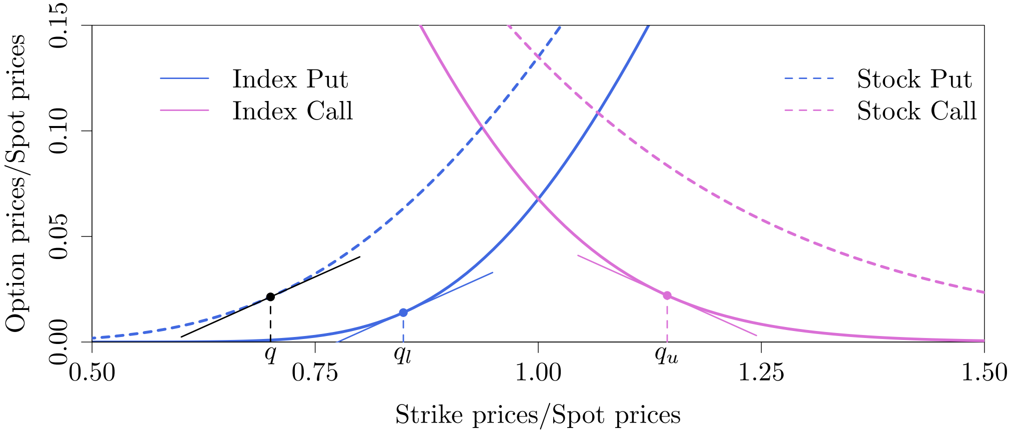
$$\mathbb{P}[R_i \leq q] = \frac{\mathbb{E}^* \left[R_m^\gamma \boldsymbol{I}(R_i \leq q)\right]}{\mathbb{E}^* \left[R_m^\gamma\right]}$$

- We do not observe the joint risk-neutral distribution, so cannot calculate the right-hand side
- But we do observe the individual (marginal) risk-neutral distributions of R_m and R_i, from options on the market and on stock i
- The Fréchet–Hoeffding theorem provides upper and lower bounds on the right-hand side, as in the 2×2 example

Theory (5)

- First, find stock-specific quantiles q_l and q_u such that

$$\mathbb{P}^*[R_m \leq q_l] = \mathbb{P}^*[R_i \leq q] = \mathbb{P}^*[R_m \geq q_u]$$



Theory (6)

Result (Bounds on the probability of a crash)

We have

$$\frac{\mathbb{E}^*\left[R_m^\gamma \boldsymbol{I}(R_m \le q_l)\right]}{\mathbb{E}^*\left[R_m^\gamma\right]} \le \mathbb{P}[R_i \le q] \le \frac{\mathbb{E}^*\left[R_m^\gamma \boldsymbol{I}(R_m \ge q_u)\right]}{\mathbb{E}^*\left[R_m^\gamma\right]}$$

- The three elements are

$$\mathbb{E}^*\left[R_m^\gamma\right] = R_f^\gamma + \gamma(\gamma-1)R_f\left[\int_0^{R_f} R^{\gamma-2}\mathrm{put}_m(R)\,\mathrm{d}R + \int_{R_f}^{\infty} R^{\gamma-2}\mathrm{call}_m(R)\,\mathrm{d}R\right]$$

$$\mathbb{E}^*\left[R_m^\gamma \boldsymbol{I}\left(R_m \le q_l\right)\right] = R_f q_l^\gamma\left[\mathrm{put}_m'(q_l) - \gamma\frac{\mathrm{put}_m\left(q_l\right)}{q_l}\right] + \gamma(\gamma-1)R_f\int_0^{q_l} R^{\gamma-2}\mathrm{put}_m\left(R\right)\,\mathrm{d}R$$

$$\mathbb{E}^*\left[R_m^\gamma \boldsymbol{I}\left(R_m \ge q_u\right)\right] = R_f q_u^\gamma\left[\gamma\frac{\mathrm{call}_m(q_u)}{q_u} - \mathrm{call}'(q_u)\right] + \gamma(\gamma-1)R_f\int_{q_u}^{\infty} R^{\gamma-2}\mathrm{call}_m\left(R\right)\,\mathrm{d}R$$

Theory (7)

- Bounds from the Fréchet–Hoeffding theorem are attainable
  - Lower bound achieved for a stock that is comonotonic with the market—i.e., whose return is a (potentially nonlinear) increasing function of the market return
  - Upper bound achieved for a stock that is countermonotonic with the market—i.e., whose return is a (potentially nonlinear) decreasing function of the market return
- Intuitively, asset prices will tend to overstate crash probabilities if crashes are scary; or understate crash probabilities if crashes occur in good times
- A priori, we expect that the scary case is the relevant one, and hence that the lower bound should be closer to the truth in practice

Theory (8)

Result (Bounds on the probability of a crash)

We have

$$\frac{\mathbb{E}^*\left[R_m^\gamma \boldsymbol{I}(R_m \leq q_l)\right]}{\mathbb{E}^*\left[R_m^\gamma\right]} \leq \mathbb{P}[R_i \leq q] \leq \frac{\mathbb{E}^*\left[R_m^\gamma \boldsymbol{I}(R_m \geq q_u)\right]}{\mathbb{E}^*\left[R_m^\gamma\right]}$$

Further results

- Both $\mathbb{P}[R_i \leq q]$ and $\mathbb{P}^*[R_i \leq q]$ lie in between the bounds
- When $\gamma = 0$, the lower and upper bounds both equal $\mathbb{P}^*[R_i \leq q]$, and $\mathbb{P}^*$ and $\mathbb{P}$ coincide
- As γ increases, the bounds widen monotonically, so higher γ is more conservative
- As $\gamma \to \infty$, the bounds become trivial: the lower bound tends to zero and the upper bound tends to one

Is this just "correlation must lie in $[-1, 1]$"?

- Here's another approach that does not work as well. Write

$$\mathbb{P}\left[R_i \leq q\right] = \mathbb{P}^*\left[R_i \leq q\right] + \frac{\text{cov}^*\left[R_m^\gamma, \boldsymbol{I}(R_i \leq q)\right]}{\mathbb{E}^*\left[R_m^\gamma\right]}$$

- As correlation must lie between plus and minus one, it follows that

$$\mathbb{P}^*\left[R_i \leq q\right] - \frac{\sigma^*\left[R_m^\gamma\right]\sigma^*\left[\boldsymbol{I}(R_i \leq q)\right]}{\mathbb{E}^*\left[R_m^\gamma\right]} \leq \mathbb{P}\left[R_i \leq q\right] \leq \mathbb{P}^*\left[R_i \leq q\right] + \frac{\sigma^*\left[R_m^\gamma\right]\sigma^*\left[\boldsymbol{I}(R_i \leq q)\right]}{\mathbb{E}^*\left[R_m^\gamma\right]}$$

  where $\sigma^*\left[\cdot\right]$ denotes risk-neutral volatility

- These bounds depend only on univariate risk-neutral expectations, so can be calculated from observable option prices. But they are much wider than our bounds

Fréchet–Hoeffding vs. Cauchy–Schwarz

- If, say, returns were jointly lognormal, then it *could* in principle be the case that log returns were perfectly positively or negatively correlated
- In this special case, our results *would* reduce to the bound on the previous slide
- But observed properties of option prices rule out lognormality
- And, using our approach, we can empirically bound correlations away from ± 1
- Are we just using the fact that correlations lie in $[-1, 1]$? No!

Width of FH bounds relative to correlation-based bounds

crash size	mo.	mean	sd	median	q25	q75	min	max
20%	1	0.291	0.170	0.269	0.160	0.411	0.000	0.800
20%	3	0.565	0.119	0.592	0.491	0.648	0.000	0.813
20%	6	0.659	0.071	0.662	0.623	0.706	0.000	0.811
20%	12	0.706	0.052	0.712	0.674	0.745	0.002	0.842
10%	1	0.544	0.107	0.565	0.487	0.618	0.000	0.848
10%	3	0.679	0.059	0.678	0.642	0.723	0.000	0.828
10%	6	0.727	0.043	0.733	0.698	0.761	0.000	0.812
10%	12	0.751	0.031	0.758	0.737	0.772	0.004	0.842
5%	1	0.615	0.083	0.630	0.578	0.668	0.000	0.849
5%	3	0.716	0.047	0.717	0.681	0.757	0.007	0.828
5%	6	0.751	0.033	0.761	0.735	0.775	0.098	0.812
5%	12	0.766	0.024	0.771	0.757	0.781	0.138	0.842

Data

- S&P 500 index and stock constituents from **Compustat**
- Option implied volatilities from **OptionMetrics**
  - Underlying stocks belonging to the S&P 500 index
  - Monthly from 1996/01 to 2022/12
  - Maturing in $1, 3, 6$ and 12 months
  - 1044 firms in total
  - On average around 492 firms each month
  - Over $155,000$ firm-month observations per maturity
- Risk-free rates from **OptionMetrics**
- Firm characteristics from **Compustat**
- Price, return, and volume data from **CRSP**

Computation

- We set γ equal to 2
- Focus on “crashes” of 5%, 10% and 20%: $q = 0.95, 0.9$ and 0.8
- For stock i in month t, compute upper and lower bounds at horizons $\tau = 1, 3, 6$ and 12 months, $\mathbb{P}^U_{i,t}(\tau, q)$ and $\mathbb{P}^L_{i,t}(\tau, q)$
- We focus on the case of 20% decline over one month

Summary statistics

		averaged across firms (number of obs. $T = 324$)				averaged across time (number of obs. $N = 1044$)			
	maturity	1	3	6	12	1	3	6	12
$q = 0.80$, down by over 20%									
realized	mean	0.021	0.069	0.111	0.152	0.029	0.084	0.130	0.173
	s.d.	0.048	0.107	0.141	0.160	0.059	0.092	0.129	0.166
lower bound	mean	0.022	0.073	0.102	0.123	0.027	0.079	0.110	0.133
	s.d.	0.020	0.029	0.028	0.027	0.029	0.046	0.052	0.056
upper bound	mean	0.038	0.144	0.233	0.339	0.044	0.152	0.242	0.350
	s.d.	0.040	0.071	0.082	0.098	0.042	0.069	0.079	0.089
risk-neutral	mean	0.031	0.113	0.173	0.236	0.037	0.120	0.181	0.245
	s.d.	0.031	0.050	0.053	0.059	0.036	0.058	0.065	0.072

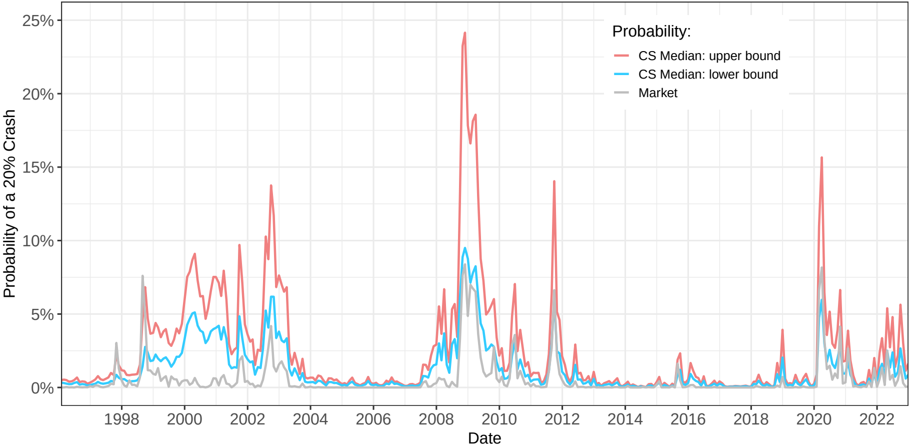
Summary statistics

		averaged across firms (number of obs. $T = 324$)				averaged across time (number of obs. $N = 1044$)			
	maturity	1	3	6	12	1	3	6	12
$q = 0.90$, down by over 10%									
realized	mean	0.096	0.173	0.211	0.236	0.110	0.191	0.231	0.252
	s.d.	0.124	0.170	0.185	0.196	0.089	0.119	0.152	0.183
lower bound	mean	0.109	0.168	0.196	0.210	0.118	0.179	0.206	0.219
	s.d.	0.037	0.031	0.028	0.023	0.050	0.055	0.056	0.056
upper bound	mean	0.156	0.277	0.366	0.466	0.166	0.289	0.378	0.475
	s.d.	0.064	0.074	0.081	0.087	0.062	0.070	0.074	0.074
risk-neutral	mean	0.136	0.228	0.286	0.341	0.145	0.239	0.297	0.350
	s.d.	0.051	0.051	0.051	0.050	0.056	0.061	0.063	0.063

Summary statistics

		averaged across firms (number of obs. $T = 324$)				averaged across time (number of obs. $N = 1044$)			
	maturity	1	3	6	12	1	3	6	12
$q = 0.95$, down by over 5%									
realized	mean	0.216	0.271	0.288	0.289	0.230	0.287	0.306	0.306
	s.d.	0.187	0.200	0.204	0.210	0.101	0.122	0.155	0.185
lower bound	mean	0.215	0.264	0.277	0.271	0.228	0.275	0.286	0.279
	s.d.	0.036	0.024	0.020	0.020	0.052	0.049	0.047	0.048
upper bound	mean	0.281	0.393	0.465	0.541	0.294	0.404	0.474	0.548
	s.d.	0.064	0.064	0.066	0.074	0.059	0.058	0.056	0.057
risk-neutral	mean	0.251	0.332	0.375	0.408	0.264	0.343	0.383	0.415
	s.d.	0.049	0.041	0.038	0.040	0.055	0.052	0.049	0.050

Probability of a 20% crash in one month



Empirical tests

- $\boldsymbol{I}(R_i \leq q) = 0 + 1 \times \underbrace{\mathbb{E}[\, \boldsymbol{I}(R_i \leq q) \,]}_{\mathbb{P}[R_i \leq q]} + \varepsilon$
- So a regression of the realized crash indicator $\boldsymbol{I}(R_i \leq q)$ onto an ideal crash probability measure $\mathbb{P}[R_i \leq q]$ would yield zero constant term and a unit regression coefficient
- If the lower bound is close to the truth, then in a regression

$$\boldsymbol{I}[R_{i,t \to t+\tau} \leq q] = \alpha^L + \beta^L \, \mathbb{P}^L_{i,t}(\tau, q) + \varepsilon_{i,t+\tau},$$

we should find $\alpha^L \approx 0$ and $\beta^L \approx 1$ at any horizon τ and threshold q

Empirical tests: results (1)

Down by 20% ($q = 0.80$)

	lower bound				upper bound				risk neutral			
maturity	1	3	6	12	1	3	6	12	1	3	6	12
α	0.00	−0.01	−0.01	0.02	0.00	0.00	0.00	0.01	0.00	−0.01	−0.02	0.00
	(0.00)	(0.01)	(0.01)	(0.01)	(0.00)	(0.01)	(0.01)	(0.02)	(0.00)	(0.01)	(0.01)	(0.02)
	[0.00]	[0.01]	[0.01]	[0.01]	[0.00]	[0.01]	[0.02]	[0.03]	[0.00]	[0.01]	[0.01]	[0.03]
β	0.92	1.03	1.15	1.08	0.55	0.51	0.50	0.41	0.68	0.69	0.73	0.66
	(0.11)	(0.09)	(0.09)	(0.08)	(0.08)	(0.06)	(0.06)	(0.06)	(0.09)	(0.07)	(0.07)	(0.08)
	[0.11]	[0.15]	[0.14]	[0.12]	[0.08]	[0.08]	[0.09]	[0.10]	[0.08]	[0.10]	[0.12]	[0.13]
R^2	5.66%	5.17%	4.78%	3.76%	5.29%	4.13%	3.26%	2.33%	5.45%	4.51%	3.91%	3.00%

Empirical tests: results (1)

with time fixed effects

Down by 20% ($q = 0.80$)

	lower bound				upper bound				risk neutral			
maturity	1	3	6	12	1	3	6	12	1	3	6	12
β	0.93	1.04	1.13	1.11	0.62	0.68	0.74	0.72	0.73	0.81	0.89	0.87
	(0.09)	(0.07)	(0.06)	(0.06)	(0.06)	(0.04)	(0.04)	(0.04)	(0.07)	(0.05)	(0.05)	(0.05)
	[0.10]	[0.10]	[0.10]	[0.08]	[0.06]	[0.09]	[0.04]	[0.06]	[0.08]	[0.05]	[0.05]	[0.06]
R^2-proj	4.45%	4.66%	4.56%	4.11%	4.29%	4.46%	4.41%	4.08%	4.35%	4.54%	4.49%	4.10%

Empirical tests: results (2)

Down by 10% ($q = 0.90$)

	lower bound				upper bound				risk neutral			
maturity	1	3	6	12	1	3	6	12	1	3	6	12
α	−0.02	−0.01	−0.01	0.02	−0.02	0.00	0.01	0.05	−0.02	−0.02	−0.02	0.00
	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)	(0.02)	(0.02)	(0.04)	(0.01)	(0.02)	(0.02)	(0.03)
	[0.01]	[0.01]	[0.02]	[0.03]	[0.01]	[0.03]	[0.03]	[0.06]	[0.01]	[0.02]	[0.03]	[0.04]
β	1.05	1.07	1.12	1.03	0.75	0.63	0.54	0.41	0.88	0.83	0.81	0.69
	(0.08)	(0.07)	(0.07)	(0.08)	(0.07)	(0.07)	(0.07)	(0.08)	(0.08)	(0.08)	(0.08)	(0.09)
	[0.08]	[0.10]	[0.11]	[0.13]	[0.07]	[0.11]	[0.10]	[0.13]	[0.08]	[0.11]	[0.12]	[0.14]
R^2	5.47%	3.73%	3.43%	2.54%	5.36%	3.06%	2.20%	1.25%	5.47%	3.41%	2.84%	1.88%

Empirical tests: results (2)

with time fixed effects

Down by 10% ($q = 0.90$)

	lower bound				upper bound				risk neutral			
maturity	1	3	6	12	1	3	6	12	1	3	6	12
β	0.99	1.00	1.06	1.06	0.81	0.79	0.83	0.83	0.89	0.89	0.95	0.95
	(0.06)	(0.05)	(0.06)	(0.06)	(0.05)	(0.04)	(0.04)	(0.05)	(0.05)	(0.05)	(0.05)	(0.06)
	[0.07]	[0.08]	[0.06]	[0.06]	[0.05]	[0.05]	[0.06]	[0.07]	[0.06]	[0.05]	[0.08]	[0.09]
R^2-proj	4.02%	3.17%	3.18%	2.97%	3.96%	3.11%	3.14%	2.96%	3.98%	3.14%	3.17%	2.95%

Empirical tests: results (3)

Down by 5% ($q = 0.95$)

	lower bound				upper bound				risk neutral			
maturity	1	3	6	12	1	3	6	12	1	3	6	12
α	0.00	0.02	−0.01	0.05	0.00	0.05	0.06	0.11	0.00	0.01	−0.01	0.04
	(0.01)	(0.02)	(0.02)	(0.03)	(0.02)	(0.03)	(0.04)	(0.05)	(0.02)	(0.03)	(0.03)	(0.05)
	[0.01]	[0.03]	[0.03]	[0.06]	[0.02]	[0.05]	[0.06]	[0.09]	[0.02]	[0.04]	[0.05]	[0.07]
β	0.98	0.95	1.06	0.88	0.76	0.56	0.49	0.33	0.88	0.77	0.80	0.61
	(0.07)	(0.07)	(0.08)	(0.10)	(0.08)	(0.09)	(0.09)	(0.10)	(0.08)	(0.09)	(0.10)	(0.12)
	[0.06]	[0.10]	[0.11]	[0.18]	[0.08]	[0.13]	[0.14]	[0.17]	[0.08]	[0.12]	[0.14]	[0.17]
R^2	3.01%	1.85%	1.86%	1.36%	3.02%	1.35%	0.94%	0.49%	3.08%	1.64%	1.45%	0.93%

Empirical tests: results (3)

with time fixed effects

Down by 5% ($q = 0.95$)

	lower bound				upper bound				risk neutral			
maturity	1	3	6	12	1	3	6	12	1	3	6	12
β	0.87	0.86	0.97	0.97	0.77	0.76	0.85	0.86	0.83	0.82	0.93	0.93
	(0.05)	(0.05)	(0.07)	(0.07)	(0.04)	(0.05)	(0.06)	(0.06)	(0.05)	(0.05)	(0.06)	(0.07)
	[0.04]	[0.06]	[0.12]	[0.07]	[0.02]	[0.08]	[0.07]	[0.06]	[0.03]	[0.09]	[0.10]	[0.09]
R^2-proj	2.21%	1.62%	1.75%	1.84%	2.19%	1.60%	1.74%	1.85%	2.20%	1.61%	1.74%	1.84%

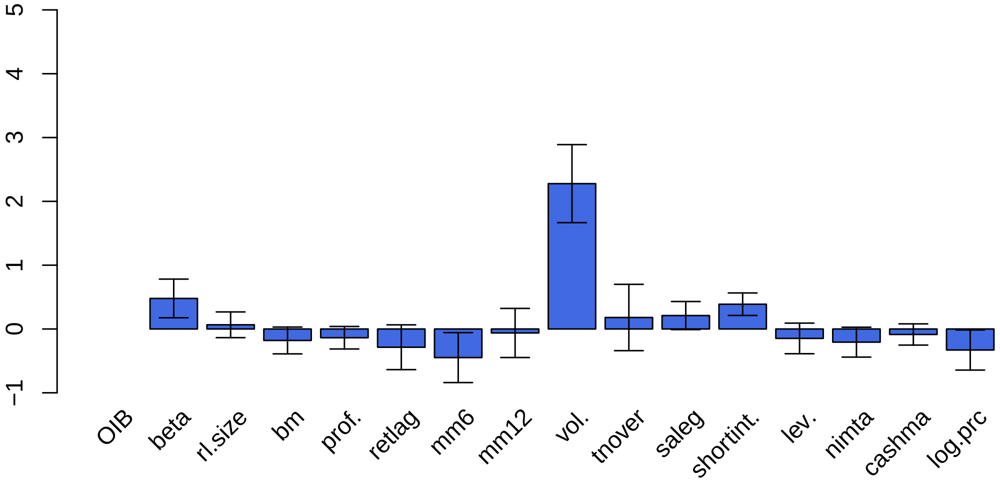
Empirical tests: competitor variables from the literature

- We compare against 15 variables drawn from the literature
  - Stock characteristics: CAPM beta, (log) relative size, book-to-market, profitability (gross profit/assets), momentum (prior 2-6 and 2-12 month returns), lagged return
  - Chen–Hong–Stein, 2001: realized volatilities (standard deviations of daily market-adjusted returns over the last six months) and monthly turnover (shares traded scaled by shares outstanding)
  - Greenwood–Shleifer–You, 2019: sales growth
  - Asquith–Pathak–Ritter, 2005; Nagel, 2005: short interest (shares shorted/shares held by institutions)
  - Campbell–Hilscher–Szilagyi, 2008: leverage (debt/total assets), net income/total assets, cash/total assets, log price per share (winsorized from above at $15)
- All variables are standardized (to unit standard deviation) for comparability

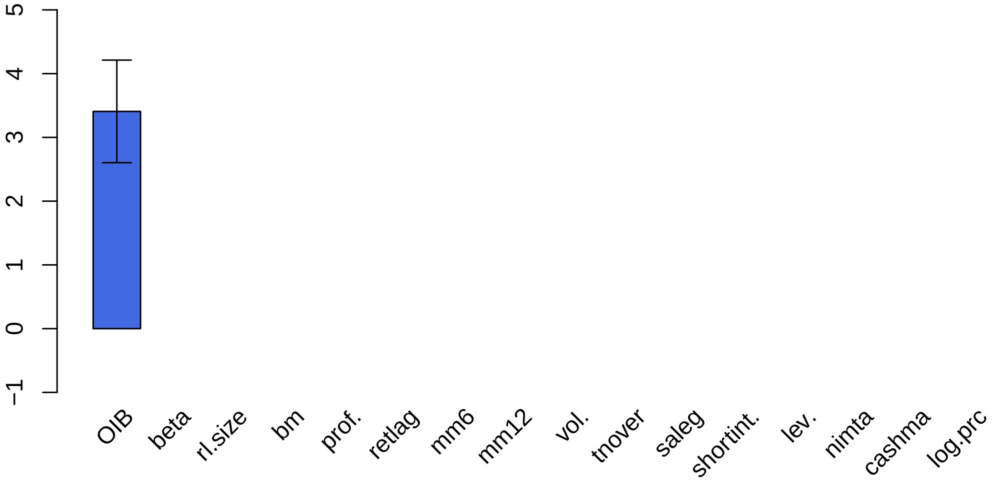
Empirical tests: results (4)

	$\boldsymbol{I}(R_{t\to t+1} \leq 0.8)$					
$\mathbb{P}^L[R_{t\to t+1} \leq 0.8]$		3.41^*	3.05^*		4.44	2.74^*
		(0.41)	(0.59)		(3.08)	(0.33)
$\mathbb{P}^*[R_{t\to t+1} \leq 0.8]$				2.83^*	-1.40	
				(0.67)	(3.37)	
CHS-volatility	2.28^*		0.30	0.43	0.31	0.50
	(0.31)		(0.38)	(0.45)	(0.39)	(0.18)
short int.	0.39^*		0.33^*	0.36^*	0.32^*	0.27^*
	(0.09)		(0.08)	(0.08)	(0.08)	(0.06)
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
R^2/R^2-proj.	4.51%	5.66%	5.85%	5.72%	5.87%	4.74%

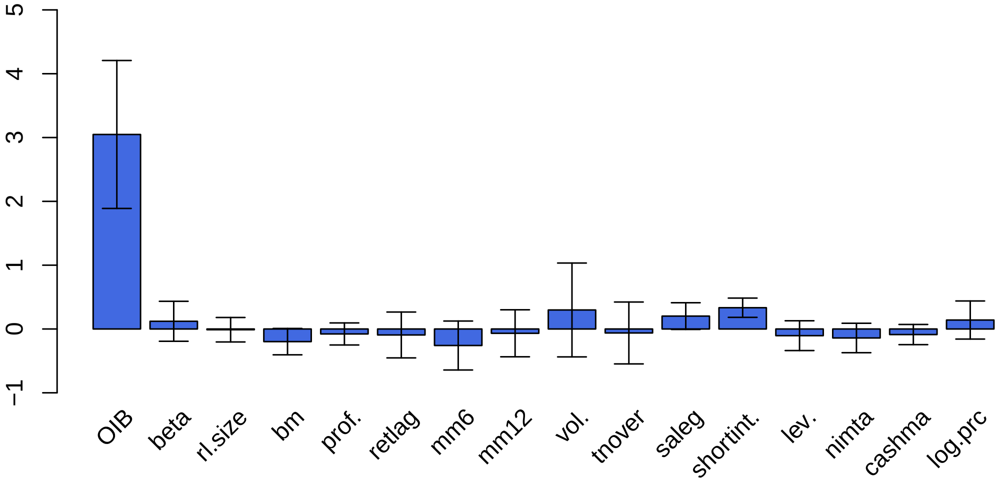
Empirical tests: results (4)



Empirical tests: results (4)



Empirical tests: results (4)



Empirical tests: results (5)

	$\boldsymbol{I}(R_{t\to t+6} \leq 0.8)$					
$\mathbb{P}^L[R_{t\to t+6} \leq 0.8]$		6.87^*	5.53^*		11.93^*	4.17^*
		(0.53)	(0.80)		(1.89)	(0.36)
$\mathbb{P}^*[R_{t\to t+6} \leq 0.8]$				2.76	-6.89	
				(0.88)	(1.91)	
sales growth	1.19^*		1.08	1.19^*	0.97	0.50
	(0.27)		(0.26)	(0.27)	(0.26)	(0.19)
short int.	1.68^*		1.56^*	1.67^*	1.44^*	1.29^*
	(0.28)		(0.27)	(0.28)	(0.27)	(0.20)
	⋮	⋮	⋮	⋮	⋮	⋮
R^2/R^2-proj.	4.94%	4.79%	5.66%	5.14%	5.99%	5.35%

Empirical tests: results (6)

	$I(R_{t\to t+12} \leq 0.8)$					
$\mathbb{P}^L[R_{t\to t+12} \leq 0.8]$		6.96^*	5.28^*		9.17^*	4.37^*
		(0.54)	(0.73)		(2.09)	(0.42)
$\mathbb{P}^*[R_{t\to t+12} \leq 0.8]$				2.55	-4.56	
				(0.95)	(2.22)	
sales growth	2.00^*		1.86^*	1.99^*	1.78^*	0.93
	(0.34)		(0.34)	(0.34)	(0.34)	(0.23)
short int.	2.42^*		2.28^*	2.40^*	2.22^*	2.04^*
	(0.41)		(0.40)	(0.41)	(0.40)	(0.32)
	⋮	⋮	⋮	⋮	⋮	⋮
R^2/R^2-proj.	4.59%	3.76%	5.16%	4.74%	5.31%	4.99%

Empirical tests: Back–Crotty–Kazempour (2022)

- GMM-based tests for the validity and tightness of bounds, applied to Martin (2017), Martin–Wagner (2019), Kadan–Tang (2020), Chabi-Yo–Loudis (2020)
- Our upper and lower bounds are valid
- Our upper bound is (with very high confidence) not tight
- Mixed evidence on tightness of the lower bound

Empirical tests: results (7)

p-values for BCK tests of validity and tightness

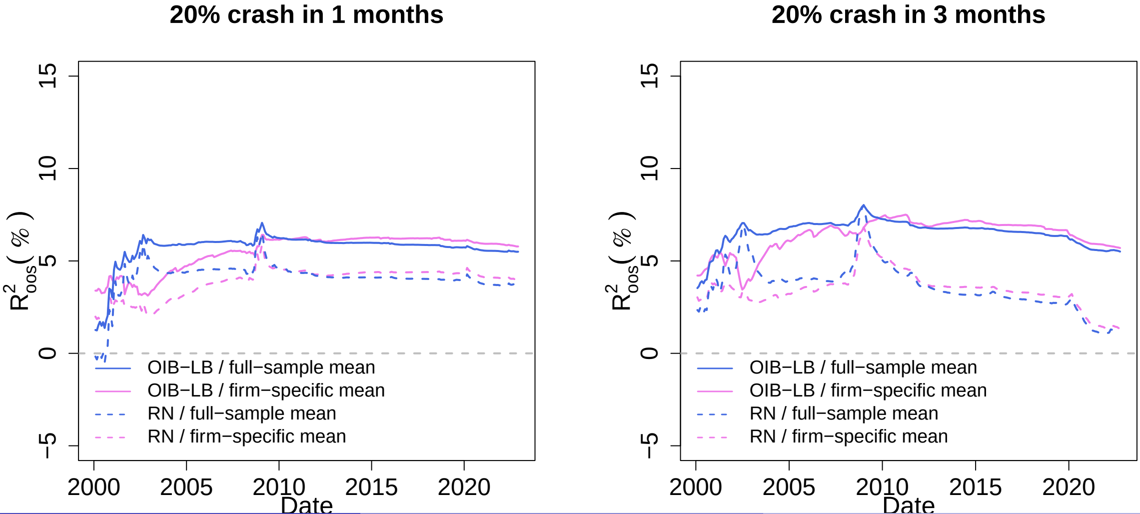
	lower bound				upper bound			
horizon	1	3	6	12	1	3	6	12
Panel A: $q = 0.80$, down by over 20%								
validity	0.452	0.381	0.621	0.487	0.769	1.000	0.754	1.000
tightness	0.352	0.022	0.043	0.164	0.011	0.000	0.000	0.018
Panel B: $q = 0.90$, down by over 10%								
validity	0.069	0.626	0.683	0.505	0.780	0.768	0.755	0.755
tightness	0.133	0.059	0.057	0.114	0.000	0.000	0.000	0.020
Panel C: $q = 0.95$, down by over 5%								
validity	0.552	0.629	0.563	0.486	1.000	0.779	0.760	1.000
tightness	0.176	0.043	0.048	0.096	0.001	0.000	0.000	0.019

Out-of-sample forecasts

- Forecasting crashes: $\boldsymbol{I}(R_i \leq q)$
- First, compare to in-sample mean crash probabilities, either full-sample or firm-specific
- Next, a more challenging competitor
  - include all additional variables in adjusted regressions
  - train predictive models using data from 1996 to 2006/2011/2016
    - ⋆ variable selection using Lasso
    - ⋆ tuning parameters for sparsity: 10-fold cross validation based on the training sample
  - out-of-sample forecasts for the rest of our sample
- Option-implied bounds are directly used to forecast with fixed $\alpha = 0$ and $\beta = 1$

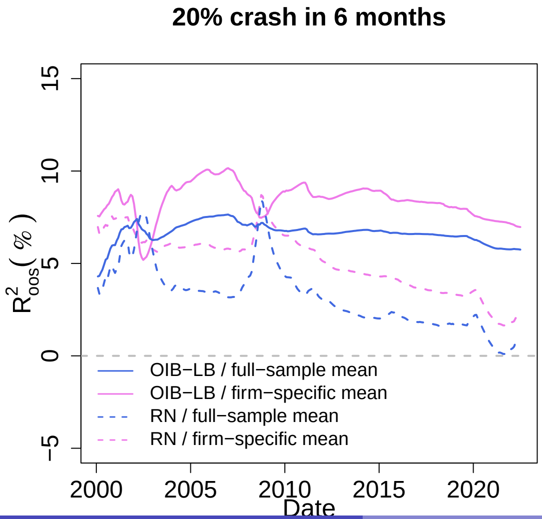
Out-of-sample R^2s

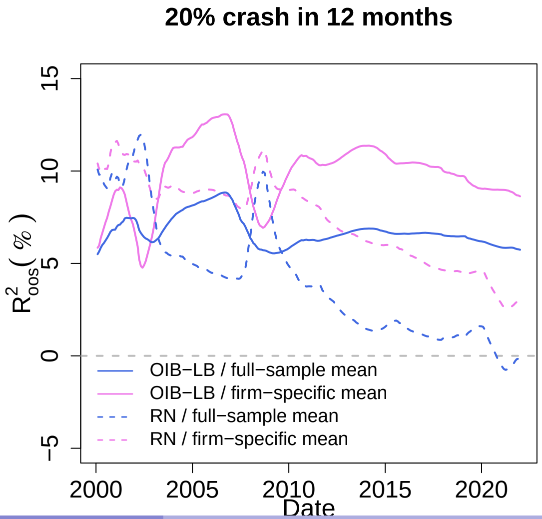
Expanding window



Out-of-sample R^2s

Expanding window

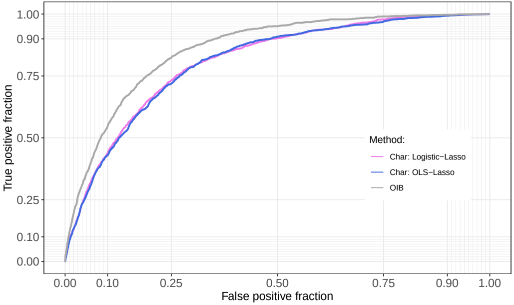




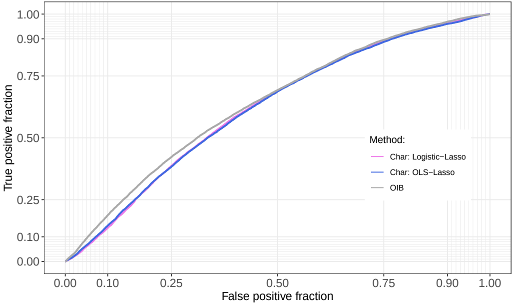
Out-of-sample forecasts: ROC curves

Training Sample 1996 – 2006

Down by over 20% in one month



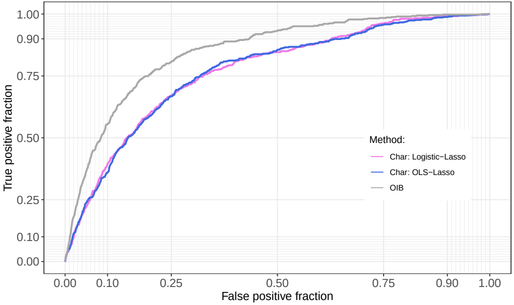
Down by over 20% in one year



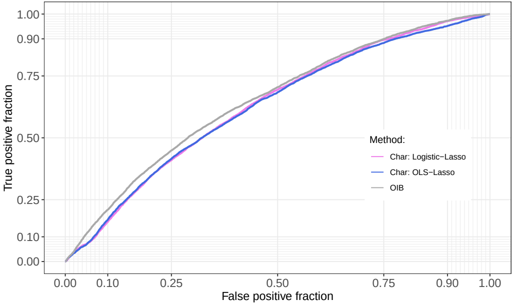
Out-of-sample forecasts: ROC curves

Training Sample 1996 – 2011

Down by over 20% in one month



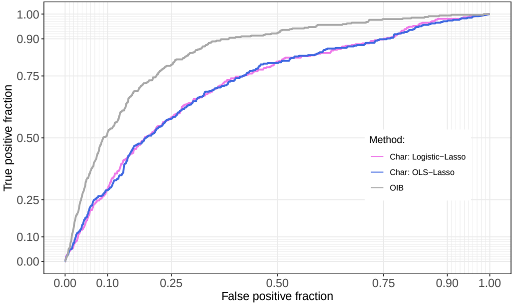
Down by over 20% in one year



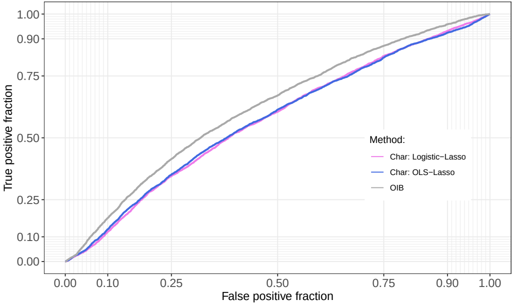
Out-of-sample forecasts: ROC curves

Training Sample 1996 – 2016

Down by over 20% in one month

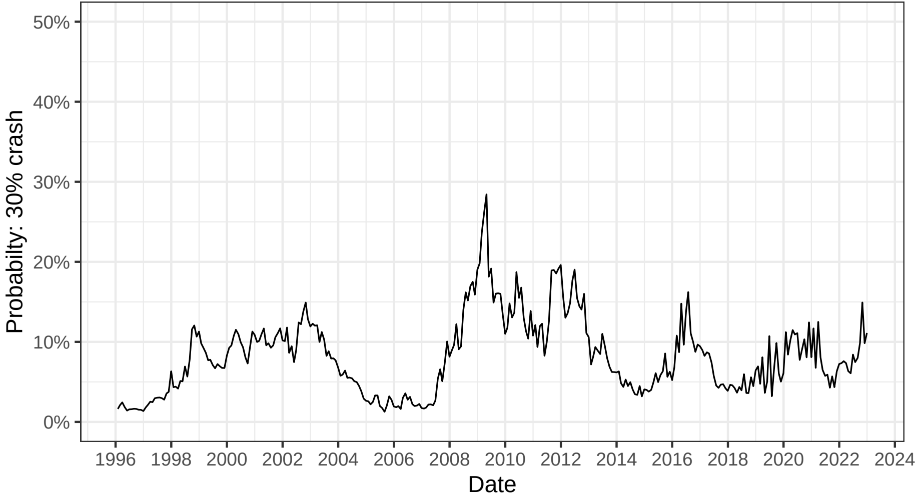


Down by over 20% in one year



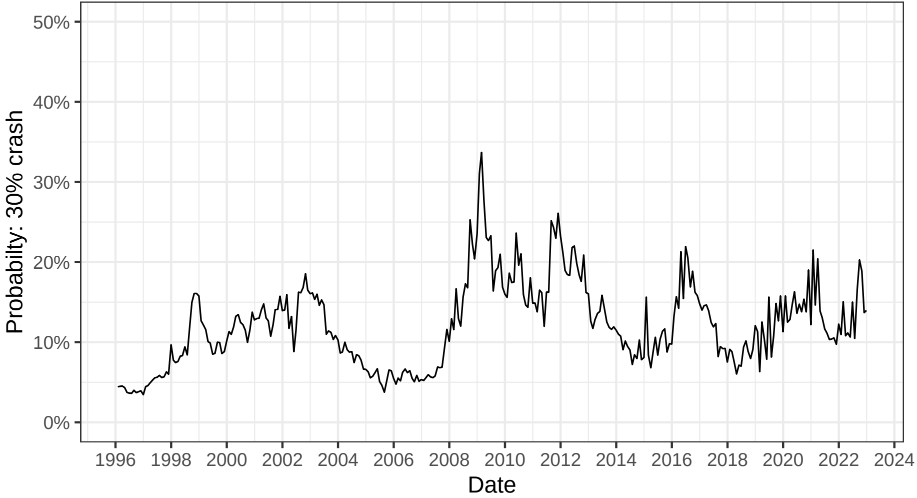
A financial risk indicator

Probability of 30% crash in six months (mean of 5 worst global systemically important banks (G-SIBs))



A financial risk indicator

Probability of 30% crash in twelve months (mean of 5 worst global systemically important banks (G-SIBs))



Summary

- We derive bounds on crash probabilities based on option prices
- A priori, we expect the lower bound to perform better, as stocks tend to move with, rather than against, the market
- The lower bound performs well in and out of sample
- It is highly statistically and economically significant, with and without other characteristics emphasized in prior literature
  - t-stats in the range 5 to 13
  - estimated coefficient is 10 times larger than the next most important competitor variable
- Our approach works in real time, so can be used for policy, or to assess reactions to news at high frequency

Probability of a rise of at least 20%

	lower bound				upper bound				risk neutral			
maturity	1	3	6	12	1	3	6	12	1	3	6	12
α	0.00	0.01	0.09	0.34	0.00	0.00	0.03	0.20	0.00	0.00	0.04	0.23
	(0.00)	(0.00)	(0.01)	(0.02)	(0.00)	(0.01)	(0.01)	(0.03)	(0.00)	(0.01)	(0.01)	(0.03)
	[0.00]	[0.01]	[0.01]	[0.03]	[0.00]	[0.01]	[0.02]	[0.04]	[0.00]	[0.01]	[0.02]	[0.03]
β	1.35	1.58	1.30	0.10	0.85	0.91	0.82	0.44	1.03	1.17	1.08	0.49
	(0.13)	(0.11)	(0.11)	(0.14)	(0.09)	(0.08)	(0.08)	(0.09)	(0.11)	(0.09)	(0.09)	(0.12)
	[0.13]	[0.16]	[0.19]	[0.19]	[0.10]	[0.11]	[0.10]	[0.13]	[0.11]	[0.12]	[0.14]	[0.17]
R^2	7.01%	5.78%	2.44%	0.01%	7.42%	6.86%	4.24%	0.80%	7.35%	6.70%	3.80%	0.43%

- For rises, the upper bound would be tight in the comonotonic case
- The sign flip reflects the fact that the crash indicator function $\boldsymbol{I}(R_i \leq q)$ is a decreasing function of R_i, whereas $\boldsymbol{I}(R_i \geq q)$ is an increasing function of R_i