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Sentiment and Speculation in a Market with Heterogeneous Beliefs†

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We present a model featuring risk-averse investors with heterogeneous beliefs. Individuals who are correct in hindsight—whether through luck or judgment—get rich, so sentiment is bullish following good news and bearish following bad news. Sentiment makes extreme outcomes far more important for pricing and has asymmetric effects on left- and right-skewed assets. Investors take speculative positions that can conflict with their fundamental views. Moderate investors are contrarian: they trade against excess volatility created by extremists. All investors view speculation as socially costly; but they also think it is in their self-interest, and the market can collapse entirely if speculation is banned. (*JEL* D81, D83, G11, G12, G41)

In this paper, we study the effect of heterogeneity in beliefs on asset prices. We work with a frictionless dynamically complete market populated by a continuum of risk-averse agents who differ in their beliefs about the probability of good news.

As a result, agents position themselves differently in the market. Optimistic investors make leveraged bets on the market; pessimists go short. If the market rallies, the wealth distribution shifts in favor of the optimists, whose beliefs become overrepresented in prices. If there is bad news, money flows to pessimists and prices more strongly reflect their pessimism going forward. At any point in time, one can define a representative agent who chooses to invest fully in the risky asset, with no borrowing or lending—our analog of Benjamin Graham's "Mr. Market"—but the identity of the representative agent changes every period, with his or her beliefs becoming more optimistic following good news and more pessimistic following bad news. Thus market sentiment shifts constantly despite the stability of individual beliefs.

All agents understand the importance of sentiment and take it into account in the risk premia that they demand, as they correctly foresee that either good or bad news will be amplified by a shift in sentiment. The idea that sentiment itself is a source of systematic price risk appears in De Long et al. (1990), but in our model sentiment

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emerges endogenously rather being modeled as random noise. The presence of sentiment induces speculation: agents take temporary positions, at prices they do not perceive as justified by fundamentals, in anticipation of adjusting their positions in the future.

We start in discrete time, providing a general pricing formula for arbitrary, exogenously specified, terminal payoffs. We find the wealth distribution, prices, and agents' investment decisions at every point in time, together with their subjective perceptions of expected returns, volatilities, and Sharpe ratios; and other quantities of interest, such as aggregate volume, leverage, and the level of the VIX index.

For most of the paper, we focus our attention on heterogeneity in beliefs by working in the limit in which investors have dogmatic priors, as is broadly consistent with the findings of Giglio et al. (2021) and Meeuwis et al. (2019). Although individual investors do not learn in this limit, the market exhibits "the wisdom of the crowd," in that the redistribution of wealth over time causes the market to behave as if it is learning as a whole. That said, our most general formulation allows the agents to learn over time by updating their heterogeneous priors according to Bayes' rule. Following good news, not only do optimists become relatively wealthier, as described above, but also every individual updates his or her beliefs in an optimistic direction. Formalizing this intuition, we show a precise sense in which learning amplifies the effect of belief heterogeneity.

We explore the properties of the model in a series of examples. The first makes the point that extreme states are much more important than they are in a homogeneous-belief economy. A risky bond matures in 50 days, and will default, paying \$30 rather than the par value of \$100, only in the "bottom" state of the world—that is, only if there are 50 consecutive pieces of bad news. Investors' beliefs about the probability, h, of an up-move are uniformly distributed between 0 and 1. Initially, the representative investor is the median agent, $h = 0.5$, who thinks the default probability is less than 10^{-15}. And yet we show that the bond trades at what might seem (given that the riskless interest rate is zero) the remarkably low price of \$95.63. Moreover, almost half the agents—all agents with beliefs h below 0.48—initially go short at this price. Most of these think the asset is fundamentally undervalued. Nonetheless, they go short initially because if there is bad news next period, pessimists' trades will have been profitable, so their views will become overrepresented in the market and the bond's price will decline sharply *in the short run*. Most of the investors who are initially short will therefore reverse their position to go long if there are two periods of bad news. Only extreme pessimists with $h < 0.006$ stay short to the bitter end.

Our second example modifies the first by considering an asset with a *high* payoff in the "top" state of the world. The asset is bubbly—sentiment inflates its price relative to the homogeneous-belief benchmark—and there are several interesting differences in the dynamics relative to the risky bond case. First, sentiment becomes increasingly important as time passes: if repeated good news arrives, the asset becomes more and more bubbly. (By contrast, sentiment has most impact early in the life of the risky bond.) Second, the risk premium perceived by the median investor is initially positive; becomes increasingly negative as the bubble grows; but then starts to rise and ultimately turns positive again at the height of the bubble, just before the terminal date. As a result, the median investor reverses position *twice* during the lifetime of the bubble. Finally, implied volatility, as measured by the VIX index, rises as the bubble grows, whereas the reverse happens in the risky bond example.

In our third example, we construct a stark situation in which there is *no* volatility if beliefs are homogeneous: the asset is (until the final period) totally riskless. When beliefs are heterogeneous, however, our investors speculate on short-run sentiment. The resulting volatility is socially costly, in the sense that average realized utility is lower than it would be if investors were prevented from speculating.

We then consider two continuous-time limits that model information as arriving continuously over time in small pieces (formally, as driven by a Brownian motion), or as arriving infrequently in lumps (formally, as driven by a Poisson process).

In the *Brownian limit*, sentiment drives up true and implied volatility, particularly in the short run, and hence also risk premia; both types of volatility are lower at long horizons due to the moderating influence of the terminal date at which pricing is dictated by fundamentals. Extremists speculate increasingly aggressively as the market moves in their favor, whereas moderate investors trade in contrarian fashion. Among moderates, there is a particularly interesting *gloomy investor*. This Eeyore-like figure (Milne 1926) is somewhat pessimistic, and has the lowest expected utility of all investors. Despite believing that the risky asset earns zero instantaneous risk premium, he thinks that sizable Sharpe ratios can be attained by selling in the face of irrational exuberance on the up side and buying in response to irrational pessimism on the down side. The gloomy investor can therefore be thought of as supplying liquidity to the extremists.

Every investor thinks sufficiently out-of-the-money options are overvalued due to the presence of investors with extreme views. As a result, every investor has a U-shaped stochastic discount factor and a target price—the ideal outcome for that investor, given his or her beliefs, and hence trading strategy—that can usefully be compared to what the investor expects to happen. An extremist is happy if the market moves even more than he or she expected. The gloomy investor, in contrast, hopes to be proved right; in a sense that we make precise, the best outcome for him is the one that he expects.

In the *Poisson limit*, news arrives infrequently. The jumps that occur at such times represent bad news, perhaps driven by credit or catastrophe risk. Optimistic investors sell insurance against jumps to pessimists: as long as things are quiet, wealth flows smoothly from pessimists to optimists, but at the time of a jump there is a sudden shift in the pessimists' favor. Optimists are in the position in which derivative traders inside major financial institutions have traditionally found themselves: short volatility, making money in quiet times but occasionally subject to severe losses at times of market turmoil. As a result, even though all individuals perceive constant jump arrival rates, the risk-neutral jump arrival rate—which can be interpreted as a CDS rate—declines smoothly in the absence of jumps, but spikes sharply after a jump occurs. Similar patterns have been documented in catastrophe insurance markets by Froot and O'Connell (1999) and Born and Viscusi (2006), among others.

Related Literature.—Our paper intersects with several strands of the large literature on the effects of disagreement in financial markets. The closest antecedent of—and the inspiration for—our paper is Geanakoplos (2010), whose paper studies disagreement among risk-neutral investors (as do Harrison and Kreps 1978;

Scheinkman and Xiong 2003). Risk neutrality simplifies the analysis in some respects but complicates it in others. For example, short sales must be restricted for equilibrium to exist. This is natural in some settings, but not if one thinks of the risky asset as representing, say, a broad stock market index; and the resulting kinked indirect utility functions are not very tractable. Moreover, the aggressive trading behavior of risk-neutral investors leads to extreme predictions: every time there is a down-move in the Geanakoplos model, all agents who are invested in the risky asset go bankrupt.

In a variation on the Geanakoplos model, Simsek (2013) emphasizes the importance of the type of disagreement: for example, an agent might be considered relatively optimistic either because she perceives a high chance of some good outcome, or because she perceives a low chance of a bad outcome. In our binomial setting there is no distinction between these alternatives, as an agent who perceives a high chance of "up" must also perceive a low chance of "down," but Simsek allows for more than two—in fact, for a continuum of—possible outcomes. (In the other direction, we have a continuum of investor types whereas Simsek has two.) The mechanisms in the two papers are complementary: Simsek's model features just one period, so his agents (who are risk neutral) do not speculate in our sense.

Other strands of the literature have focused on the role of disagreement in the efficiency of the market (Figlewski 1978), in the amplification of volatility and trading volume (Basak 2005; Banerjee and Kremer 2010; Atmaz and Basak 2018), in the evolution of the wealth distribution (Zapatero 1998; Jouini and Napp 2007; Bhamra and Uppal 2014), in the underreaction of prices to public information (Ottaviani and Sørensen 2015), in amplifying the importance of extremely unlikely states (Kogan et al. 2006), and in the pricing of options (Buraschi and Jiltsov 2006). Other papers generate similar asset-pricing effects by allowing for heterogeneity in risk aversion rather than beliefs (Dumas 1989; Chan and Kogan 2002), though of course they do not account for the direct evidence from surveys that individuals have heterogeneous beliefs (Shiller 1987; Ben-David, Graham, and Harvey 2013).

A related literature addresses the question of which agents will survive into the infinite future (Sandroni 2000; Jouini and Napp 2007; Borovička 2020). Our paper does not directly bear on this question, as we fix a finite terminal horizon. But as the truth lies in the support of every agent's prior in our extended model with learning, all agents would in principle survive to infinity (Blume and Easley 2006).

Most of the prior literature restricts to the diffusion setting (of the papers mentioned, Dumas 1989; Zapatero 1998; Chan and Kogan 2002; Scheinkman and Xiong 2003; Basak 2005; Buraschi and Jiltsov 2006; Kogan et al. 2006; Jouini and Napp 2007; Dumas, Kurshev, and Uppal 2009; Cvitanić et al. 2012; Atmaz and Basak 2018; Borovička 2020); while Banerjee and Kremer (2010) work with a CARA–Normal model, and Geanakoplos (2010) and Simsek (2013) with one- or two-period models. (A notable exception is Chen, Joslin, and Tran (2012), who present a model with heterogeneity in beliefs about disaster risk.)

Our model is extremely tractable, which allows us to study all these issues analytically—together with new results on the implied volatility surface, the variance risk premium, individual investors' trading strategies and attitudes to speculation and so forth—in a simple framework that allows for learning and for general terminal payoffs. This tractability is due in part to our use of log utility, which we view as a reasonable benchmark given the results of Martin (2017); Kremens and Martin (2019); and Martin and Wagner (2019), and which implies (even in a nondiffusion setting) that the representative investor's perceived risk premium is equal to risk-neutral variance so that our model generates empirically plausible first and second moments of returns. It also reflects the fact that we work with a continuum of beliefs, like Geanakoplos (2010) and Atmaz and Basak (2018) but unlike the two-type models of Harrison and Kreps (1978); Scheinkman and Xiong (2003); Basak (2005); Buraschi and Jiltsov (2006); Kogan et al. (2006); Dumas, Kurshev, and Uppal (2009); Banerjee and Kremer (2010); Simsek (2013); Bhamra and Uppal (2014); Borovička (2020). Aside from the evident desirability of having a realistic belief distribution, the identities of the representative investor and of the investor who chooses to sit out of the market entirely then become smoothly varying equilibrium objects that are determined endogenously in an intuitive and tractable way.

I. The Model

We work in discrete time, $t = 0, \ldots, T$. Uncertainty evolves on a binomial tree, so that whatever the current state of the world, there are two possible successor states next period: "up" and "down." There is a risky asset with exogenously specified payoffs at the terminal date T. As our agents will have log utility, these payoffs must be strictly positive so that utility is finite at every node, but they can otherwise be arbitrary. We will assume that the binomial tree is recombining—i.e., that the terminal payoffs depend on the number of total up- and down-moves rather than on the path by which the terminal node is reached—but our approach generalizes to the non-recombining case.

We normalize the net interest rate to zero. This implies that any variation in expected returns, across agents or over time, reflects variation in risk premia.

There is a unit mass of agents indexed by $h \in (0,1)$. Each agent has log utility over terminal wealth, zero time-preference rate, and is initially endowed with one unit of the risky asset, which we will think of as representing "the market," so the risky asset is in unit supply and the riskless asset is in zero net supply. Agent h believes that the probability of an up-move is h; we often refer to h as the agent's *belief*, for short. In most of our examples, payoffs will be higher at nodes closer to the "top" of the tree, so we will think of an up-move as good news, and of agents with higher h as being more optimistic. By working with the open interval $(0,1)$, as opposed to the closed interval $[0,1]$, we ensure that the investors agree on *what events can possibly happen* (more formally, their beliefs are absolutely continuous with respect to each other). As our investors have log utility, they will not allow their wealth to go to zero in any state of the world; there are endogenous limits on the amount of leverage taken by optimists and on the extent of short-selling by pessimists.

The mass of agents with belief h is $f(h)$. We allow $f(h)$ to be an arbitrary probability density function (PDF) throughout Section IA and in our main pricing Result 1. But for much of the paper, we find it convenient to specify that the cross-section of beliefs obeys a beta distribution, so that the PDF is

$$f(h) = \frac{h^{\alpha-1}(1-h)^{\beta-1}}{B(\alpha,\beta)}, \tag{1}$$

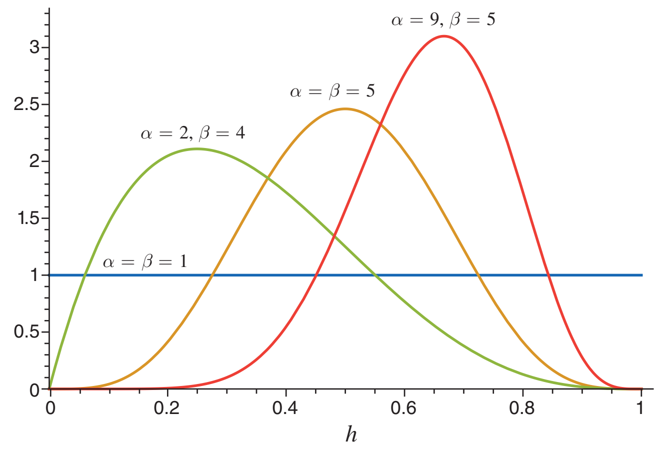

Figure 1. Beta Distributions for Various Choices of α and β

where $\alpha > 0$ and $\beta > 0$ are parameters and $B(\alpha,\beta) = \int_{h=0}^{1} h^{\alpha-1}(1-h)^{\beta-1}dh$ is the beta function.[1] The beta distribution is the conjugate prior for the binomial distribution, which makes the analysis tractable. The flexibility and tractability of the family of beta distributions is particularly important in the continuous-time limits considered in Sections III and IV.

Figure 1 illustrates beta distributions for a range of choices of α and β. If $\alpha = \beta$ then the distribution of beliefs is symmetric with mean $1/2$. In particular, if $\alpha = \beta = 1$ then $f(h) = 1$, so that beliefs are uniformly distributed over $(0,1)$; this is a useful case to keep in mind as one works through the algebra. More generally, the case $\alpha \neq \beta$ allows for asymmetric distributions with mean $\alpha/(\alpha+\beta)$ and variance $\alpha\beta/\left[(\alpha+\beta)^2(\alpha+\beta+1)\right]$. Thus the distribution shifts toward 1 if $\alpha > \beta$ and toward 0 if $\alpha < \beta$, and there is little disagreement when α and β are large: if, say, $\alpha = 90$ and $\beta = 10$ then beliefs are concentrated around a mean of 0.9, with standard deviation 0.03.

A. *Equilibrium*

As agents have log utility over terminal wealth, they behave myopically; we can therefore consider each period in isolation. We start by taking next-period prices at the up- and down-nodes as given, and use these prices to determine the equilibrium price at the current node. This logic will ultimately allow us to solve the model by backward induction, and to express the price at time 0 in terms of the exogenous terminal payoffs.[2]

[1] The beta function is related to the gamma function by $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$. If α and β are integers, then $B(\alpha,\beta) = (\alpha-1)!(\beta-1)!/(\alpha+\beta-1)!$.

[2] A referee pointed out to us that we could also exploit dynamic completeness to solve our model as a static Arrow–Debreu equilibrium. We lay out this approach, which gives a shorter proof of Result 1, in the online Appendix. We take the approach in the body of the paper for expositional reasons, as it allows us to introduce quantities that will be important for understanding the dynamics of the model.

Suppose, then, that the price of the risky asset will be either p_d or p_u next period. Our problem, for now, is to determine the equilibrium price, p, at the current node; we assume that $p_d \neq p_u$ so that this pricing problem is nontrivial. (If $p_d = p_u$ then the asset is riskless so $p = p_d = p_u$.) Suppose also that agent h has wealth w_h at the current node. If he chooses to hold x_h units of the asset, then his wealth next period is $w_h - x_h p + x_h p_u$ in the up-state and $w_h - x_h p + x_h p_d$ in the down-state. So the portfolio problem is

$$\max_{x_h} h \log \left[w_h - x_h p + x_h p_u\right] + \left(1 - h\right) \log \left[w_h - x_h p + x_h p_d\right].$$

The agent's first-order condition is therefore

$$x_h = w_h \left(\frac{h}{p - p_d} - \frac{1 - h}{p_u - p}\right). \tag{2}$$

The sign of x_h is that of $p - p_u$ for $h = 0$ and that of $p - p_d$ for $h = 1$. These must have opposite signs to avoid an arbitrage opportunity, so at every node there are some agents who are short and others who are long. The most optimistic agent[3] levers up as much as possible without risking default. From the perspective of an extreme optimist, p_d can be thought of as the liquidation value: when it is large, the optimist can get more leverage. For, the first-order condition (2) implies that as $h \to 1$, agent h holds $w_h / (p - p_d)$ units of the risky asset. This is the largest possible position that does not risk default: to acquire it, the agent must borrow $w_h p / (p - p_d) - w_h = w_h p_d / (p - p_d)$. If the unthinkable (to this most optimistic agent!) occurs and the down state materializes, the agent's holdings are worth $w_h p_d / (p - p_d)$, which is precisely what the agent owes to his creditors. Correspondingly, the most pessimistic agent takes on the largest short position possible that does not risk default if the good state occurs.

It will often be convenient to think in terms of the risk-neutral probability of an up-move, h^*, defined by the property that the price can be interpreted as a risk-neutral expected payoff, $p = h^* p_u + (1 - h^*) p_d$. (There is no discounting, as the riskless rate is zero.) Hence,

$$h^* = \frac{p - p_d}{p_u - p_d}.$$

In these terms, the first-order condition (2) becomes

$$x_h = \frac{w_h}{p_u - p_d} \frac{h - h^*}{h^*\left(1 - h^*\right)},$$

for example. An agent whose h equals h^* will have zero position in the risky asset: by the defining property of the risk-neutral probability, such an agent perceives that the risky asset has zero expected excess return.

[3] This is an abuse of terminology: there is no "most optimistic agent" since h lies in the *open* set $(0,1)$. More formally, this discussion relates to the behavior of agents in the limit as $h \to 1$. An agent with $h = 1$ would want to take arbitrarily large levered positions in the risky asset, so there is a behavioral discontinuity at $h = 1$ (and similarly at $h = 0$).

Agent h's wealth next period is therefore $w_h + x_h(p_u - p) = w_h(h/h^*)$ in the up-state, and $w_h - x_h(p - p_d) = w_h[(1-h)/(1-h^*)]$ in the down-state. In either case, all agents' returns on wealth are linear in their beliefs. Moreover, this relationship applies at every node. It follows that person h's wealth at the current node is $\lambda_{path} h^m(1-h)^n$, where λ_{path} is a constant that is independent of h but which can depend on the path travelled to the current node, which we have assumed has m up and n down steps.

As aggregate wealth is equal to the value of the risky asset—which is in unit supply—we must have

$$\int_0^1 \lambda_{path} h^m(1-h)^n f(h)dh = p.$$

This enables us to solve for the value of λ_{path}:

$$\lambda_{path} = \frac{p}{\int_0^1 h^m(1-h)^n f(h)dh}.$$

Substituting back, agent h's wealth equals

$$w_h = \frac{h^m(1-h)^n p}{\int_0^1 h^m(1-h)^n f(h)dh}.$$

This is maximized by $h \equiv m/(m+n)$: the agent whose beliefs turned out to be most accurate ex post ends up richest.

The wealth distribution—that is, the fraction of aggregate wealth held by type-h agents—satisfies

$$\frac{w_h f(h)}{p} = \frac{h^m(1-h)^n f(h)}{\int_0^1 h^m(1-h)^n f(h)dh}. \tag{3}$$

The *wealth-weighted cross-sectional average belief*, H, therefore equals

$$H = \int_0^1 h \frac{w_h f(h)}{p} dh = \frac{\int_0^1 h^{m+1}(1-h)^n f(h)dh}{\int_0^1 h^m(1-h)^n f(h)dh} \tag{4}$$

at time t, after m up-moves and n down-moves.[4]

These expressions take a convenient form if $f(h) = h^{\alpha-1}(1-h)^{\beta-1}/B(\alpha,\beta)$ is the density function of the Beta(α,β) distribution. In that case equation (3) implies

[4] If there is a fixed true probability of an up-move, h_{true}, then investors with $h \neq h_{true}$ will eventually become irrelevant. Let us refer to the share of wealth (3) held by type-h agents as $\Omega(h,m,n)$. As the elapsed number of periods, t, tends to infinity, $m/t \to h_{true}$ and $n/t \to 1 - h_{true}$ almost surely, by the strong law of large numbers, and the asymptotic rate of exponential decay in the type-h wealth share is

$$\lim_{t\to\infty} -\frac{1}{t} \log \Omega[h, h_{true}t, (1-h_{true})t] = h_{true} \log \frac{h_{true}}{h} + (1-h_{true}) \log \frac{1-h_{true}}{1-h}.$$

This holds for any belief distribution $f(h)$. The decay rate is strictly positive when $h \neq h_{true}$, so the wealth share of any incorrect investor declines exponentially fast. But investors who are roughly correct will retain a substantial share of wealth for many periods. For example, the half-life for investor $h = 0.5$—that is, $\log 2/$decay rate, the time required for the investor's wealth to halve—is more than 34 periods for all values of h_{true} between 0.4 and 0.6. See the online Appendix for a proof and more discussion.

that the wealth distribution is also a beta distribution with parameters $\alpha + m$ and $\beta + n$, so that

$$\frac{w_h f(h)}{p} = \frac{h^{\alpha+m-1}(1-h)^{\beta+n-1}}{B(\alpha+m,\beta+n)}, \tag{5}$$

and equation (4) simplifies to

$$H = \frac{m+\alpha}{t+\alpha+\beta}.$$

Example.—Let us now revisit Figure 1. For the sake of argument, suppose that $f(h)$ describes a beta distribution with $\alpha = \beta = 1$ so that investor beliefs $h \in (0,1)$ are uniformly distributed. If, by time 4, there have been $m = 1$ up-moves and $n = 3$ down-moves, then equation (5) implies that the new wealth distribution follows the line labeled $\alpha = 2$, $\beta = 4$. (Investors with h close to 0 or to 1 have been almost wiped out by their aggressive trades; the best performers are moderate pessimists with $h = 1/4$, whose beliefs happen to have been vindicated ex post.) At time 8, following three more up-moves and one down-move, the new wealth distribution is indicated by the line labeled $\alpha = \beta = 5$. And if by time 12 there have been a further four up-moves, then the wealth distribution is given by the line labeled $\alpha = 9$, $\beta = 5$. These shifts in the wealth distribution are central to our model. They reflect the fact that money flows, over time, toward investors whose beliefs appear correct in hindsight.

Now we solve for the equilibrium price using the first-order condition described in (2). The price p adjusts to clear the market, so that aggregate demand for the asset by the agents equals the unit aggregate supply:

$$\int_0^1 x_h f(h)dh = \frac{p\left[H(p_u-p) - (1-H)(p-p_d)\right]}{(p_u-p)(p-p_d)} = 1.$$

This has a unique solution with respect to p:

$$p = \frac{p_d p_u}{Hp_d + (1-H)p_u}. \tag{6}$$

In equilibrium, therefore, the risk-neutral probability of an up-move is

$$h^* = \frac{Hp_d}{Hp_d + (1-H)p_u}. \tag{7}$$

It follows that

$$\frac{p_u}{p} = \frac{H}{h^*} \quad \text{and} \quad \frac{p_d}{p} = \frac{1-H}{1-h^*}. \tag{8}$$

Hence, h^* is smaller than H if $p_u > p_d$ and larger than H if $p_u < p_d$: in either case, risk-neutral beliefs are more pessimistic than the wealth-weighted average belief.

The share of wealth an agent of type h invests in the risky asset is

$$\frac{x_h p}{w_h} = \frac{h-h^*}{H-h^*}, \tag{9}$$

using equations (2) and (8). We can use this equation to calculate the leverage of investor h, which we define as the ratio of funds borrowed to wealth:

$$\frac{x_h p - w_h}{w_h} = \frac{h - H}{H - h^*}. \tag{10}$$

The agent with $h = H$ can be thought of as *the representative agent*, Benjamin Graham's Mr. Market: by equation (9), this is the agent who chooses to invest his wealth fully in the market, with no borrowing or lending. Similarly, the individual with $h = h^*$ is an all-cash investor who chooses to hold his or her wealth fully in the bond. Pessimistic investors with $h < h^*$ choose to short the risky asset; moderate investors with $h^* < h < H$ hold a balanced portfolio with long positions in both the bond and the risky asset; and optimistic investors with $h > H$ take on leverage, shorting the bond to go long the risky asset.

For comparison, in a homogeneous economy in which all agents agree that the up-probability is $\bar{H}$, it is easy to check that

$$h^* = \frac{\bar{H} p_d}{\bar{H} p_d + \left(1 - \bar{H}\right) p_u}.$$

Comparing this expression with equation (7), we see that for *short-run* pricing purposes our heterogeneous economy looks the same as a homogeneous economy featuring a representative agent with belief H. But as the identity, H, of the representative agent changes over time in our model, the similarity will disappear when we study the pricing of multiperiod claims.

Investors disagree on risk premia: for example, agent h perceives that the risk premium is

$$\frac{h p_u + (1 - h) p_d}{p} - 1 = \frac{(h - h^*)(H - h^*)}{h^*(1 - h^*)}. \tag{11}$$

By contrast, they agree on objective quantities such as the risk-neutral variance of the asset return, which is

$$h^* \left(\frac{p_u}{p}\right)^2 + (1 - h^*) \left(\frac{p_d}{p}\right)^2 - 1 = \frac{(H - h^*)^2}{h^*(1 - h^*)}. \tag{12}$$

In particular, notice that the representative agent's perceived risk premium, as given in equation (11) with $h = H$, equals risk-neutral variance (12). Inside our model, this quantity (which Martin (2017) argues is a good empirical proxy for the equity premium) is a natural measure of "the market's" risk premium; it equals the wealth-weighted cross-sectional average risk premium, and (12) shows that it is always positive. Conversely, the *equally* weighted cross-sectional average risk premium—the analog of an average survey expectation, as in Ben-David, Graham and Harvey (2013)—may be negative. In the beta case (1), it is given by equation (11) with $h = \alpha/(\alpha + \beta)$.

We can use these expressions to understand the expression for the proportion of wealth invested by an arbitrary agent h in the risky asset,

$$h\text{'s risky share (9)} = \frac{h\text{'s subjective risk premium (11)}}{\text{objective risk-neutral variance (12)}}.$$

This can be compared to the Merton (1969) formula, risky share $= \mu/(\gamma\sigma^2)$, derived in a homogeneous setting with risk premium μ, arbitrary risk aversion $\gamma > 0$, and no distinction between true and risk-neutral variance, σ^2. In our model, agents disagree on true variance but agree on risk-neutral variance; and it turns out that the latter is the quantity that influences the risky share.

We can calculate the level of the VIX index on similar lines.[5] To do so, we use the model-free relationship $\mathrm{VIX}^2_{t\to t+1} = 2(\log \mathbb{E}^*_t R_{t\to t+1} - \mathbb{E}^*_t \log R_{t\to t+1})$, where $R_{t\to t+1}$ is the gross return on the risky asset from t to $t+1$(see, e.g., Martin 2017). As the net riskless rate is zero, $\mathbb{E}^*_t R_{t\to t+1} = 1$; together with the equilibrium relationship (8), this implies that

$$\mathrm{VIX}^2_{t\to t+1} = 2\left[h^*\log\frac{h^*}{H} + (1-h^*)\log\frac{1-h^*}{1-H}\right]. \tag{13}$$

Thus, the VIX index squared equals twice the relative entropy (or Kullback-Leibler divergence) of the beliefs of the representative agent with respect to the beliefs of the all-cash agent. When VIX is high, the two agents have very different beliefs.

The left panel of Figure 2 shows a numerical example with uniformly distributed beliefs and $T = 2$. Sentiment in the heterogeneous belief economy is initially the same as it would be in a homogeneous economy—the representative investor's belief is $H = 0.5$ at the initial node—but the price is lower because of sentiment risk. If bad news arrives, money flows to pessimists, the price declines further than it would in a homogeneous economy, and the previously representative investor with $h = 0.5$ takes a levered long position. Conversely, if good news arrives, money flows to optimists who drive the price up, to the extent that the previously representative investor $h = 0.5$ exits the market entirely and keeps all his money in cash.

The right panel plots the Sharpe ratios perceived by different investors in each of the possible states. As person h's subjectively perceived variance of the asset's return is

$$h\left(\frac{p_u}{p}\right)^2 + (1-h)\left(\frac{p_d}{p}\right)^2 - \left(\frac{hp_u + (1-h)p_d}{p}\right)^2 = \frac{h(1-h)(H-h^*)^2}{h^{*2}(1-h^*)^2},$$

his or her perceived Sharpe ratio is

$$\frac{h-h^*}{\sqrt{h(1-h)}},$$

which is increasing in h for all h^*. Extremists perceive extreme Sharpe ratios, reflecting their perception that true volatility is close to zero. This might seem surprising, given the heuristic that second moments of returns are relatively easy to measure empirically, and hence relatively difficult to disagree upon. Indeed this is to some extent an artifact of the two-period setting of the present example: in the Brownian limit of Section III, all agents disagree about expected returns, but agree on the volatility of returns.

[5] By definition, $\mathrm{VIX}^2_{t\to t+1} \equiv 2R_{f,t}\left(\int_0^{F_t}\frac{1}{K^2}put_t(K)dK + \int_{F_t}^{\infty}\frac{1}{K^2}call_t(K)dK\right)$, where $R_{f,t}$ is the gross one-period interest rate, F_t is the one-period-ahead forward price of the risky asset, and $\mathrm{put}_t(K)$ and $\mathrm{call}_t(K)$ are time t prices of one-period put and call options with strike K.

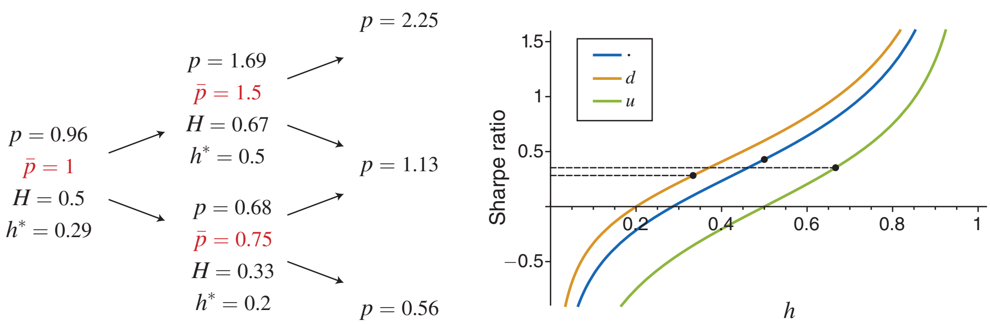

FIGURE 2

Notes: Left: the price in a homogeneous economy with $\overline{H} = 1/2$ is denoted by $\overline{p}$; the price in a heterogeneous economy with $\alpha = \beta = 1$ is denoted by p; and h^* and H indicate the risk-neutral probability of an up-move and the identity of the representative agent in the heterogeneous economy. Right: the Sharpe ratio perceived by different agents in the initial state $(\cdot)$, down state (d), and up state (u).

The figure also shows that all investors believe that Sharpe ratios are high in bad times and low in good times. Thus the model does not generate extrapolative beliefs (as studied empirically by Greenwood and Shleifer (2014) and theoretically by Barberis et al. 2015) on the part of individual investors. But the *representative* investor (whose identity in each state is indicated by dots in the right panel) is more optimistic, and perceives a higher Sharpe ratio, in good times than in bad times. Our model generates extrapolative behavior in a dollar-weighted sense: Mr. Market disagrees with *every* individual investor about the behavior of Sharpe ratios in good and bad states.

B. *The General Case*

From now on we will keep track of the current node by writing subscripts to indicate the number of up-moves to date and total time elapsed. Thus, for example, $p_{m,t}$ is the price at time t after m up-moves and $n = t - m$ down-moves, and $H_{m,t}$ and $h^*_{m,t}$ represent the identities of the representative investor and of the investor who is fully invested in cash, respectively. Translating the notation of the last section into this new notation, we have $p = p_{m,t}$, $p_u = p_{m+1,t+1}$, $p_d = p_{m,t+1}$, $H = H_{m,t}$, and $h^* = h^*_{m,t}$. We also sometimes write p_0 for $p_{0,0}$.

Writing $z_{m,t} = 1/p_{m,t}$, equation (6) implies that the following recurrence relation holds at each node:

$$z_{m,t} = H_{m,t} z_{m+1,t+1} + (1 - H_{m,t}) z_{m,t+1}. \tag{14}$$

That is, the price at each node is the weighted harmonic mean of the next-period prices, with weights given by the beliefs of the currently representative agent. This leads to our main pricing result, whose proof is in the Appendix.

RESULT 1: *If the risky asset has terminal payoffs* $p_{m,T}$ *at time* T (*for* $m = 0, \ldots, T$), *then its initial price is*[6]

$$p_0 = \frac{1}{\sum_{m=0}^{T} \frac{c_m}{p_{m,T}}}, \quad where \quad c_m = \binom{T}{m} \int_0^1 h^m (1-h)^{T-m} f(h) dh.$$

The coefficient c_m can be interpreted as the cross-sectional average perceived probability of reaching node m. For reference, note that if the cross-section of beliefs obeys a beta distribution, as in equation (1), then c_m satisfies

$$c_m = \binom{T}{m} \frac{B(\alpha + m, \beta + T - m)}{B(\alpha, \beta)}.$$

Conversely, if beliefs are homogeneous and all agents perceive that the probability of an up-move is $\bar{H}$, then $c_m = \binom{T}{m} \bar{H}^m (1 - \bar{H})^{T-m}$.

In our setting, pricing is the same as it would be if a single representative investor with appropriately chosen prior beliefs learned over time about the probability of an up-move. Although such a model is inconsistent with the evidence that individuals have different beliefs, the link reveals a sense in which the market exhibits "the wisdom of the crowd," in that the redistribution of wealth between agents causes the market to behave as if it is learning as a whole.[7]

RESULT 2 (The wisdom of the crowd): *Pricing in the heterogeneous-agent economy is identical to pricing in an economy with a representative agent with log utility whose prior belief, as of time 0, about the probability of an up-move has distribution* $f(h)$, *and who updates his or her beliefs over time via Bayes' rule.*

The next result characterizes the effect of belief heterogeneity on prices for a broad class of assets. In this result, and for the rest of the paper, we restrict attention to the beta family of belief distributions, so that $f(h)$ is the PDF given in equation (1).

RESULT 3: *If the risky asset has terminal payoffs such that* $1/p_{m,T}$ *is convex when viewed as a function of* m, *then the asset's time 0 price decreases as beliefs become more heterogeneous* (*that is, as the cross-sectional variance of* h *increases with its mean held constant*). *In particular, it is sufficient, though not necessary, that* $\log p_{m,T}$ *be weakly concave for the asset's price to be decreasing in the degree of belief heterogeneity.*

Conversely, if $1/p_{m,T}$ *is concave in* m *then the asset's time* 0 *price increases as beliefs become more heterogeneous.*

Result 3 implies that if the risky asset's terminal payoff $p_{m,T}$ is concave in m, then its price declines as heterogeneity increases. But the same may be true even

[6] It is of course also possible to represent the price as $p_0 = \sum_{m=0}^{T} q_m^* p_{m,T}$ for appropriate risk-neutral probabilities, q_m^*, that we provide in the Appendix. In equilibrium, these risk-neutral probabilities are such that the formula in Result 1 holds.

[7] The existence of a representative investor in this sense is guaranteed by the results of Rubinstein (1976). Our result makes explicit what the beliefs of such an investor must be. See Blume and Easley (1993) and Blume and Easley (2010) for further discussion.

for assets with convex payoffs—for example, if the asset's payoffs are exponential in m then the log payoff is linear, and hence weakly concave, in m. The examples of Sections 3 and 4 fall into this category. On the other hand,[8] if the risky asset has highly convex payoffs—as might be the case for a "growth" asset with a large payoff in some extreme state of the world—then its price increases with heterogeneity. We explore the two cases further via concrete examples in Sections IIA and IIB.

Notice that if there were only one period, then—with only two possible terminal payoffs—$1/p_{m,T}$ would be both convex and concave in m, and hence the asset's price would be independent of the degree of heterogeneity. Thus the dynamic aspect of our model is critical for Result 3 to be interesting and nontrivial. It is therefore complementary to Simsek (2013, Theorems 4 and 5), who presents results with a superficially similar flavor in a static, one-period, model: whereas Result 3 characterizes the impact of heterogeneity in terms of the concavity or convexity of the asset's payoffs, Simsek emphasizes the importance of the degree of skewness of beliefs in the minds of investors.

C. *Bayesian Learning*

We can extend our model to allow the heterogeneous individuals to update their beliefs over time using Bayes' rule. We continue to assume that each investor has a type $h \in (0,1)$, and that types follow a beta distribution with parameters α and β, as in equation (1). Now, however, investor h's prior belief is that the probability of an up-move is $\tilde{h} \sim \text{Beta}(\zeta h, \zeta(1-h))$. This prior has mean h and variance $h(1-h)/(1+\zeta)$, so is sharply peaked around h when the (positive) constant ζ is large. This structure allows us to calibrate the disagreement across individuals in the population, which is controlled by α and β, separately from the uncertainty in the mind of a fixed individual, which is controlled by ζ. In the limit as ζ tends to infinity, we recover the dogmatic limiting case considered in the rest of the paper.

RESULT 4 (*Pricing with belief heterogeneity and learning*): *If the risky asset has terminal payoffs $p_{m,T}$ at time T (for $m = 0,\ldots,T$), then its initial price is*

$$p_0 = \frac{1}{\sum_{m=0}^{T}\frac{\tilde{c}_m}{p_{m,T}}}, \quad \textit{where} \quad \tilde{c}_m = \binom{T}{m}\int_0^1 \frac{B\big(\zeta h + m, \zeta(1-h) + T - m\big)}{B\big(\zeta h, \zeta(1-h)\big)} f(h)\,dh,$$

where $f(h)$ is the PDF of a beta distribution, as defined in equation (1).

If the risky asset has terminal payoffs such that $1/p_{m,T}$ is convex when viewed as a function of m, then for any level of belief heterogeneity the asset's time 0 *price decreases as investors' prior uncertainty increases* (*i.e., as ζ decreases, with $\alpha/(\alpha+\beta)$ held constant so that the mean investor type is held constant*). *Conversely, if $1/p_{m,T}$ is concave in m then the asset's time 0 price increases as investors' prior uncertainty increases.*

[8] The empirical evidence concerning the effect of belief dispersion on prices is mixed. Johnson (2004) considers levered firms with option-like payoffs and finds that price is increasing in belief dispersion (see also Chen, Hong, and Stein 2002; Yu, 2011) while Avramov et al. (2009), Banerjee (2011) and others find the opposite result.

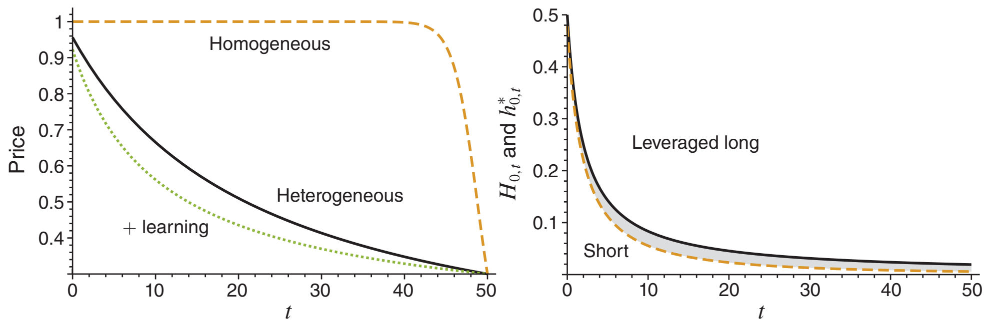

FIGURE 3

Notes: Left: the risky bond's price over time in the heterogeneous and homogeneous economies following consistently bad news. Right: the identity, at time t, following consistently bad news, of the representative agent, $H_{t,t}$; and of the investor who is fully invested in the riskless bond at time t, with zero position in the risky bond, $h^*_{0,t}$.

This result generalizes Results 1 and 2 in the case where $f(h)$ is the beta PDF. (To recover the former, let ζ tend to infinity; to recover the latter, set $\alpha = aN$, $\beta = bN$, and $\zeta = a + b$, and let N tend to infinity.) It shows that the effect of learning compounds the effect of sentiment, thereby putting Result 3 into a broader context. In the online Appendix, we show how the initial price of the risky asset depicted in the left panel of Figure 2 varies when agents learn, for a range of values of $\zeta < \infty$; and we illustrate the effect of learning in Figure 3 of the next section. Elsewhere, we focus on the dogmatic limit case $\zeta \to \infty$.

II. Three Examples

We now explore a series of examples. These are highly stylized even by the standards of our highly stylized model, but they allow us to emphasize some important features of the framework.

A. *A Risky Bond*

The dynamic that drives our model is particularly stark in the risky bond example outlined in the introduction. Suppose that the terminal payoff is 1 in all states apart from the very bottom one, in which the bond defaults with payoff ε. The price of the asset is therefore 1 as soon as an up-move occurs. But if bad news keeps coming, then the price at time t following t consecutive down-moves satisfies[9]

$$p_{0,t} = \frac{1}{1 + \frac{1-\varepsilon}{\varepsilon} \frac{\Gamma(\beta + T)\Gamma(\alpha + \beta + t)}{\Gamma(\beta + t)\Gamma(\alpha + \beta + T)}}, \tag{15}$$

[9]In this special case, we could argue directly: from equation (6), $p_{0,t} = \frac{\alpha p_{0,t+1} + (t+\beta)p_{0,t+1}}{\alpha p_{0,t+1} + t + \beta}$. Defining $y_t \equiv 1/p_{0,t} - 1$, this can be rearranged as $y_t = \frac{\beta + t}{\alpha + \beta + t} y_{t+1}$. Solving forward, imposing the terminal condition that $y_T = (1-\varepsilon)/\varepsilon$, and using the fact that $\Gamma(z+1)/\Gamma(z) = z$ for any $z > 0$, we have (15).

by Result 1. If the belief distribution is uniform, $\alpha = \beta = 1$, we can simplify further, to

$$p_{0,t} = \frac{1}{1 + \frac{1-\varepsilon}{\varepsilon}\frac{1+t}{1+T}}. \tag{16}$$

We can determine the identity of the all-cash investor, following t consecutive down-moves, by applying (8) with $p = p_{0,t}$ and $p_u = 1$ to find that

$$h^*_{0,t} = H_{0,t}p_{0,t} = \frac{\alpha p_{0,t}}{\alpha + \beta + t}.$$

Figure 3 shows how the bond price and the identities of the representative agent and of the all-cash investor evolve, assuming bad news arrives each period, in an example with uniform beliefs ($\alpha = \beta = 1$), $T = 50$ periods, and a recovery value of $\varepsilon = 0.30$. (The left panel also shows how the price evolves if investors have heterogeneous priors and learn about the probability of a down-move as in Section IC. We set $\zeta = 24$ so that the standard deviation of the median investor's prior belief about the probability of an up-move is 10 percent.)

For comparison, in a homogeneous economy with $H = 1/2$ the price and risk-neutral probability would be

$$p_{0,t} = \frac{1}{1 + \frac{1-\varepsilon}{\varepsilon}2^{-(T-t)}} \tag{17}$$

and $h^*_{0,t} = p_{0,t}/2$, respectively. Thus with homogeneous beliefs the bond price is approximately 1 and the risk-neutral probability of an up-move, $h^*_{0,t}$, is approximately $1/2$ until shortly before the bond's maturity.

From the perspective of time 0, the risk-neutral probability of default, δ^*, satisfies $p_0 = 1 - \delta^* + \delta^*\varepsilon$, so $\delta^* = (1 - p_0)/(1 - \varepsilon)$. In the homogeneous case, therefore, $\delta^* = 1/\big(1 + \varepsilon(2^T - 1)\big) = O(2^{-T})$, whereas in the heterogeneous case with uniform belief distribution we have $\delta^* = 1/(1 + \varepsilon T) = O(1/T)$. There is a qualitative difference between the homogeneous economy, in which default is exponentially unlikely under the risk-neutral distribution, and the heterogeneous economy, in which default is only polynomially unlikely. More generally, it is straightforward to show that $\delta^* = O(T^{-\alpha})$ for any α and β, using Stirling's formula. And the result remains true if $\varepsilon > 1$, as in the bubbly asset example of the next section: the risk-neutral probability of the bubbly asset having a large payoff is exponentially small in the homogeneous economy but only polynomially small in the heterogeneous belief economy.

To understand pricing in the heterogeneous economy, it is helpful to think through the portfolio choices of individual investors. The median investor, $h = 0.5$, thinks the probability that the bond will default—i.e., that the price will follow the path shown in Figure 3 all the way to the end—is $2^{-50} < 10^{-15}$. Even so, he believes the price is right at time zero (in the sense that he *is* the representative agent) because of the short-run impact of sentiment. Meanwhile, a modestly pessimistic agent with $h = 0.25$ will choose to short the bond at the price of 0.9563—and will remain

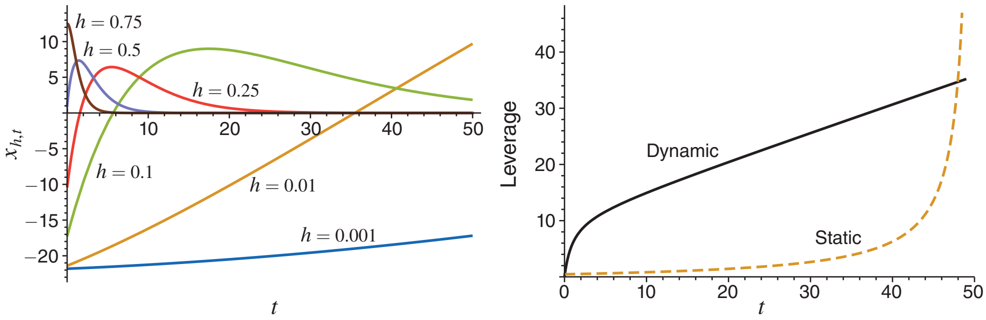

Figure 4

Notes: Left: the number of units of the risky bond held by different agents, $x_{h,t}$, plotted against time. Right: the evolution of leverage for the median investor under the optimal dynamic and static strategies. Both panels assume bad news arrives each period.

short at time $t = 1$ before reversing her position at $t = 2$—despite believing that the bond's default probability is less than 10^{-6}. (Recall from equation (9) that $h^*_{0,t}$ is the belief of the agent who is neither long nor short the asset. More optimistic agents, $h > h^*_{0,t}$, are long, and more pessimistic agents, $h < h^*_{0,t}$, are short.) Following a few periods of bad news, almost all investors are long; but the most pessimistic investors are *rich*.

The left panel of Figure 4 shows the holdings of the risky asset for a range of investors with different beliefs, along the trajectory in which bad news keeps on coming. The optimistic investor $h = 0.75$ starts out highly leveraged so rapidly loses almost all his money. The median investor, $h = 0.5$, initially invests fully in the risky bond without leverage. If bad news arrives, he levers up to increase the size of his position despite his losses; if bad news keeps coming, he is almost completely wiped out after about 10 periods. Moderately bearish investors start out short: investor $h = 0.25$ starts out short about 10 units of the bond, despite believing that the probability it defaults is less than one in a million, but reverses her position after two down-moves. Investor $h = 0.01$, who thinks that there is more than a 60 percent chance of default, is initially extremely short but eventually reverses position as still more bearish investors come to dominate the market.

The right panel of Figure 4 shows how the median investor's leverage (10) evolves over time. If forced to trade statically, his leverage ratio is initially 0.457—a seemingly modest number dictated by the requirement that the investor avoid bankruptcy at the bottom node. If he can trade dynamically, the optimal strategy starts out fully invested in the risky bond, with no leverage. Subsequently, however, leverage rises fast. Thus the median investor "keeps his powder dry," investing cautiously at first but levering up after further selloffs. (We report further results on the evolution of aggregate leverage and volume in the online Appendix.)

Having seen these results, it might still seem surprising that an investor with (say) $h = 0.25$, who perceives less than a one-in-a-million chance of default, goes short when the risky bond is almost certain to deliver an excess return on the order of five per cent. But it is important to bear in mind that there is a sharp distinction between investors' speculative trading strategies and what one might call their "fundamental

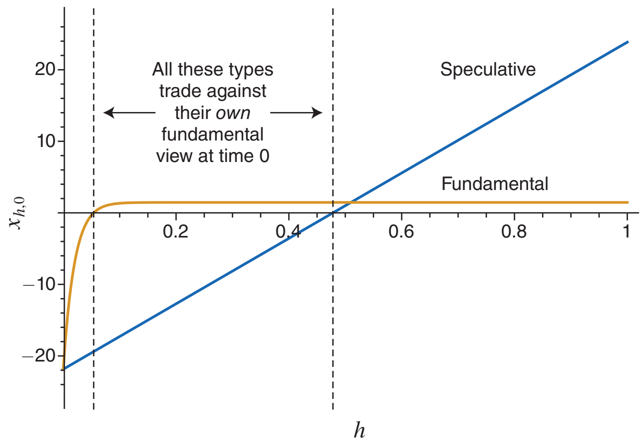

Figure 5. Positions of Different Investors at Time 0 under Dynamic ("Speculative") and Static ("Fundamental") Trade

view." We define the latter as the position that an investor would choose, at time 0, *if forced to hold the position statically*.[10] Figure 5 shows how investors of different types, h, position themselves when they can speculate (calculated from equation (2)), and compares with the position they would choose in the static case (as calculated in the online Appendix). More than 40 percent of the investor population—all investors with h between 0.054 and 0.48—trade in the opposite direction to their own perception of fundamental value. If forced to hold a static position, they would go long at time 0. But when speculation is possible, they choose to go *short* initially due to the anticipated short-run impact of sentiment, before reversing the trade to go long if sentiment worsens. To an observer, this behavior might superficially appear to be manipulative, but here it arises in a perfectly liquid and frictionless market.

Figure 5 illustrates a second important point. When dynamic trade is possible, optimists trade aggressively because they can reduce their position sizes to avoid bankruptcy if events start to turn against them. In the static equilibrium, by contrast, an investor must confront the possibility of eventual default from the start. This makes the endogenous leverage limit tighter, so most investors have almost exactly the same position: they take (approximately) the largest position that avoids the possibility of bankruptcy in the event of default. In the Brownian limit of Section 3, we will see that related logic causes the market to collapse entirely if dynamic trade is ruled out.

B. *A Bubbly Asset*

We now modify the example of the previous section by considering the case in which the extreme payoff ε is greater than 1. This seemingly trivial modification will reveal the differing effects of sentiment on assets with left- and right-skewed

[10] In this particular example, the price in a static economy would be the same as it is in the dynamic economy, as we show in the online Appendix; but this is not true in general.

payoffs. In this case we refer to the asset as *bubbly* because—in contrast with the risky bond case—sentiment will now inflate its price, by Result 3.

As the extreme payoff now corresponds to a good, rather than bad, outcome, we will think of the asset as paying $\varepsilon > 1$ in the "top" state, i.e. if there are T consecutive up moves, and 1 otherwise. Hence, the price of the asset is 1 if ever there is a down move.

But our interest now is in the evolution of the price if there is repeated good news. For general $\alpha > 0$ and $\beta > 0$, the price at time t, following t up moves, is

$$p_{t,t} = 1 \Big/ \left[1 + \tfrac{1-\varepsilon}{\varepsilon} \frac{\Gamma(\alpha + T)}{\Gamma(\alpha + t)} \frac{\Gamma(\alpha + \beta + t)}{\Gamma(\alpha + \beta + T)} \right]. \tag{18}$$

Thus the price rises with each successive piece of good news. In part, of course, these rises simply reflect good news about fundamentals, which would also cause price rises in a homogeneous-belief economy.

To isolate the influence of belief heterogeneity on pricing, we therefore define the *sentiment multiplier* as the ratio of the price (18) to the price that would prevail in a homogeneous economy in which all agents perceive $h = \alpha/(\alpha + \beta)$. Along the path on which good news keeps on coming,

$$\text{sentiment multiplier}_t = \frac{p_{t,t}}{\bar{p}_{t,t}} = \frac{1 + \tfrac{1-\varepsilon}{\varepsilon}\left(\frac{\alpha}{\alpha+\beta}\right)^{T-t}}{1 + \tfrac{1-\varepsilon}{\varepsilon} \frac{\Gamma(\alpha + T)}{\Gamma(\alpha + t)} \frac{\Gamma(\alpha + \beta + t)}{\Gamma(\alpha + \beta + T)}}. \tag{19}$$

We define the sentiment multiplier for the risky bond analogously. Heterogeneity in beliefs drives the bubbly asset's price up but the risky bond's price down, by Result 3, so the sentiment multiplier is initially greater than 1 for the bubbly asset and less than 1 for the risky bond. In either case it equals 1 at the terminal time T (as the price is then equal to the payoff whether or not there is heterogeneity in beliefs).

The left panel of Figure 6 shows the evolution of the sentiment multiplier over time along the path in which the extreme outcome—the "top" outcome in the case of the bubbly asset, and the "bottom" outcome in the case of the risky bond—remains possible. Beliefs are uniformly distributed between zero and one, that is, $\alpha = \beta = 1$. We set $\varepsilon = 25$ for the bubbly asset and (symmetrically, from the point of view of a log investor) $\varepsilon = 1/25$ for the risky bond, and $T = 20$.

Sentiment has little effect on the pricing of the bubbly asset early in its life, but becomes much more important following repeated good news: the multiplier is initially only slightly greater than one, but accelerates—it is convex, even on a log scale—toward a peak shortly before time T. Conversely, sentiment has a substantial effect on the price of the risky bond early in its life. We provide a formal result along these lines for arbitrary α and β below.

The risk premium perceived by the median investor is positive at first—though small because, as we will see, volatility is initially very low. As the bubble emerges, the median investor's perceived risk premium turns negative and declines as sentiment drives the asset's price up. But it then starts to rise, and ultimately turns positive

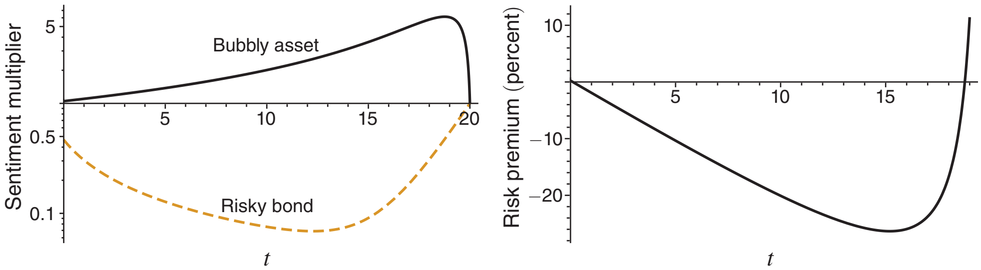

Figure 6

Notes: Left: sentiment multipliers along the extreme paths. Log scale. Right: the risk premium on the bubbly asset, as perceived by the median investor.

toward the height of the bubble, as the terminal date T approaches (Figure 6, right panel).[11]

These facts have a striking implication: the median investor reverses his position *twice* over the lifetime of the bubble. He starts out long, as the representative investor at time 0. Following good news, he goes short as optimists drive the price higher than he thinks reasonable. But if good news keeps coming, he reverses position a second time to go long again at time $T - 1$.[12]

Implied volatility, as measured by the VIX index, rises as the bubbly asset experiences repeated good news (Figure 7, left panel).[13] Conversely, the VIX index declines as the risky bond experiences repeated bad news. The central asymmetry that separates the two examples is that risk considerations drive the price down and *toward* the extreme payoff for the risky bond, but down and *away* from the extreme payoff for the bubbly asset.

We can view the behavior of the median investor through this lens. As shown in equation (13), movements in VIX measure the difference in beliefs between the representative agent $(h = H_{t,t})$ and the investor who is out of the market and on the boundary between the longs and shorts $(h = h^*_{t,t})$. As there is a limit to how optimistic the representative agent can become, $h^*_{t,t}$ must eventually decline to open the gap as VIX rises along the bubble path (Figure 7, right panel, which should be contrasted with Figure 3, right panel)—to the extent that the median agent ends up long, i.e., $h^*_{t,t} < 1/2$.

We close this section with a result that applies for arbitrary values of α and β. In order to formulate a clean statement, we consider the two extreme cases in which ε tends to zero or to infinity. We refer to these as the *risky bond limit* and *bubbly asset*

[11] By contrast, in the risky bond example, the median investor perceives that the risk premium rises monotonically along the extreme price path.

[12] Somewhat more optimistic agents are long for a more extended period at both the beginning and the end. For example, investor $h = 0.80$ starts out long, goes short in period 5, and reverses position to go long again in period 15, as can be seen in Figure 7, right panel.

[13] The relationship between volatility and bubbles has been widely noted. For example, Cochrane (2003) links high volatility to the high prices of growth stocks around the turn of the millennium; more recently, Gao and Martin (2021) argue empirically that the high and rising level of implied volatility in the late 1990s points to bubbliness in the stock market at the time.

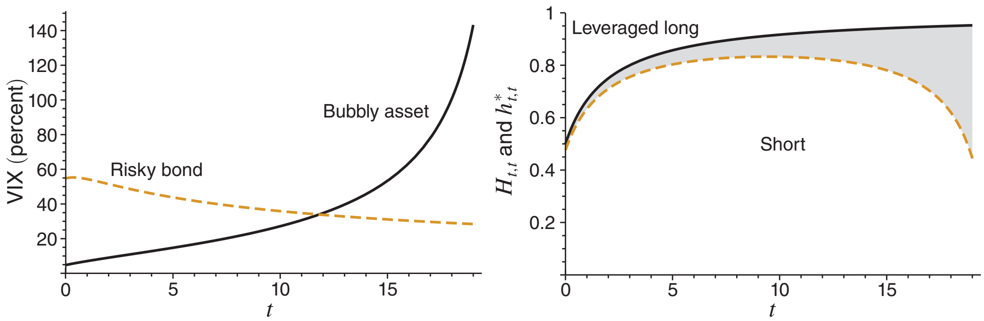

FIGURE 7

Notes: Left: the evolution of the one-period VIX index along the extreme paths. Right: the identity, t, following consistently good news, of the representative agent, $H_{t,t}$ (solid); and of the investor who is fully invested in the riskless bond, $h^*_{t,t}$ (dashed).

limit, respectively. The result formalizes a sense in which sentiment has most impact early in the life of a left-skewed asset, but late in the life of a right-skewed asset.

RESULT 5: *Let $\alpha > 0$ and $\beta > 0$ be arbitrary.*

In the risky bond limit, the sentiment multiplier is less than 1 *for* $0 \le t \le T-1$. *Sentiment becomes less important over time along the extreme path: the multiplier increases monotonically from a minimum at time 0 to 1 at time T.*

In the bubbly asset limit, the sentiment multiplier is greater than 1 *for* $0 \le t \le T-1$. *Sentiment becomes more important over time along the extreme path: the multiplier increases from time* 0 *toward a maximum at time* $T-1$, *before dropping back to* 1 *at time T.*

C. *Speculation on Sentiment*

Many authors have noted that markets exhibit more volatility than seems justified by fundamental news; classic references include Shiller (1981) and Roll (1984). In our setting, investors may speculate on sentiment even if there is essentially no news arriving about fundamentals. In doing so, they generate excess volatility.

Consider an example in which the asset pays off $1/(1+\varepsilon)$, where $0 < \varepsilon < 1$, if there have been an even number of up-moves by the terminal date, and $1/(1+\varepsilon)$ if there have been an odd number of up-moves. Suppose further that T is odd, so that there are an even number of terminal nodes.

In a homogeneous economy with $\bar{H} = 1/2$, the asset trades at price $\bar{p} = 1$ at every node until the terminal payoff: there is therefore *no* volatility, and the asset is riskless until the final period.

With heterogeneity in beliefs, it follows from Result 1 that the initial price is 1 if the distribution of beliefs is symmetric around $1/2$ (in the sense that $f(h) = f(1-h)$ for all h).[14] But sentiment creates volatility and time-varying risk

[14] This implies that $c_m = c_{T-m}$, and hence as T is odd, $\sum_{m=0}^{T}(-1)^m c_m = 0$. Together with the fact that $\sum_{m=0}^{T} c_m = 1$, this gives the result. (The initial price is also 1 if there is learning, as $\tilde{c}_m$, defined in Result 3, also has the properties $\sum_{m=0}^{T}(-1)^m \tilde{c}_m = 0$ and $\sum_{m=0}^{T} \tilde{c}_m = 1$.)

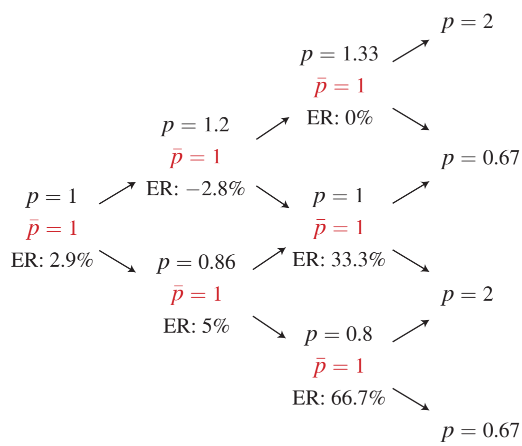

Figure 8

Notes: Speculation on sentiment. Heterogeneous-economy price (p), homogeneous-economy price ($\bar{p}$), and the cross-sectional average expected excess return in the heterogeneous economy (ER).

premia in the minds of all investors. Suppose that beliefs are uniformly distributed, as they are in Figure 8, which illustrates with $T = 3$ and $\varepsilon = 1/2$. Then, at time 1, the risky asset's price rises to $1/(1 - \varepsilon/T)$ if there is an up-move but drops to $1/(1 + \varepsilon/T)$ if there is a down-move.[15]

In equilibrium, our investors use the risky asset to speculate against each other. This cannot end well for all of them. If we write $\tilde{\mathbb{E}}$ to indicate a cross-sectional mean and $R^{(h)}_{0\to T}$ to denote the return on agent h's chosen strategy, then we have $\tilde{\mathbb{E}}\, R^{(h)}_{0\to T} = R_{0\to T}$, because (by market clearing) the cross-sectional average return on investors' strategies equals the return on aggregate wealth. It follows, by Jensen's inequality, that cross-sectional average realized utility is lower than it would be if all agents held the risky asset statically:

$$\tilde{\mathbb{E}}\ \log R^{(h)}_{0\to T} < \log \tilde{\mathbb{E}}\, R^{(h)}_{0\to T} = \log R_{0\to T}.$$

In this sense, speculation is socially costly. But every investor believes that speculation is in his or her self-interest: the ability to speculate raises expected utility above what is attainable by statically holding the risky asset. We return to this point in a more conventional example in Sections IIIC and IIID.

III. A Brownian Limit

In this section, we consider a natural continuous time limit by allowing the number of periods to tend to infinity and specifying geometrically increasing terminal payoffs. This is the setting of Cox, Ross, and Rubinstein (1979), in which the option

[15] This follows from Lemma 1 in the Appendix. There is also an equilibrium in which the asset's price is 1 until time $T - 1$, as in the homogeneous economy. Then the market is incomplete, and agents have no means of betting against one another. But this equilibrium is not robust to vanishingly small generic perturbations of the terminal payoffs, which would restore market completeness.

price formula of Black and Scholes (1973) emerges in the corresponding limit with homogeneous beliefs.

We divide the time interval from time 0 to time T into $2N$ periods of length $T/(2N)$. (The choice of an even number of periods is unimportant, but it simplifies the notation in some of our proofs.) Terminal payoffs are $p_{m,T} = \exp\left[2\sigma\sqrt{\frac{T}{2N}}(m-N)\right]$, as in the Cox, Ross, and Rubinstein (1979) model. As we will see, σ can be interpreted as the volatility of log terminal payoffs, on which all agents agree.

As the number of steps increases, the extent of disagreement over any individual step must decline to generate sensible limiting results. We achieve this by setting $\alpha = \theta N + \eta\sqrt{N}$ and $\beta = \theta N - \eta\sqrt{N}$ in (1), which makes the distribution of h—that is, of investors' beliefs about the probability of a single up-move—increasingly spiky as N increases. Small values of θ correspond to substantial belief heterogeneity, while the limit $\theta \to \infty$ represents the homogeneous case. The parameter η allows for asymmetry in the distribution of beliefs. Using tildes to denote cross-sectional means and variances, the cross-sectional mean of h satisfies $\tilde{\mathbb{E}}[h] = \frac{1}{2} + \frac{\eta}{2\theta\sqrt{N}}$ and $\widetilde{\text{var}}[h] = \frac{1}{8\theta N} + O\left(\frac{1}{N^2}\right)$.

Given that, by design, the cross-sectional variance of h shrinks toward zero, it becomes convenient to parametrize an agent by the number of standard deviations, $z = \left(h - \tilde{\mathbb{E}}h\right)/\sqrt{\widetilde{\text{var}}h}$, by which his or her belief deviates from the mean. Thus an agent with $z = 2$ is two standard deviations more optimistic than the mean agent. Standard results on the beta distribution imply that the cross-sectional distribution of z is asymptotically standard Normal. When we use this parametrization, we write superscripts z rather than h: for example, $\mathbb{E}^{(z)}$ rather than $\mathbb{E}^{(h)}$.

Using Result 1 to price the asset, then taking the limit as $N \to \infty$, we have the following result.

RESULT 6: *The price of the asset at time 0 is*

$$p_0 = \exp\left(\frac{\eta}{\theta}\sigma\sqrt{2T} - \frac{\theta + 1}{2\theta}\sigma^2 T\right).$$

Consistent with Result 3, the price declines as beliefs become more heterogeneous (i.e., as θ decreases with η/θ, and hence the mean level of optimism, held constant).

We now study agents' return expectations.

RESULT 7: *The return of the asset from time 0 to time t is lognormally distributed from the perspective of agent z, with*

$$\mathbb{E}^{(z)} \log R_{0\to t} = \frac{\theta + 1}{\theta + \frac{t}{T}}\left(\frac{z\sigma}{\sqrt{\theta T}} + \frac{\theta + 1}{2\theta}\sigma^2\right)t$$

$$\text{var}^{(z)} \log R_{0\to t} = \left(\frac{\theta + 1}{\theta + \frac{t}{T}}\right)^2 \sigma^2 t.$$

Thus agents agree on the second moment but disagree on the first moment of log returns. Agents also agree that log returns are negatively autocorrelated, with more negative autocorrelations at longer horizons: for $t \leq T/2$,

$$\operatorname{corr}^{(z)}\left(\log R_{0\to t}, \log R_{t\to 2t}\right) = -\frac{1}{\sqrt{1 + \left(1 + \frac{\theta T}{t}\right)^2}}.$$

The annualized expected return of the asset from 0 to t is

$$\frac{1}{t} \log \mathbb{E}^{(z)} R_{0\to t} = \frac{\theta + 1}{\theta + \frac{t}{T}} \left[\frac{z\sigma}{\sqrt{\theta T}} + \frac{\theta + 1}{2\theta} \frac{2\theta + \frac{t}{T}}{\theta + \frac{t}{T}} \sigma^2 \right].$$

The cross-sectional mean (*or median*) *expected return is*

$$\tilde{\mathbb{E}}\left[\frac{1}{t} \log \mathbb{E}^{(z)} R_{0\to t}\right] = \frac{(\theta + 1)^2\left(\theta + \frac{t}{2T}\right)}{\theta\left(\theta + \frac{t}{T}\right)^2} \sigma^2.$$

Disagreement (*that is*, *the cross-sectional standard deviation of* $\frac{1}{t} \log \mathbb{E}^{(z)} R_{0\to t}$) *is*

$$\sqrt{\widetilde{\operatorname{var}}\left[\frac{1}{t} \log \mathbb{E}^{(z)} R_{0\to t}\right]} = \frac{\theta + 1}{\theta + \frac{t}{T}} \frac{\sigma}{\sqrt{\theta T}}.$$

Our next result characterizes option prices. The unusual feature of the result is not that option prices can be quoted in terms of the Black–Scholes formula, as this is always possible, but that the associated implied volatilities $\tilde{\sigma}_t$ can be expressed in a simple yet nontrivial closed form. (We denote risk-neutral variance with an asterisk in Result 8 and throughout the paper.)

RESULT 8: *The time 0 price of a call option with maturity t and strike price K obeys the Black-Scholes formula with maturity-dependent implied volatility* $\tilde{\sigma}_t$*:*

$$C(t, K) = p_0 \Phi\left(\frac{\log \frac{p_0}{K} + \frac{1}{2}\tilde{\sigma}_t^2 t}{\tilde{\sigma}_t \sqrt{t}}\right) - K\Phi\left(\frac{\log \frac{p_0}{K} - \frac{1}{2}\tilde{\sigma}_t^2 t}{\tilde{\sigma}_t \sqrt{t}}\right),$$

$$\text{where} \quad \tilde{\sigma}_t = \frac{\theta + 1}{\sqrt{\theta\left(\theta + \frac{t}{T}\right)}} \sigma,$$

and Φ *is the standard Normal cumulative distribution function. The VIX index* (*at time 0*, *for settlement at time t*) *is* $VIX_{0\to t} = \tilde{\sigma}_t$, *and there is a variance risk premium*, *on which all agents agree:*

$$\frac{1}{T}\left(\operatorname{var}^* \log R_{0\to T} - \operatorname{var} \log R_{0\to T}\right) = \frac{\sigma^2}{\theta}.$$

In the limit as $\theta \to \infty$, implied and physical volatility are each equal to σ and there is no variance risk premium, as in Black and Scholes (1973). But with heterogeneity, $\theta < \infty$, speculation boosts implied and physical volatility, particularly in the short run, and opens up a gap between the two in the long run. The existence of such a variance risk premium is a robust feature of the data; see, for example, Bakshi

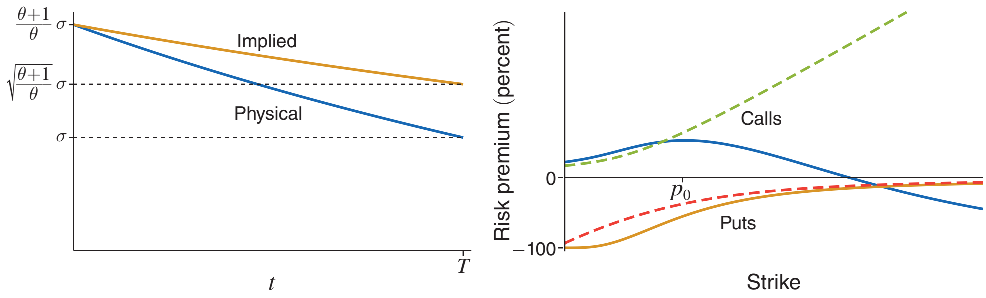

FIGURE 9

Notes: Left: the term structures of implied and physical volatility. Right: expected excess returns on options of different strikes, K, as perceived by the median investor, $z = 0$. Solid/dashed lines denote heterogeneous/homogeneous beliefs.

and Kapadia (2003); Carr and Wu (2009); and Bollerslev, Gibson, and Zhou (2011). Holding option maturity fixed, implied volatility is constant across strikes (though this is not a general property of our framework: the Poisson limit of Section 4 generates a volatility "smirk"). Note, however, that although belief heterogeneity increases implied volatility by the same amount at all strikes, this implies that its proportional impact on *prices* is greater for out-of-the-money options.

To understand intuitively why there is a variance risk premium, note that for any tradable payoff X and stochastic discount factor (SDF) M, one has the identity

$$\text{var}^* X - \text{var}\, X = R_f \,\text{cov}\left[M, (X - \kappa)^2\right], \tag{20}$$

where R_f is the gross riskless rate and $\kappa = \left(\mathbb{E}X + \mathbb{E}^*X\right)/2$ is a constant.[16] We apply this identity in our setting with $X = \log R_{0\to T}$ and $R_f = 1$. Different people agree on physical variance, as shown in Result 7, but the SDF and $\kappa = z\sigma\sqrt{T}/\left(2\sqrt{\theta}\right)$ are person-specific, so (20) specializes to

$$\text{var}^* \log R_{0\to T} - \text{var} \log R_{0\to T} = \text{cov}^{(z)}\left[M_{0\to T}^{(z)}, \left(\log R_{0\to T} - \frac{z\sigma\sqrt{T}}{2\sqrt{\theta}}\right)^2\right].$$

From the perspective of the median agent ($z = 0$), for example, the presence of a variance risk premium indicates that the SDF is positively correlated with $\left(\log R_{0\to T}\right)^2$, i.e., that bad times are associated with extreme values of $\log R_{0\to T}$.

To see why this is the case, we will study individual agents' trading strategies in the next section. For now, as a suggestive indication, the right panel of Figure 9 shows the risk premia on options perceived by the median investor. In a homogeneous economy, out-of-the-money call options have—as levered claims on the risky asset—high expected excess returns. With heterogeneous beliefs, the median

[16] We are not aware of any prior references to the identity (20) in the literature, and it may be of independent interest. It requires only that there is no arbitrage.

Table 1—Observables in the Model with $\theta = 1.8$ (Baseline) and $\theta = 0.2$ (Crisis) and, Time-Averaged, in the Data (in Percent)

	Data	Model ($\theta = 1.8$)	Model ($\theta = 0.2$)
1 month implied volume	18.6	18.6	70.5
1 year implied volume	18.1	18.2	58.8
2 year implied volume	17.9	17.7	50.9
1 year cross-sectional mean risk premium	3.8	3.2	28.8
1 year disagreement	4.8	4.2	33.9
10 year cross-sectional mean risk premium	3.6	1.8	5.0
10 year disagreement	2.9	2.8	8.5

investor perceives that deep out-of-the-money calls are so overvalued due to the presence of extremists that they earn *negative* expected excess returns.[17]

A Calibration.—We illustrate the predictions of the model in a simple calibration. We do so with the obvious (but important) caveat that our model is highly stylized; moreover, the results above show that the parameter θ, which controls belief heterogeneity, simultaneously dictates several quantities that a priori need not be linked. The goal of the exercise is merely to point out that a single value of θ can generate predictions of broadly the right order of magnitude across multiple dimensions.

We set the horizon over which disagreement plays out to $T = 10$ years, and we set σ, which equals the volatility of log fundamentals (i.e., payoffs), to 12 percent. In our baseline calibration, we set $\theta = 1.8$, which implies that one-month, one-year, and two-year implied volatilities are 18.6 percent, 18.2 percent, and 17.7 percent, respectively, as shown in Table 1. These numbers are close to their empirically observed counterparts: in the data of Martin and Wagner (2019), mean implied volatility is 18.6 percent, 18.1 percent, and 17.9 percent at the one-month, one-year, and two-year horizons.

The model-implied cross-sectional mean expected returns are 3.2 percent and 1.8 percent at the one- and 10-year horizons. For comparison, in the survey data of Ben-David, Graham, and Harvey (2013), the corresponding time-series average levels of cross-sectional average expected returns are 3.8 percent and 3.6 percent. The cross-sectional standard deviations of expected returns ("disagreement") at the one- and 10-year horizons are 4.2 percent and 2.8 percent in the model and 4.8 percent and 2.9 percent, on average, in the data of Ben-David, Graham, and Harvey (2013).

An alternative interpretation of our model would interpret time 0 as a time when the market is preoccupied by some new phenomenon over which there is considerable disagreement. With 2008 in mind, one might imagine agents disagreeing about the implications of the Lehman Brothers default and the likely severity of the ensuing recession; in early 2020, the COVID-19 coronavirus was sweeping the world. On both occasions, short-term measures of implied volatility rose to extraordinarily high levels. Within our model, heightened belief heterogeneity (low θ) generates steeply downward-sloping term structures of volatility and

[17] This perception is qualitatively consistent with the findings of Coval and Shumway (2001), who "find considerable evidence that both call and put contracts earn exceedingly low expected returns. A strategy of buying zero-beta straddles has an average return of around −3 percent per week."

of risk premia. To capture scenarios such as these, the table also reports results for a "crisis" calibration with $\theta = 0.2$. (For comparison, the implied volatility measure SVIX, introduced in Martin (2017), rose to 74.1 percent at the one-month horizon and 45.9 percent at the one-year horizon in late November 2008.) We plot the term structures of physical and implied volatilities, and of the average risk premium and disagreement, in the two calibrations in the online Appendix.

A. *Speculation in Equilibrium*

Our investors speculate using complicated dynamic trading strategies. These determine, for each investor, an equilibrium return on wealth that is a function of the return on the underlying risky asset. To express this in a convenient form, we make two definitions. First, we refer to the investor

$$z = z_g = -\frac{\theta + 1}{\sqrt{\theta}}\sigma\sqrt{T} \tag{21}$$

as the *gloomy investor*. There are, of course, more pessimistic investors $(z < z_g)$, but they are less gloomy in the sense that they perceive attractive opportunities associated with short positions in the risky asset. Second, we introduce the notion of an investor-specific *target return* $K^{(z)}$ defined via

$$\log K^{(z)} = \frac{\theta + 1}{\sqrt{\theta}} z\sigma\sqrt{T} + \frac{(\theta + 1)(2\theta + 1)}{2\theta}\sigma^2 T. \tag{22}$$

The target return represents the ideal outcome for investor z: it is the realized return on the risky asset that maximizes wealth, and hence utility, ex post.

RESULT 9: *Agent z's equilibrium return on wealth, $R^{(z)}_{0\to T}$, can be expressed as a function of the return on the risky asset, $R_{0\to T}$, as*

$$R^{(z)}_{0\to T} = \sqrt{\frac{\theta + 1}{\theta}}\exp\left\{\frac{1}{2}(z - z_g)^2 - \frac{1}{2(1 + \theta)\sigma^2 T}\left[\log\left(R_{0\to T}/K^{(z)}\right)\right]^2\right\}. \tag{23}$$

Thus agent z's terminal wealth is maximized when $R_{0\to T} = K^{(z)}$, and as

$$\mathbb{E}^{(z)}\log R^{(z)}_{0\to T} = \frac{1}{2}\log\frac{\theta + 1}{\theta} + \frac{(z - z_g)^2 - 1}{2(1 + \theta)},$$

the gloomy investor has the lowest expected utility of all investors.

The left panel of Figure 10 shows how different investors' outcomes depend on the realized return on the market. The best-case scenario for investor z is that the target return is attained, $R_{0\to T} = K^{(z)}$, in which case $R^{(z)}_{0\to T} = \sqrt{\frac{\theta + 1}{\theta}}\exp\left\{\frac{1}{2}(z - z_g)^2\right\}$. An extremist's best-case scenario is better than that of a moderate investor, because it is cheap to purchase claims to states of the world that extremists consider likely, as few people are extremists. Furthermore, the best case scenario for an optimistic agent $z > 0$ is better than that of the symmetrically pessimistic investor—agent

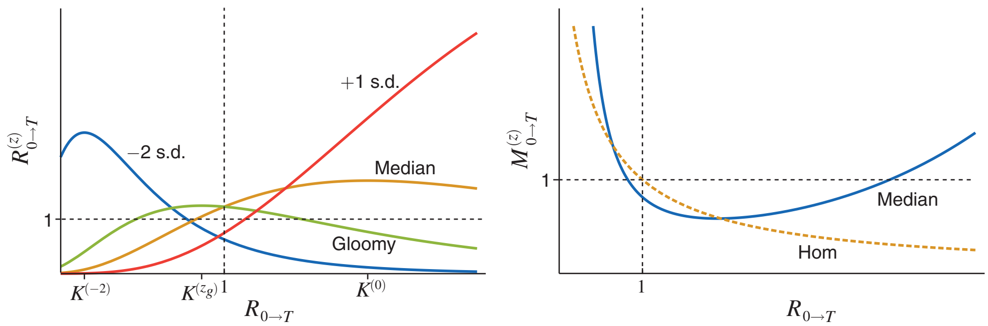

Figure 10

Notes: Left: return on wealth, as a function of the realized return on the risky asset, for different agents. Dots indicate the expected return on the risky asset perceived by each investor. Right: all investors have U-shaped SDFs when beliefs are heterogeneous. The figure shows the SDF of the median investor, together with the SDF that would prevail in a homogeneous economy.

$-z < 0$—because there is more aggregate wealth to go around in good states than in bad states. There is a useful distinction between what investors expect to happen and what they would like to happen. (The distinction also exists, but is uninteresting, in representative-agent models, as the target return is then infinite.) Using Result 7 and equations (21) and (22), we can write

$$\log K^{(z)} = \mathbb{E}^{(z)} \log R_{0\to T} + \left(z - z_g\right)\sigma\sqrt{\theta T}. \tag{24}$$

The gloomy investor would like to be proved right: his target log return equals his expected log return. Targets and expectations differ for all other investors.[18] More optimistic investors have a target return that exceeds their expectations—i.e., they are best off if the risky asset modestly outperforms their expectations—while more pessimistic investors are best off if the risky asset modestly underperforms their expectations. But any investor does very poorly if the asset performs far better or worse than he or she anticipated, consistent with the discussion surrounding identity (20).

As our investors—who perceive different SDFs because they disagree on true probabilities but agree on asset prices—have log utility, their SDFs satisfy $M_{0\to T}^{(z)} = 1/R_{0\to T}^{(z)}$. It follows that every investor's SDF is a U-shaped function of the market return, as shown in the right panel of Figure 10. Thus our model is consistent with the seemingly puzzling empirical evidence documented by a large literature starting from Aït-Sahalia and Lo (2000) and Jackwerth (2000). By contrast, if beliefs were homogeneous the SDF $M_{0\to T} = 1/R_{0\to T}$ would be a downward-sloping function of the market return, as in conventional models.

In our baseline calibration, the median investor's speculative strategy performs well, in the baseline calibration, even if the market itself "does nothing": she earns a positive excess return, $R_{0\to T}^{(0)} > 1$, even if the market's realized return equals the

[18] Note, however, that maximizing (23) with respect to z, we see that the richest investor for fixed $R_{0\to T}$ is the investor whose expectations are met in the sense that $\log R_{0\to T} = \mathbb{E}^{(z)} \log R_{0\to T}$.

riskless rate, $R_{0\to T} = 1$. There are various ways to understand this fact. In dynamic terms, the median investor trades in contrarian fashion, increasing her position in the risky asset when its price falls and reducing her position when its price rises, as was the case in the simple example shown in the left panel of Figure 2. If the risky asset's price ends up close to where it started, her speculative "buy low, sell high" trades are collectively profitable. Alternatively—as the model is dynamically complete—the strategy can be implemented statically via a portfolio of options, along the lines of Breeden and Litzenberger (1978).

RESULT 10: *Investor z can implement her optimal strategy by holding a position in the riskless bond together with put options at strikes $K < p_0 K^{(z)}$ and call options at strikes $K > p_0 K^{(z)}$, with position size at strike K proportional to $\frac{\partial^2 R_{0\to T}^{(z)}}{\partial R_{0\to T}^2}(K)$. Thus she is long (short) options in regions in which $R_{0\to T}^{(z)}$ is convex (concave) as a function of $R_{0\to T}$. In particular, if she is a moderate investor, in the sense that $z \in [z_g, 0]$, then she will be short options with strikes close to $\exp \mathbb{E}^{(z)} \log p_T$.*

B. *Maximum-Sharpe-Ratio Strategies: A Cautionary Tale*

As the log return on the risky asset is perceived as Normally distributed by all agents, we can use equation (23) to calculate the first and second (subjectively perceived) moments of each agent's chosen return. These are

$$\mathbb{E}^{(z)} R_{0\to T}^{(z)} = \frac{1+\theta}{\sqrt{\theta(2+\theta)}} \exp\left[\frac{(z-z_g)^2}{2+\theta}\right]$$

$$\text{and} \quad \mathbb{E}^{(z)}\left[R_{0\to T}^{(z)2}\right] = \frac{1+\theta}{\theta}\sqrt{\frac{1+\theta}{3+\theta}} \exp\left[\frac{3(z-z_g)^2}{3+\theta}\right],$$

and together they pin down the Sharpe ratio of agent z's chosen investment strategy. Similarly, the Sharpe ratio of a static investment in the risky asset can be calculated using Result 7.

We can contrast these with the *maximum* Sharpe ratios that investors perceive as attainable. We use the Hansen and Jagannathan (1991) bound to compute the latter; the bound can be attained as the market is dynamically complete.

RESULT 11: If $\theta > 1$, *the maximum Sharpe ratio (MSR) perceived by investor z is $MSR_{0\to t}^{(z)} = \sqrt{\mathrm{var}^{(z)} M_{0\to t}^{(z)}}$, where*

$$\mathrm{var}^{(z)} M_{0\to t}^{(z)} = \frac{\theta}{\sqrt{\theta^2 - (t/T)^2}} \exp\left[\frac{(z-z_g)^2 t/T}{\theta - t/T}\right] - 1. \tag{25}$$

Hence, the gloomy investor perceives the minimal maximum Sharpe ratio.

If $\theta \leq 1$ then all investors perceive infinite Sharpe ratios at sufficiently long horizons.

It follows that the annualized MSR perceived by agent z over very short horizons is

$$\lim_{t\to 0}\frac{1}{\sqrt{t}}\mathrm{MSR}^{(z)}_{0\to t} \;=\; \frac{\left|z - z_g\right|}{\sqrt{\theta T}}. \tag{26}$$

(We annualize, here and in the figures below, by scaling the Sharpe ratio by $1/\sqrt{t}$.) This equals the instantaneous Sharpe ratio of the risky asset. But over longer horizons, all agents believe that there are dynamic strategies with Sharpe ratios strictly exceeding that of the risky asset.

Although the gloomy investor perceives that it is impossible to earn positive Sharpe ratios in the very short run, as is clear from equation (26), he perceives that positive Sharpe ratios are attainable at longer horizons: by Result 11,

$$\mathrm{MSR}^{(z_g)}_{0\to T} \;=\; \sqrt{\frac{\theta}{\sqrt{\theta^2 - 1}} - 1}.$$

The left panel of Figure 11 shows that the maximum attainable Sharpe ratio exceeds the Sharpe ratio on a static position in the risky asset, indicating that all investors must trade dynamically (that is, must speculate) to achieve their perceived MSR. But the figure also shows that the Sharpe ratios that investors perceive on their own optimally chosen strategy are not in general close to the maximum Sharpe ratio or to the Sharpe ratio of the market.

More strikingly, Result 11 implies that if there is substantial disagreement, $\theta \leq 1$, all investors perceive that *arbitrarily* high Sharpe ratios are attainable at long horizons. At first sight, this might seem obviously inconsistent with equilibrium. But our investors are not mean-variance optimizers so Sharpe ratios do not adequately summarize investment opportunities.[19]

To see why, we can study the strategies that achieve these maximal Sharpe ratios. By the work of Hansen and Richard (1987), a MSR strategy for investor z must take the form $a - bM^{(z)}_{0\to T}$ for some constants $a > 0$ and $b > 0$, where $a = 1 + b\mathbb{E}^{(z)}\left[M^{(z)2}_{0\to T}\right]$. As the return on wealth chosen by investor z, which we derived in Result 9, reveals the investor's SDF, $M^{(z)}_{0\to T} = 1/R^{(z)}_{0\to T}$, we can write an MSR return as

$$R^{(z)}_{\mathrm{MSR},0\to T} \;=\; 1 + b\left(\mathrm{var}^{(z)}M^{(z)}_{0\to T} + 1\right) - \frac{b}{R^{(z)}_{0\to T}}, \tag{27}$$

where $\mathrm{var}^{(z)}M^{(z)}_{0\to T}$ is provided in equation (25) and b can be any positive constant (the free parameter reflecting the fact that any strategy can be combined with a position in the riskless asset without altering its Sharpe ratio). The right panel of Figure 11 plots the realized return $R^{(z)}_{\mathrm{MSR},0\to T}$ as a function of the risky return $R_{0\to T}$, for investors $z = 0$ and $z = 1$ in the baseline calibration. The MSR strategies could be implemented dynamically via a contrarian market-timing strategy that

[19] For our investors, the correct risk-adjusted measure of the attractiveness of investment opportunities is the expected log return. As shown in Result 9, this is finite for all investors for any $\theta > 0$.

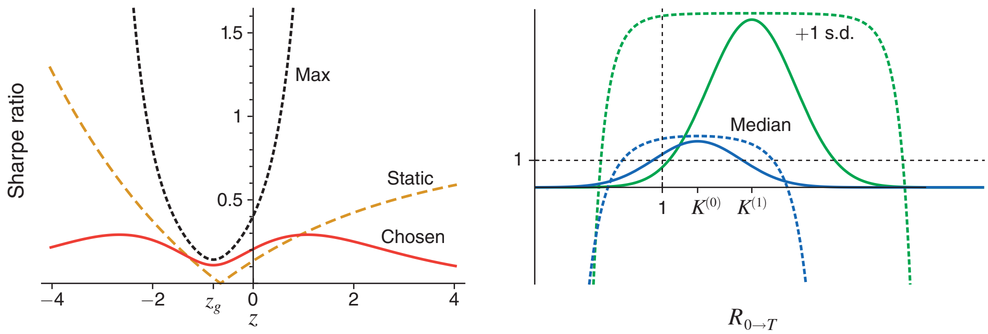

FIGURE 11

Notes: Left: the annualized Sharpe ratios, from 0 to T, that investors perceive on their own chosen strategies (solid) and on a static position in the risky asset (dashed); and the perceived maximum Sharpe ratio attainable through dynamic trading (dotted). Baseline calibration. Right: realized returns on the strategies chosen by investors $z = 0$ and 1 (solid) and the realized returns on their MSR strategies (dotted) as a function of the realized return on the risky asset. Log scale on x-axis.

goes long if the market sells off and short if the market rallies, thereby exploiting what investors view as irrational exuberance on the upside and irrational pessimism on the downside; or statically—by the logic of Result 10—via extremely short positions in out-of-the-money call and put options.

We view this as a cautionary tale. If betas are calculated with respect to the market return, or to any investor's optimally chosen return, then MSR strategies—or factors that load up on tail risk—will earn large alphas. But our investors do not do mean–variance analysis, so alphas are not useful or interesting measures for them, and although it is possible to earn high Sharpe ratios via short option positions, these strategies are not remotely attractive: our investors prefer to choose strategies that lie well inside the mean–variance efficient frontier. Indeed, as MSR strategies feature the possibility of unboundedly negative gross returns, our investors would prefer to invest fully in (say) cash than to rebalance, even slightly, toward a MSR strategy.

Note, finally, that this is all true even in an ostensibly well-behaved setting in which investors have log utility and the risky asset's return is universally agreed to be lognormally distributed. In Section 4, we will show that in a limit featuring jumps, all investors perceive that arbitrarily high Sharpe ratios are attainable in *any* calibration.

C. *Ex Ante Attitudes to Speculation*

We have seen that investors believe that substantial gains in Sharpe ratio can be achieved by speculating. But Sharpe ratios do not adequately capture our investors' attitudes to speculation. A better measure is provided by agent z's perceived gain from speculation, $\xi^{(z)}$, which satisfies the equation $\mathbb{E}^{(z)} \log R^{(z)}_{0\to T} = \mathbb{E}^{(z)} \log \left[\left(1 + \xi^{(z)}\right) R_{0\to T}\right]$. This is the proportional increase in wealth that would leave investor z as happy, holding the market, as he or she would have been when allowed to speculate. More generally, we can ask what investor z

thinks investor x's gain from speculation is. When we do so, we assume that investor z uses his or her own beliefs in assessing investor x's expected utility, and we assume that other investors continue to trade, so that prices are unaffected by investor x's absence: thus we wish to solve

$$\mathbb{E}^{(z)} \log R_{0\to T}^{(x)} = \mathbb{E}^{(z)} \log \left[\left(1 + \xi^{(z,x)}\right) R_{0\to T}\right] \tag{28}$$

for $\xi^{(z,x)}$. (Note that $\xi^{(z)} = \xi^{(z,z)}$.) As $\xi^{(z,x)}$ is a dollar measure of the gain from speculation, we can then aggregate over x to determine agent z's view of the impact of speculation on social welfare. In doing so, we are committing to the utilitarian idea of cardinal utility that can be compared across people.

RESULT 12 (Ex ante attitudes to speculation): *Investor z's perception of investor x's gain from speculation, $\xi^{(z,x)}$, is*

$$\xi^{(z,x)} = \sqrt{\frac{\theta + 1}{\theta}} \exp\left[\frac{z^2 - 1}{2(1+\theta)} - \frac{(z-x)^2}{2\theta}\right] - 1.$$

This is positive for investor types x that are sufficiently close to z and negative otherwise. Aggregating over x, investor z's perception of the aggregate gain to speculation is

$$\xi = \exp\left[-\frac{1}{2(1+\theta)}\right] - 1,$$

which is independent of z and negative for all $\theta > 0$.

Ex ante, all investors perceive that the ability to speculate is in their own interest and in the interest of investors with beliefs sufficiently similar to their own, as $\xi^{(z,z)} > 0$. But they all also think that speculation is socially costly, as $\xi < 0$. In the terminology of Brunnermeier, Simsek, and Xiong (2014), speculation is *belief-neutral inefficient*, despite every investor finding it attractive.[20] Moreover, as heterogeneity increases (i.e., as θ decreases) the degree of dissonance increases, in the sense that all investors perceive that speculation is increasingly beneficial for them personally but increasingly costly for the population as a whole.

One might wonder whether a sufficiently enlightened collection of individuals might agree to ban speculation. But if dynamic trade were shut down entirely, so that all agents had to trade once at time 0 and then hold their positions statically to time T, then equilibrium would not exist in the limit. To see this, write ψ_z for the share of wealth invested by agent z in the risky asset. Given any positive time 0 price, $R_{0\to T}$ is lognormal from every agent's perspective by Result 7 (which applies even in the static case at horizon T, because the terminal payoff is specified exogenously). Confronted with a lognormally distributed return, any agent z will choose $\psi_z \in [0,1]$ to avoid the possibility of terminal wealth becoming negative. Market clearing requires that $\psi_z = 1$ on average across agents, so we must in fact have $\psi_z = 1$ for

[20] Brunnermeier, Simsek, and Xiong (2014) present some examples of economies with inefficient speculation in the presence of heterogeneous beliefs, but their examples have no aggregate risk.

all z, which is impossible: there is no positive price at which all agents choose to invest fully in the risky asset.

D. *Ex Post Regret and Inequality*

Ex post, there will always be some investors who are happy to have speculated—because their chosen return $R^{(z)}_{0\to T}$ turned out to be higher than the static return $R_{0\to T}$ —and others who are regretful (that is, whose realized utility is lower as a result of speculating than it would have been if they had held the risky asset statically).

On average, however, people are regretful, in the utilitarian sense that average realized utility is lower than it would have been had all agents held their original position, without trading. This is a direct consequence of inequality in the presence of risk aversion. To see this, we can measure inequality at time T using the Atkinson (1970) inequality index, A_T, which satisfies[21]

$$\log(1 - A_T) = \tilde{\mathbb{E}} \log R^{(z)}_{0\to T} - \log R_{0\to T}. \tag{29}$$

Average ex post regret, $\log R_{0\to T} - \tilde{\mathbb{E}} \log R^{(z)}_{0\to T}$, is therefore a function of ex post inequality, A_T. Equation (29) shows that the Atkinson index can be interpreted as the fraction of wealth that could be sacrificed while holding social welfare constant, if wealth were redistributed equally across the population ex post.

The extent of ex post inequality depends on how surprising the realized outcome is, in the mind of the median investor—specifically, on the number of standard deviations by which the realized log return on the risky asset exceeds the median investor's expectation,

$$s_T = \frac{\log R_{0\to T} - \mathbb{E}^{(0)} \log R_{0\to T}}{\sqrt{\operatorname{var}^{(0)} \log R_{0\to T}}}.$$

RESULT 13 (Ex post inequality): *At time T, the Atkinson inequality index satisfies*

$$A_T = 1 - \sqrt{\frac{\theta + 1}{\theta}} \exp\left\{ -\frac{s_T^2}{2(1+\theta)} - \frac{1}{2\theta} \right\}.$$

Thus inequality is minimized if the realized log return on the risky asset meets the expectations of the median investor, and is high if the realized log return is far from the median investor's expectations.

IV. A Poisson Limit

We now consider an alternative continuous time limit in which the risky asset is subject to jumps that arrive at times dictated (in the limit) by a Poisson process. We

[21] Atkinson (1970) defined a family of indices indexed by an inequality aversion parameter, ε. In equation (29) we are considering the case $\varepsilon = 1$, which is widely used in practice and which has a natural interpretation in our equilibrium.

think of this setting as representing a stylized model of insurance or credit markets in which credit events or catastrophes arrive suddenly and cause large losses.

We divide the period from 0 to T into N steps, and we will let N tend to infinity. We want the mean agent to perceive a jump arrival rate of λ, and the cross-sectional standard deviation to be of a similar order of magnitude. These considerations dictate that the distribution of agent types h should be concentrated around a mean of $1 - \lambda dt$ (so that the mean perceived probability of a down-move is λdt, where we write $dt = T/N$) and should have standard deviation $\omega\lambda dt$ (so that a higher ω corresponds to a higher degree of disagreement). Exploiting the flexibility of the beta family (1), we therefore set

$$\alpha_N = \frac{N}{\omega^2 \lambda T} \quad \text{and} \quad \beta_N = \frac{1}{\omega^2},$$

which achieves the desired mean and standard deviation in the limit as $N \to \infty$.

If there are no down-moves, the terminal payoff is one; we assume that each down-move causes the same proportional loss to the terminal payoff, so that the payoffs are $p_{m,T} = e^{-(N-m)J}$ for some constant J. This setup might be viewed as a stylized model of a risky bond, for example. Our next result applies Result 1 to characterize pricing in the limit as $N \to \infty$. In the limit, a down-move corresponds to a Poisson jump. The price is only defined under an assumption that jumps are not too frequent or severe, and that there is not too much disagreement:

$$\omega^2 \lambda T\left(e^J - 1\right) < 1. \tag{30}$$

(We will treat J as positive, so that jumps represent bad news, but our results go through for negative J, in which case a jump represents good news and (30) is always satisfied.) As before, we parametrize investors by z, which indexes the number of standard deviations more optimistic than the mean a given investor is; thus person z thinks that the Poisson process has jump arrival rate $\lambda(1 - z\omega)$. In contrast to the Brownian limit, investors now disagree about *all* moments of returns. We also modify our previous notation by writing $p_{q,t}$ for the price at time t if q jumps have occurred.

RESULT 14: *The price at time t, if q jumps have occurred, is*

$$p_{q,t} = e^{-qJ}\left(1 - \frac{\omega^2 \lambda (T-t)}{1 + \omega^2 \lambda t}\left(e^J - 1\right)\right)^{q + \frac{1}{\omega^2}}. \tag{31}$$

Investor z's SDF at time t is a function of q, the number of jumps that have occurred:

$$M_{0\to t}^{(z)} = \frac{\Gamma\left(q + \frac{1}{\omega^2}\right)}{\Gamma\left(\frac{1}{\omega^2}\right)}\left[1 - \omega^2 \lambda T\left(e^J - 1\right)\right]^{\frac{1}{\omega^2}}$$
$$\times \left[1 - \omega^2 \lambda T\left(e^J - 1\right) + \omega^2 \lambda t\, e^J\right]^{-q-\frac{1}{\omega^2}}\left[\frac{\omega^2 e^J}{1 - z\omega}\right]^q e^{\lambda(1 - z\omega)t}.$$

Expected utility is finite for all investors because $\mathbb{E}^{(z)}\log R^{(z)}_{0\to T} = -\mathbb{E}^{(z)}\log M^{(z)}_{0\to T}$ is finite. But as $M^{(z)}_{0\to t}$ has infinite variance, all investors perceive that arbitrarily high Sharpe ratios are attainable.

Agent z's return on wealth is $R^{(z)}_{0\to t} = 1/M^{(z)}_{0\to t}$, so the richest agent at time t can be identified by minimizing $M^{(z)}_{0\to t}$ with respect to z, giving $z_{richest} = (\lambda t - q)/(\omega\lambda t)$. This agent perceives arrival rate $\lambda_{richest} = q/t$, so has beliefs that appear correct in hindsight.

We can calculate the risky asset share of agent z by comparing the return on wealth with the return on the risky asset (which can be computed using the price (31)):

$$\text{risky share}^{(z)}_t = 1 + \frac{\omega}{e^J - 1}\underbrace{\left[1 - \frac{\omega^2\lambda}{1+\omega^2\lambda T}e^J(T-t)\right]}_{>0 \text{ by assumption (30)}}\frac{1+\omega^2\lambda t}{1+\omega^2 q}\left[\frac{\omega(q-\lambda t)}{1+\omega^2\lambda t} + z\right].$$

The representative agent (whose risky share equals one) is therefore $z = -\frac{\omega(q-\lambda t)}{1+\omega^2\lambda t}$, with perceived jump arrival rate $\lambda_{\text{rep},t} = \lambda + \frac{\omega^2\lambda t}{1+\omega^2\lambda t}\left(\frac{q}{t} - \lambda\right)$. Thus initially the mean investor is representative. Subsequently, the representative investor's perceived arrival rate grows if the realized jump arrival rate is higher than expected $(q/t > \lambda)$ and declines otherwise. For large t, the representative investor perceives an arrival rate close to the historically realized arrival rate q/t.

If q jumps have occurred by time t, the investor who is out of the market perceives arrival rate

$$\lambda^*_t = \frac{1+q\omega^2}{1-\omega^2\lambda T(e^J-1)+\omega^2\lambda t e^J}e^J\lambda. \tag{32}$$

Agents who are more pessimistic, perceiving arrival rates higher than λ^*_t, are short the risky asset. They lose money while nothing happens, but experience sudden gains if a jump arrives. Conversely, agents who are more optimistic are long, so do well if nothing happens but are exposed to jump risk; one can think of the pessimists as having purchased jump insurance from the optimists.

It follows from equation (32) that if jumps are sufficiently large—if $e^J - 1 \geq 1$—then $\lambda^*_t \geq \lambda$ for all t and q. In this case, the mean investor is never short the risky asset, no matter what happens. By contrast, in any calibration of the Brownian limit there are sample paths on which the mean investor goes short the risky asset.

The risk-neutral arrival rate measures the cost of insuring against a jump. We will refer to it as the CDS rate, λ^*_t, as it equals the price (scaled by the length of contract horizon) of a very short-dated CDS contract that pays \$1 if there is a jump:

$$\lambda^*_t = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{P}^*_t\big(\text{at least one jump occurs in } [t, t+\varepsilon]\big). \tag{33}$$

We have already used λ^*_t to denote the arrival rate (32) perceived by the investor who is out of the market, but the next result shows that the two quantities coincide.

RESULT 15: *The risk-neutral arrival rate, or CDS rate, is λ^*_t as defined in equation* (32).

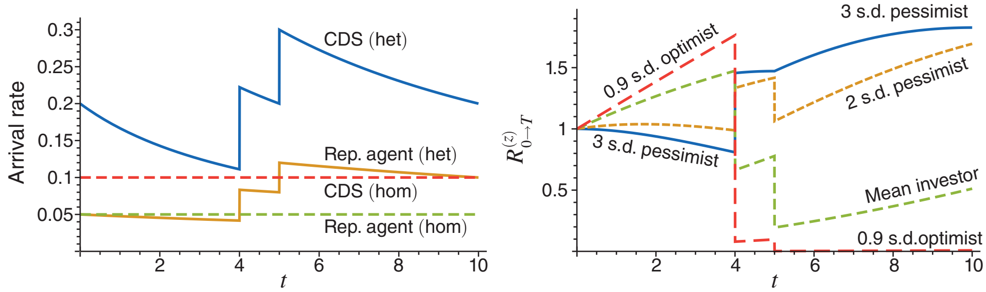

Figure 12

Notes: Left: the evolution of the representative agent's subjectively perceived arrival rate, and of the CDS rate (i.e., risk-neutral arrival rate), in the heterogeneous and homogeneous economies, on a sample path with jumps occurring at times $t = 4$ and $t = 5$. Right: the evolution over time of the wealth of four agents ($z = -3, -2, 0, 0.9$) on the same sample path.

The CDS rate jumps when there is a Poisson arrival and declines smoothly as time passes during periods where there are no arrivals. (For comparison, the CDS rate is constant over time in the homogeneous case: $\lambda^*_{hom} = e^J\lambda$.) Initially, when $t = q = 0$, the CDS rate is unambiguously higher in the presence of belief heterogeneity:

$$\lambda_0^* = \underbrace{\frac{1}{1 - \omega^2\lambda T\left(e^J - 1\right)}}_{\in(0,1)\ by\ assumption\ (30)} e^J\lambda > \lambda^*_{hom}.$$

By the terminal date, $t = T$, we have $\lambda_T^* = \frac{1 + q\omega^2}{1 + \lambda T\omega^2}\lambda^*_{hom}$. Thus λ_T^* may be larger or smaller than λ^*_{hom}, depending on whether the realized number of jumps exceeded the mean agent's expectations ($q > \lambda T$) or not.

Figure 12 shows how the equilibrium evolves along a particular sample path on which two jumps occur in quick succession, at times $t = 4$ and $t = 5$. We set $\omega = 1$, $\lambda = 0.05$ and $T = 10$ and assume that half of the fundamental value is destroyed every time there is a jump, that is, $e^{-J} = 1/2$, or $e^J - 1 = 1$. The figure shows a relatively unlucky sample path, on which the expectations of the pessimistic agent $z = -3$ are realized; for comparison, the mean agent only expected 0.5 jumps over the ten years.

The left panel shows the evolution of the representative agent's subjectively perceived arrival rate, and of the CDS rate. These two quantities decline smoothly during quiet periods with no jumps, but spike immediately after a jump arrives. (Similar patterns have been documented in catastrophe insurance markets by Froot and O'Connell 1999 and Born and Viscusi 2006, and have also been studied theoretically by Duffie 2010.) By contrast, in a homogeneous economy, each would be constant over time.

As we have seen, the CDS rate reveals the identity of the investor who is out of the market. More optimistic investors hold long positions in the risky asset, analogous to selling insurance or shorting CDS contracts. They accumulate wealth in quiet times, but experience sudden losses when bad news arrives. Pessimistic investors, who perceive higher arrival rates than the CDS rate, are short the risky asset, which is analogous to buying insurance or going long CDS. Their wealth bleeds away during quiet times, but they experience sudden windfalls if bad news arrives.

The right panel plots the cumulative return on wealth for four different agents over the same sample path. The figure shows two pessimists, who are two and three standard deviations below the mean, and who therefore perceive arrival rates of 0.15 and 0.20, respectively; the mean investor, with perceived arrival rate 0.05; and an optimist who is 0.9 standard deviations above the mean, with perceived arrival rate 0.005. (All agents must perceive a positive arrival rate, and this imposes a limit on how optimistic an agent can be: as $\omega = 1$ in our calibration, we must have $z < 1$.)

The optimist and the mean investor are both long the asset (i.e., short jump insurance) throughout the sample path. The two pessimists buy or sell insurance depending on whether the CDS rate is above or below their subjectively perceived arrival rates. By the time of the first jump, both are short the asset—long jump insurance—so experience sudden increases in wealth at $t = 4$. In this example, the positions of the four investors in the wealth distribution are reversed as a result of the first jump. As the CDS rate then spikes, the two pessimists reverse their positions temporarily, and are short jump insurance between times 4 and 5. At the instant the jump occurs at time 5, the $z = -3$ pessimist is out of the market, so her wealth is unaffected by the jump. The $z = -2$ pessimist is still selling insurance, however, so experiences a loss.

For completeness, we present an option-pricing formula for the Poisson limit in the Online Appendix. Notably, the model generates a volatility smirk, with high volatility at low strikes, and a hump-shaped term structure of implied volatility.

V. Conclusion

We have presented a dynamic model in which individuals have heterogeneous beliefs. Short sales are allowed; all agents are risk averse; and all agents are marginal. Wealth shifts toward agents whose beliefs are correct in hindsight, whether through luck or judgment, so the identity of the representative investor, "Mr. Market," changes constantly over time, becoming more optimistic following good news and more pessimistic following bad news. These shifts in sentiment drive up volatility and induce speculation: that is, agents take on positions they would not wish to hold to maturity. Indeed, they may even temporarily trade in the *opposite* direction to their view of fundamental value.

We would expect these dynamics to be particularly important at times when investors are preoccupied by some phenomenon—a plane hits the World Trade Center, Lehman Brothers goes bankrupt, a novel coronavirus emerges, a war breaks out—around which there is considerable disagreement. At such times, markets are typically volatile, with steeply downward-sloping term structures of implied volatility, as our model predicts.

Agents anticipate the future impact of sentiment, so payoffs in extreme states of the world acquire more importance. If payoffs are right-skewed, as in our bubbly asset example, sentiment drives the price up. Conversely, if payoffs are left-skewed, as in our risky bond example, sentiment drives the price down. The dynamics of sentiment are quite different in the two cases. In the risky bond example, the possibility of bad news, and hence negative sentiment, in future drives prices sharply down *today*. By contrast, in the bubbly asset example, good news has little effect at first, but positive sentiment gathers pace and volatility (as measured by the VIX index) rises over time as the bubble develops. Remarkably, the median investor reverses his position *twice* over the course of the bubble: he starts out bullish; becomes bearish as more optimistic investors pump up the asset price; and finally turns bullish again at the height of the bubble.

The Brownian and Poisson continuous-time limits embed these phenomena in a richer setting. We characterize how all agents speculate, and various insights emerge. Extremists drive up both true ("$\mathbb{P}$") and implied ("$\mathbb{Q}$") volatility, and this induces moderates to trade in a contrarian way, or equivalently—in our dynamically complete model—to take positions that are "short volatility," i.e. short options. Moderate agents perceive deep out-of-the-money options as extraordinarily overvalued; as a result, their SDFs are U-shaped. Indeed, strategies that short these options very aggressively can earn extremely high (or even, in some settings, infinite) Sharpe ratios. Such strategies are exposed to tail risk, however, and their unattractive higher-moment properties mean that our investors would not want to invest even a tiny fraction of their wealth in a maximum-Sharpe-ratio strategy. We view this as a cautionary tale. The use of alphas and Sharpe ratios as performance measures is pervasive in the finance literature; but they are economically meaningless in our setting.

Each investor perceives speculation as being in his or her own interest; but also thinks that the average investor would be better off (at prevailing market prices) simply holding their endowment statically instead of speculating. One might therefore wonder whether a sufficiently enlightened collection of individuals should agree to ban speculation in order to move to an equilibrium in which investors trade just once, at time zero. But, as we show in the Brownian limit, doing so can cause the market to collapse entirely when agents have heterogeneous beliefs. The ability to trade dynamically is therefore a mixed blessing. It makes speculation possible, thereby creating inequality and ex post regret for the average investor; but it also enables investors to rebalance to avoid bankruptcy if the market starts to move against them. This makes it possible for investors to lever up and to short-sell, and hence permits the existence of equilibrium.

Appendix A. Proofs of Results

PROOF OF RESULT 1:

Observe from the recurrence relation (14) that $z_{0,0}$ is a linear combination of the reciprocals of the terminal payoffs, $z_{0,0} = \sum_{m=0}^{T} c_m z_{m,T}$. Each coefficient c_m is a sum of products of terms of the form $H_{j,s}$ and $1 - H_{j,s}$ over appropriate j and s. In order to better handle these products it will be helpful to introduce $J_{m,t}(h) = h^m(1-h)^{t-m} f(h) \propto w_h f(h)$. Then $\int_0^1 J_{m,t}(h)dh \propto \int_0^1 w_h f(h)dh = p$, and hence

$$H_{m,t} = \frac{\int_0^1 h w_h f(h)dh}{p} = \frac{\int_0^1 h J_{m,t}(h)dh}{\int_0^1 J_{m,t}(h)dh} = \frac{\int_0^1 J_{m+1,t+1}(h)dh}{\int_0^1 J_{m,t}(h)dh} \tag{A1}$$

and

$$\text{(A2)} \qquad 1 - H_{m,t} = \frac{\int_0^1 J_{m,t+1}(h)dh}{\int_0^1 J_{m,t}(h)dh}.$$

We first show that path independence holds, so that all the possible ways of getting from the initial node to node m at time T make an equal contribution to c_m. It suffices to show that, starting from any node, the risk-neutral probability of down-up equals the risk-neutral probability of up-down. That is, for any m and t, $h^*_{m,t}(1 - h^*_{m+1,t+1}) = (1 - h^*_{m,t})h^*_{m,t+1}$. Rewriting equation (8) to insert subscripts, we have

$$\text{(A3)} \qquad h^*_{m,t} = H_{m,t}\frac{p_{m,t}}{p_{m+1,t+1}} \quad \text{and} \quad 1 - h^*_{m,t} = (1 - H_{m,t})\frac{p_{m,t}}{p_{m,t+1}}.$$

It follows that we have path independence if and only if $H_{m,t}(1 - H_{m+1,t+1}) = (1 - H_{m,t})H_{m,t+1}$. But this follows immediately from equations (A1) and (A2).

By path independence, we have

$$c_m = \binom{T}{m}(1 - H_{0,0})\cdots(1 - H_{0,T-m-1})H_{0,T-m}H_{1,T-m+1}\cdots H_{m-1,T-1}.$$

Equations (A1) and (A2) allow us to write c_m as a telescoping product:

$$\begin{aligned} c_m &= \binom{T}{m}\frac{\int_0^1 J_{m,T}(h)dh}{\int_0^1 J_{0,0}(h)dh} \\ &= \binom{T}{m}\int_0^1 h^m(1-h)^{T-m}f(h)dh. \end{aligned}$$

If $f(h)$ is the PDF of a Beta(α, β) distribution, we can evaluate the integral explicitly to give

$$c_m = \binom{T}{m}\frac{B(\alpha + m, \beta + T - m)}{B(\alpha, \beta)}.$$

This quantity is the probability that a random variable with beta-binomial distribution with parameters (T, α, β) equals m. Thus the price at time zero satisfies $p_{0,0}^{-1} = \mathbb{E}[p_{\tilde{m},T}^{-1}]$, where the expectation is over $\tilde{m} \sim$ BetaBinomial(T, α, β). For future reference, we note that the mean and variance of a BetaBinomial(T, α, β) random variable are $T\alpha/(\alpha + \beta)$ and $T\alpha\beta(\alpha + \beta + T)/[(\alpha + \beta)^2(\alpha + \beta + 1)]$, respectively.

The risk-neutral probability of ending at node (m, T), q^*_m, can be determined using (A3) and path-independence:

$$\begin{aligned} q^*_m &= \binom{T}{m}(1 - h^*_{0,0})\cdots(1 - h^*_{0,T-m-1}) \cdot h^*_{0,T-m}h^*_{1,T-m+1}\cdots h^*_{m-1,T-1} \\ &= \binom{T}{m}(1 - H_{0,0})\frac{p_{0,0}}{p_{0,1}}\cdots(1 - H_{0,T-m-1})\frac{p_{0,T-m-1}}{p_{0,T-m}} \\ &\quad \cdot H_{0,T-m}\frac{p_{0,T-m}}{p_{1,T-m+1}}\cdots H_{m-1,T-1}\frac{p_{m-1,T-1}}{p_{m,T}} \\ &= c_m\frac{p_{0,0}}{p_{m,T}}. \quad \blacksquare \end{aligned}$$

We also have the following generalization of Result 1, which gives the (inverse of) the price of the risky asset at node (m,t). We omit the proof, which is essentially identical to the above.

LEMMA 1: *At node (m,t), we have $z_{m,t} = \sum_{j=0}^{T-t} c_{m,t,j} z_{m+j,T}$, where j represents the number of further up-moves after time t, and*

$$c_{m,t,j} = \binom{T-t}{j} \frac{\int_0^1 h^{m+j}(1-h)^{T-m-j} f(h)dh}{\int_0^1 h^m(1-h)^{t-m} f(h)dh}.$$

If, in particular, $f(h)$ is the PDF of a Beta (α,β) distribution, then

$$c_{m,t,j} = \binom{T-t}{j} \frac{B(\alpha+m+j, \beta+T-m-j)}{B(\alpha+m, \beta+t-m)}.$$

This is the probability that a random variable with BetaBinomial$(T-t, \alpha+m, \beta+t-m)$ distribution equals $j \in \{0,\ldots,T-t\}$.

Moreover, the risk-neutral probability of ending up at node $(m+j,T)$ starting from node (m,t) is given by

$$q^*_{m,t,j} = c_{m,t,j} \frac{p_{m,t}}{p_{m+j,T}}.$$

PROOF OF RESULT 2:

At time t, following m up-moves, let the investor's posterior belief about the probability of an up-move be denoted by $h_{m,t}$. In this general case we assume the belief h_0 of the representative agent has a density function $f(h)$. Then, using Bayes' rule, the posterior density function, $f_{m,t}(\cdot)$, satisfies

$$f_{m,t}(h) = \frac{h^m(1-h)^{t-m} f(h)}{\int_0^1 h^m(1-h)^{t-m} f(h)dh}.$$

If for instance $f(h)$ is the density function of a Beta(α,β) distribution, then $f_{m,t}(h)$ is the probability density function of a Beta$(\alpha+m, \beta+t-m)$ distribution. Thus, in particular, using equation (A1):

$$\mathbb{E}[h_{m,t}] = \frac{\int_0^1 h^{m+1}(1-h)^{t-m} f(h)}{\int_0^1 h^m(1-h)^{t-m} f(h)dh} = H_{m,t}. \tag{A4}$$

That is, the expected belief of the representative agent is the same as the wealth-weighted belief in the heterogeneous economy.

The agent's portfolio problem at time t, following m up moves, is therefore

$$\max_{x_h} \mathbb{E}\left[h_{m,t}\log(w_h - x_h p + x_h p_u) + (1-h_{m,t})\log(w_h - x_h p + x_h p_d)\right],$$

with associated first-order condition

$$x_h = w_h\left(\frac{\mathbb{E}\left[h_{m,t}\right]}{p - p_d} - \frac{1 - \mathbb{E}\left[h_{m,t}\right]}{p_u - p}\right).$$

Market clearing dictates that $x_h = 1$ and $w_h = p$. Thus

$$p = \frac{p_u p_d}{\mathbb{E}\left[h_{m,t}\right]p_d + \left(1 - \mathbb{E}\left[h_{m,t}\right]\right)p_u}.$$

By equation (37), this is equivalent to the price (6) in the heterogeneous economy. ■

PROOF OF RESULT 3:

We use the fact (noted in the proof of Result 1) that the price at time zero satisfies $p_{0,0}^{-1} = \mathbb{E}\left[p_{\tilde{m},T}^{-1}\right]$, where the expectation is over $\tilde{m} \sim \text{BetaBinomial}\left(T, \alpha, \beta\right)$. Note that an increase in belief heterogeneity corresponds to a decrease in α and β with α/β held constant. The key to the proof is then the following lemma. We presume it is well known but have not found a reference, so we include a proof in the online Appendix.

LEMMA 2: *If* $\tilde{m}_1 \sim$ *BetaBinomial*$\left(T, \bar{\alpha}, \lambda\bar{\alpha}\right)$ *and* $\tilde{m}_2 \sim$ *BetaBinomial*$\left(T, \underline{\alpha}, \lambda\underline{\alpha}\right)$, *where* $\bar{\alpha} > \underline{\alpha}$ *and* $\lambda > 0$, *then* $\tilde{m}_1$ *second order stochastically dominates* $\tilde{m}_2$.

If $\tilde{m}_1$ second order stochastically dominates $\tilde{m}_2$ then $\mathbb{E}\left[u\left(\tilde{m}_1\right)\right] \geq \mathbb{E}\left[u\left(\tilde{m}_2\right)\right]$ for any concave function $u\left(\cdot\right)$ (Rothschild and Stiglitz 1970). Therefore, if $1/p_{m,T}$ is convex (so that $-1/p_{m,T}$ is concave) then $\mathbb{E}\left[1/p_{\tilde{m}_1,T}\right] \leq \mathbb{E}\left[1/p_{\tilde{m}_2,T}\right]$, from which the first part of the result follows. If instead $1/p_{m,T}$ is concave, then the inequality is reversed. Finally, log-concavity of p is equivalent to $\left(p'\right)^2 \geq pp''$. This implies that $2\left(p'\right)^2 \geq pp''$, which is equivalent to $1/p$ being convex. ■

PROOF OF RESULT 4:

As the beta distribution is conjugate to the binomial distribution, investor h's posterior probability of an up move at node $\left(m,t\right)$ follows the distribution $\tilde{h}_{m,t} \sim \text{Beta}\left(\zeta h + m, \zeta(1-h) + t - m\right)$; thus $\mathbb{E}\left[\tilde{h}_{m,t}\right] = \left(h + m/\zeta\right)/\left(1 + t/\zeta\right)$. The agent's first-order condition is therefore

$$x_h = w_h\left(\frac{\frac{h + m/\zeta}{1 + t/\zeta}}{p - p_d} - \frac{1 - \frac{h + m/\zeta}{1 + t/\zeta}}{p_u - p}\right).$$

As in the main text, we have suppressed the dependence of asset demand x_h (and, below, of p and h^*) on m and t for notational convenience.

Starting from node $\left(m,t\right)$, the wealth of an investor is $w_h + x_h\left(p_u - p\right) = w_h\frac{h + m/\zeta}{h^*\left(1 + t/\zeta\right)}$ following an up-move (i.e., at node $\left(m+1, t+1\right)$) or $w_h + x_d\left(p_d - p\right) = w_h\left(1 - \frac{h + m/\zeta}{1 + t/\zeta}\right)/\left(1 - h^*\right)$ following a down-move (i.e., at node $(m, t+1)$). It follows, by induction, that the wealth of an investor at node (m,t) is $w_h = \tilde{\lambda}_{path} \cdot I_{m,t}(h)$, where

(A5)
$$I_{m,t}(h) = \underbrace{(1-h)\left(1-\frac{h}{1+\frac{1}{\zeta}}\right)\cdots\left(1-\frac{h}{1+\frac{t-m-1}{\zeta}}\right)}_{t-m \text{ down moves}}\underbrace{\left(\frac{h}{1+\frac{t-m}{\zeta}}\right)\cdots\left(\frac{h+\frac{m-1}{\zeta}}{1+\frac{t-1}{\zeta}}\right)}_{m \text{ up moves}}$$

$$= \frac{B\big(\zeta h + m, \zeta(1-h) + t - m\big)}{B\big(\zeta h, \zeta(1-h)\big)}.$$

(The ordering of up- and down-moves is immaterial because $\mathbb{E}\big[1-\tilde{h}_{m,t}\big]\mathbb{E}\big[\tilde{h}_{m,t+1}\big] = \mathbb{E}\big[\tilde{h}_{m,t}\big]\mathbb{E}\big[1-\tilde{h}_{m+1,t+1}\big]$.) As initial wealth does not depend on h, we have $I_0(h) = 1$. We can find the constant $\tilde{\lambda}_{path}$ by equating aggregate wealth to the value of the risky asset:

$$p = \tilde{\lambda}_{path}\int_0^1 I_{m,t}(h) f(h) dh. \tag{A6}$$

To clear the market, we must have

$$1 = \tilde{\lambda}_{path}\left[\int_0^1 I_{m,t}(h)\left(\frac{\frac{h+m/\zeta}{1+t/\zeta}}{p-p_d} - \frac{1-\frac{h+m/\zeta}{1+t/\zeta}}{p_u - p}\right) f(h)\, dh\right]. \tag{A7}$$

If we define

$$G_{m,t} = \frac{\int_0^1 I_{m,t}(h)(h+m/\zeta) f(h) dh}{(1+t/\zeta)\int_0^1 I_{m,t}(h) f(h) dh} = \frac{\int_0^1 I_{m+1,t+1}(h) f(h) dh}{\int_0^1 I_{m,t}(h) f(h) dh} \tag{A8}$$

then one can check that

$$1 - G_{m,t} = \frac{\int_0^1 I_{m,t+1}(h) f(h) dh}{\int_0^1 I_{m,t}(h) f(h) dh}. \tag{A9}$$

In these terms, equations (A6) and (A7) imply that

$$\frac{1}{p} = \frac{G_{m,t}}{p-p_d} - \frac{1-G_{m,t}}{p_u - p}.$$

Defining $z_{m,t} = 1/p$, $z_{m+1,t+1} = 1/p_u$, and $z_{m,t+1} = 1/p_d$, we can rewrite this as

$$z_{m,t} = G_{m,t} z_{m+1,t+1} + (1 - G_{m,t}) z_{m,t+1}.$$

By backward induction, and using the fact that $(1-G_{m,t})G_{m,t+1} = G_{m,t}(1-G_{m+1,t+1})$, we have $z_{0,0} = \sum_{m=0}^{T} \tilde{c}_m \cdot z_{m,T}$, where $\tilde{c}_m = \binom{T}{m}(1-G_{0,0})\cdots(1-G_{0,T-m-1})G_{0,T-m}\cdots G_{m-1,T-1}$. Using equations (A8) and (A9) to evaluate this as a telescoping product,

$$\tilde{c}_m = \binom{T}{m}\int_0^1 I_{m,T}(h) f(h) dh,$$

which completes the proof of the first part of the Result.

To prove the second part of the result, note from (A5) that $\binom{T}{m} I_{m,T} = \mathbb{P}(\tilde{m} = m)$ where $\tilde{m} \sim \text{BetaBinomial}\big(T, \zeta h, \zeta(1-h)\big)$, so $z_{0,0} = \int_0^1 \mathbb{E}[z_{\tilde{m}}] f(h) dh$. If $\tilde{m}_i \sim \text{BetaBinomial}\big(T, \zeta_i h, \zeta_i(1-h)\big)$ for $i = 1, 2$, where $\zeta_1 > \zeta_2$, then $\tilde{m}_1$ second order stochastically dominates $\tilde{m}_2$ by Lemma 2. It follows that if z_m is convex, $\mathbb{E}\big[z_{\tilde{m}_1}\big] < \mathbb{E}\big[z_{\tilde{m}_2}\big]$ for all h, and hence $p_{0,0}^{(1)} > p_{0,0}^{(2)}$. Also by Lemma 2, the converse is true if z_m is concave. ■

PROOF OF RESULT 5:

We will repeatedly use the fact that $\Gamma(z+1)/\Gamma(z) = z$ without comment. In the risky bond limit, $(1-\varepsilon)/\varepsilon \to \infty$, so the sentiment multiplier (19) simplifies to

$$g(t) = \left(\frac{\alpha}{\alpha+\beta}\right)^{T-t} \frac{\Gamma(\alpha+t)\Gamma(\alpha+\beta+T)}{\Gamma(\alpha+T)\Gamma(\alpha+\beta+t)}. \tag{A10}$$

It follows that $g(t)$ is increasing:

$$\frac{g(t+1)}{g(t)} = \frac{\alpha+\beta}{\alpha} \frac{\Gamma(\alpha+t+1)}{\Gamma(\alpha+t)} \frac{\Gamma(\alpha+\beta+t)}{\Gamma(\alpha+\beta+t+1)} = \frac{\alpha+\beta}{\alpha} \frac{\alpha+t}{\alpha+\beta+t} \geq 1.$$

In the bubbly asset limit, $(1-\varepsilon)/\varepsilon \to -1$, so the sentiment multiplier (19) simplifies to

$$g(t) = \frac{1 - \left(\frac{\alpha}{\alpha+\beta}\right)^{T-t}}{1 - \frac{\Gamma(\alpha+T)}{\Gamma(\alpha+t)} \frac{\Gamma(\alpha+\beta+t)}{\Gamma(\alpha+\beta+T)}}. \tag{A11}$$

Write $x(t) = \big[\alpha/(\alpha+\beta)\big]^{T-t}$ and $y(t) = \Gamma(\alpha+T)\Gamma(\alpha+\beta+t)/\big[\Gamma(\alpha+t) \times \Gamma(\alpha+\beta+T)\big]$, so that

$$g(t) = \frac{1-x(t)}{1-y(t)} \quad \text{and} \quad g(t+1) = \frac{1-x(t+1)}{1-y(t+1)} = \frac{1 - x(t)\frac{\alpha+\beta}{\alpha}}{1 - y(t)\frac{\alpha+\beta+t}{\alpha+t}}.$$

It follows that $g(t+1) > g(t)$ if and only if

$$\frac{t}{\alpha+t} x(t) y(t) + \frac{\alpha}{\alpha+t} y(t) > x(t). \tag{A12}$$

We can write

$$y(t) = \frac{\alpha+t}{\alpha+\beta+t} \frac{\alpha+t+1}{\alpha+\beta+t+1} \cdots \frac{\alpha+T-1}{\alpha+\beta+T-1},$$

which implies that

$$y(t) > \left(\frac{\alpha+t}{\alpha+\beta+t}\right)^{T-t}. \tag{A13}$$

We can use this fact to establish that inequality (A12) holds, as required:

$$\begin{aligned}
\frac{t}{\alpha+t} x(t) y(t) + \frac{\alpha}{\alpha+t} y(t) &> \frac{t}{\alpha+t} \left(\frac{\alpha}{\alpha+\beta} \frac{\alpha+t}{\alpha+\beta+t}\right)^{T-t} + \frac{\alpha}{\alpha+t} \left(\frac{\alpha+t}{\alpha+\beta+t}\right)^{T-t} \\
&> \left(\frac{t}{\alpha+t} \frac{\alpha}{\alpha+\beta} \frac{\alpha+t}{\alpha+\beta+t} + \frac{\alpha}{\alpha+t} \frac{\alpha+t}{\alpha+\beta+t}\right)^{T-t} \\
&= x(t).
\end{aligned}$$

The first inequality uses the definition of $x(t)$ and (A12); the second is Jensen's inequality. ∎

PROOF OF RESULT 6:

As discussed in the proof of Result 1, and noting that we have $2N$ periods in total, we can write $p_{0,0}^{-1} = \mathbb{E}[z_{\tilde{m},T}] = \mathbb{E}\left[e^{-\sigma\sqrt{2T}\frac{\tilde{m}-N}{\sqrt{N}}}\right]$, where $\tilde{m} \sim$ BetaBinomial $(2N, \alpha, \beta)$ and $\alpha = \theta N + \eta\sqrt{N}$ and $\beta = \theta N - \eta\sqrt{N}$. Paul and Plackett (1978) show that $\tilde{m}$, appropriately shifted by its mean and scaled by its standard deviation, converges in distribution and in moment-generating function (MGF) to a Normal random variable. The mean of $\tilde{m}$ is $2N\alpha/(\alpha + \beta)$, and its variance is $2N\alpha\beta(\alpha + \beta + 2N)/[(\alpha + \beta)^2(\alpha + \beta + 1)]$. Thus $\left(\tilde{m} - N - \frac{\eta}{\theta}\sqrt{N}\ \right)/\sqrt{\frac{1 + \theta}{2\theta}N} \longrightarrow \Psi \sim N(0,1)$ and $p_{0,0}^{-1} \rightarrow \mathbb{E}\exp\left[-\sigma\sqrt{2T}\left(\Psi\sqrt{\frac{1 + \theta}{2\theta}} + \frac{\eta}{\theta}\right)\right] = \exp\left[-\frac{\eta}{\theta}\sigma \times \sqrt{2T} + \frac{\theta + 1}{2\theta}\sigma^2 T\right]$. ∎

PROOF OF RESULT 7:

Fix t, and write $\phi = t/T$. Consider the perspective of agent h at time 0: for her, m (the number of up-moves that have occurred by time t) has a binomial distribution with mean $2\phi Nh$ and variance $2\phi Nh(1 - h)$. Hence, by the Central Limit Theorem, we can standardize m so that it converges in distribution and in MGF to a standard Normal distribution as $N \rightarrow \infty$: $\frac{(m - 2\phi Nh)}{\sqrt{2\phi Nh(1 - h)}} \rightarrow N(0,1)$. Again, we emphasize that we are taking the perspective of agent h.

When we take the limit as $N \rightarrow \infty$, it is convenient to parametrize the investor type by $z = (h - \tilde{\mathbb{E}}h)/\sqrt{\widetilde{\text{var}}h}$, as described in the main text. Similarly, we will parametrize m via $\psi = \frac{m - \mathbb{E}^{(0)}[m]}{\sqrt{\text{var}^{(0)}[m]}}$, the number of standard deviations by which m exceeds the expectations of the mean investor. Thus $m = \phi N + \left(\frac{\psi}{\sqrt{2}} + \frac{\eta}{\theta}\sqrt{\phi}\right)\sqrt{\phi N}$. (Here and throughout the proof, we neglect terms of lower order in N, which will be irrelevant in the limit.) In this notation, $\frac{m - 2\phi Nh}{\sqrt{2\phi Nh(1 - h)}} \rightarrow N(0,1)$ can be rewritten as

$$\psi - \frac{\sqrt{\phi}}{\sqrt{\theta}}z \rightarrow \xi \sim N(0,1). \tag{A14}$$

At time t, we have (by Lemma 1) $p_{m,t}^{-1} = \mathbb{E}\left[e^{-\sigma\sqrt{2T}\frac{m+\tilde{j}-N}{\sqrt{N}}}\right]$, where the expectation is over $\tilde{j} \sim$ BetaBinomial$(2(1 - \phi)N, \alpha + m, \beta + 2\phi N - m)$. After substituting in α and β, as given in the main text, together with the parametrization of m described above, we can standardize $\tilde{j}$ so that it converges in distribution and in MGF to a standard Normal random variable, as in the proof of Result 6, and we have the realized price, at time t, as a function of ψ:

$$p_t = \exp\left(-\frac{1 - \phi}{2}\frac{\theta + 1}{\theta + \phi}\sigma^2 T + \frac{\eta}{\theta}\sigma\sqrt{2T} + \frac{\theta + 1}{\phi + \theta}\sigma\sqrt{\phi T}\Psi\right). \tag{A15}$$

Thus, from (A14) and (A15), the investor perceives p_t as lognormally distributed, and

$$\mathbb{E}^{(z)} \log p_t = \frac{t(\theta+1)\frac{z}{\sqrt{\theta}}\sigma\sqrt{T} - \frac{1}{2}(T-t)(\theta+1)\sigma^2 T}{\theta T + t} + \frac{\eta}{\theta}\sigma\sqrt{2T}$$

and

$$\operatorname{var}^{(z)} \log p_t = \sigma^2 t \left(\frac{\theta+1}{\theta+\frac{t}{T}}\right)^2.$$

Her expected return is therefore

$$\begin{aligned}\mathbb{E}^{(z)}[R_{0\to t}] &= \mathbb{E}^{(z)}\left[\frac{p_t}{p_{0,0}}\right] \\ &= \exp\left\{\frac{\phi(\theta+1)}{\theta+\phi}\left[\frac{z}{\sqrt{\theta}}\sigma\sqrt{T} + \frac{\theta+1}{2}\left(\frac{1}{\theta}+\frac{1}{\theta+\phi}\right)\sigma^2 T\right]\right\}.\end{aligned}$$

The cross-sectional average expectation and disagreement follow immediately, using the fact that z has zero cross-sectional mean and unit variance.

To find the autocorrelation of returns, let m and $m+j$ be random variables representing the number of up-moves by times t_1 and t_2 respectively. As above, we have $\frac{m-2\phi_1 Nh}{\sqrt{2\phi_1 Nh(1-h)}} \to \xi_1$ and $\frac{m+j-2\phi_2 Nh}{\sqrt{2\phi_2 Nh(1-h)}} \to \xi_2$ as $N \to \infty$, where ξ_1, ξ_2 are standard Normal, $\sqrt{\phi_2}\xi_2 = \sqrt{\phi_1}\xi_1 + \sqrt{\phi_2-\phi_1}\,\Xi$, and $\Xi \sim N(0,1)$ is independent of ξ_1. We then have

$$\begin{aligned}\operatorname{cov}(\log R_{0\to t_1}, \log R_{t_1\to t_2}) &= \operatorname{cov}\left[\frac{\theta+1}{\phi_1+\theta}\sigma\sqrt{\phi_1 T}\xi_1, \frac{\theta+1}{\phi_2+\theta}\sigma\sqrt{\phi_2 T}\xi_2 - \frac{\theta+1}{\phi_1+\theta}\sigma\sqrt{\phi_1 T}\xi_1\right] \\ &= \frac{(\theta+1)^2}{(\phi_1+\theta)^2}\frac{\phi_1-\phi_2}{\phi_2+\theta}\sigma^2\phi_1 T.\end{aligned}$$

Moreover, $\operatorname{var}[\log R_{0\to t_1}] = \frac{(\theta+1)^2}{(\phi_1+\theta)^2}\sigma^2\phi_1 T$, and using the fact that $\operatorname{var}[\log R_{t_1\to t_2}] = \operatorname{var}[\log(p_{t_1})] + \operatorname{var}[\log(p_{t_2})] - 2\operatorname{cov}(\log(p_{t_1}), \log(p_{t_2}))$, we have

$$\begin{aligned}\operatorname{var}[\log R_{t_1\to t_2}] = \frac{(\theta+1)^2}{(\phi_1+\theta)^2(\theta+\phi_2)^2}\sigma^2\phi_1 T\bigg[&(\phi_2+\theta)^2 + \frac{\phi_2}{\phi_1}(\phi_1+\theta)^2 - \\ &2(\phi_1+\theta)(\phi_2+\theta)\bigg].\end{aligned}$$

Combining these, we find the expression for the autocorrelation given in the result. ∎

PROOF OF RESULT 8:

Note that $2\phi N$ is the number of periods corresponding to $t = \phi T$. Writing $q_{m,t}$ for the risk-neutral probability of going from node $(0,0)$ to node (m,t), we have (as in

Lemma 1) $q_{m,t} = \frac{p_0}{p_{m,t}} c_{m,t}$, where $c_{m,t} = \binom{2\phi N}{m} \dfrac{B(\alpha + m, \beta + 2\phi N - m)}{B(\alpha, \beta)}$. As the riskless rate is 0, the time zero price of a call option with strike K, maturing at time t, is

$$C(0,t;K) = \sum_{m=0}^{2\phi N} q_{m,t}(p_{m,t} - K)^+ = p_0 \sum_{m=0}^{2\phi N} c_{m,t}\left(1 - \frac{K}{p_{m,t}}\right)^+ = p_0 \mathbb{E}\left[\left(1 - \frac{K}{p_{m,t}}\right)^+\right].$$

The expectation is over $m \sim \text{BetaBinomial}(2\phi N, \alpha, \beta)$ which is asymptotically Normal as above. Using (A15) to substitute for the price we have, in the limit,

$$C(0,t;K) = p_0 \mathbb{E}\left[\left(1 - Ke^{\frac{1-\phi}{2}\frac{\theta+1}{\theta+\phi}\sigma^2 T - \frac{\eta}{\theta}\sigma\sqrt{2T} - \frac{\theta+1}{\theta+\phi}\sigma\sqrt{\phi T}\sqrt{\frac{\phi+\theta}{\theta}}\Psi}\right)^+\right],$$

where $\Psi \sim N(0,1)$. (Convergence in distribution implies convergence in expectation by the Helly–Bray theorem, as the function of Ψ inside the expectation is bounded and continuous.) The expectation can be evaluated as in Black and Scholes (1973), giving the result.

Lastly, we can calculate the variance risk premium at arbitrary horizons $t < T$. We have $\text{var}^* \log R_{0\to t} = \mathbb{E}^*\left[(\log R_{0\to t})^2\right] - \left[\mathbb{E}^*(\log R_{0\to t})\right]^2$. Each of the risk-neutral expectations is determined by the prices of options expiring at time t, by the logic of Breeden and Litzenberger (1978), so $\text{var}^* \log R_{0\to t}$ is as it would be in the Black–Scholes model with constant volatility $\tilde{\sigma}_t$. As is well known, this is $\tilde{\sigma}_t^2$ in annualized terms. Using the expression for $\text{var} \log R_t$ provided in Result 7, we have a generalization of the result given in the text:

$$\frac{1}{t}\left(\text{var}^* \log R_{0\to t} - \text{var} \log R_{0\to t}\right) = \frac{(\theta + 1)^2 \frac{t}{T}}{\theta\left(\theta + \frac{t}{T}\right)^2}\sigma^2. \blacksquare$$

PROOF OF RESULT 9:

As all investors have log utility, $R_{0\to T}^{(z)}$ is the growth optimal return from 0 to T as perceived by investor z, which equals $1/M_{0\to T}^{(z)}$ where $M_{0\to T}^{(z)}$ is the SDF perceived by investor z. The following lemma provides a formula for this quantity.

LEMMA 3: *The SDF of investor* z, $M_{0\to t}^{(z)}$, *is given by*

$$\text{(A16)} \quad M_{0\to t}^{(z)} = \sqrt{\frac{\theta}{\theta + \phi}} \exp\left\{\frac{\theta + \phi}{2(1 + \theta)^2 \sigma^2 T}\left[\log\left(R_{0\to t}/K_t^{(z)}\right)\right]^2 - \frac{1}{2}(z - z_g)^2\right\},$$

where $\log K_t^{(z)} = \mathbb{E}^{(z)} \log R_{0\to t} + \left[(\theta + 1)/(\theta + \phi)\right](z - z_g)\sigma\sqrt{\theta T}$.

PROOF:

$M_{0\to t}^{(z)}$ links investor z's perceived true probabilities to the objectively observed risk-neutral probabilities, which we computed in Lemma 1. Hence the value of agent z's SDF at node (m,t) is $M_{0\to t}^{(z)} = \frac{p_{0,0}}{p_{m,t}} \frac{c_{m,t}}{\pi_{m,t}^{(z)}}$, where $\pi_{m,t}^{(z)}$ is the agent's subjective probability of arriving at node (m,t) and $c_{m,t}$ was defined in the proof of Result 8. We established in the proofs of Results 7 and 8 that $c_{m,t}$ and $\pi_{m,t}^{(z)}$ correspond to the probability mass functions of a beta-binomial distribution and of a binomial distribution, which converge (after appropriate rescaling) to the pdf of the Normal distribution. For the binomial distribution we standardize m via $\psi = \frac{m - \phi N - \frac{\eta}{\theta}\phi\sqrt{N}}{\sqrt{\phi N/2}}$, as in the proof of Result 7, while for the beta-binomial distribution we set $\tilde{\psi} = \frac{\psi}{\sqrt{(\phi+\theta)/\theta}}$. Then $c_{m,t} = \mathbb{P}\left(\frac{X_N - \phi N - \frac{\eta}{\theta}\phi\sqrt{N}}{\sqrt{\frac{\phi+\theta}{2\theta}\phi N}} = \tilde{\psi}\right) \sim \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\tilde{\psi}^2} \frac{1}{\sqrt{\frac{\phi+\theta}{2\theta}\phi N}}$, where X_N is a sequence of beta-binomial random variables, and $\pi_{m,t}^{(z)} \sim \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\psi - \frac{\sqrt{\phi}}{\sqrt{\theta}}z\right)^2} \frac{1}{\sqrt{\phi N/2}}$. Thus $M_t^{(z)} \sim \sqrt{\frac{\theta}{\phi+\theta}} \frac{1}{R_{0\to t}} e^{-\frac{\theta}{2(\phi+\theta)}\psi^2 + \frac{1}{2}\left(\psi - \frac{\sqrt{\phi}}{\sqrt{\theta}}z\right)^2}$. Using the fact that $\psi - \frac{\sqrt{\phi}}{\sqrt{\theta}} z = \frac{\log R_{0\to t} - \mathbb{E}^{(z)}[\log R_{0\to t}]}{\sqrt{\mathrm{var}^{(z)} \log R_{0\to t}}}$ (which follows from equation (A15)) and rearranging, we have equation (A16) as $N \to \infty$. ∎

Equation (23) and the remaining statements of the result follow immediately. For future reference, note that as $\psi - \left(\sqrt{\phi}/\sqrt{\theta}\right)z \xrightarrow{d} \xi \sim N(0,1)$ from (A14), we have

$$M_{0\to t}^{(z)} \xrightarrow{d} \sqrt{\frac{\theta}{\phi+\theta}} e^{-\frac{\theta}{2(\phi+\theta)}\left(\xi + \frac{\sqrt{\phi}}{\sqrt{\theta}}z\right)^2 + \frac{1}{2}\xi^2 - \frac{\theta+1}{\theta+\phi}\sigma\sqrt{t}\,\xi - \mathbb{E}^{(z)}[\log R_{0\to t}]}, \tag{A17}$$

where $\xi \sim N(0,1)$. ∎

PROOF OF RESULT 10:

We exploit a lemma in the spirit of Breeden and Litzenberger (1978) whose proof we provide in the online Appendix:

LEMMA 4: *Write $W(p_T)$ for an investor's wealth at time T, as a function of the price of the risky asset p_T. Suppose that $W(0) = 0$. Then, terminal wealth $W(p_T)$ can be achieved by holding a portfolio of (i) $W'(K_0)$ units of the underlying asset, (ii) bonds with face value $W(K_0) - K_0 W'(K_0)$, (iii) $W''(K)dK$ put options on the risky asset maturing at time T with strike K, for every $K < K_0$, and (iv) $W''(K)dK$ call options maturing at time T with strike K, for every $K > K_0$. The constant $K_0 > 0$ can be chosen arbitrarily.*

We can write investor z's wealth at time T as $W^{(z)}(p_T) = p_0 R_{0\to T}^{(z)}(p_T/p_0)$, where $R_{0\to T}^{(z)}$ is given as a function of $R_{0\to T} = p_T/p_0$ in equation (23). The result follows by applying Lemma 4 with $K_0 = p_0 K^{(z)}$, and noting that $W^{(z)\prime}(K_0) = 0$ from the definition (24) of $K^{(z)}$. Furthermore, from (23) we have

$$\mathrm{sign}\left[W^{(z)\prime\prime}\left(\exp \mathbb{E}^{(z)} \log p_T\right)\right] = \mathrm{sign}\left[z^2 - z_g z - \frac{\theta+1}{\theta}\right],$$

which is negative for moderate investors (including all investors $z \in [z_g, 0]$), and positive if $|z|$ is sufficiently large. ∎

PROOF OF RESULT 11:

As the market is complete, there is a strategy that attains the MSR implied by the Hansen and Jagannathan (1991) bound. In order to be able to use equation (A17) to compute the variance of M_t—for the rest of the proof, we write $M_{0\to t}^{(z)} = M_t$ to save space—we will first need the following Lemma which we prove in the online Appendix:

LEMMA 5: *If* $\theta > 1$, *then the sequence* $\left\{\left(M_t^2\right)^{(N)}\right\}$ (*where we include superscripts to emphasize the dependence on* N) *is uniformly integrable.*

Uniform integrability implies convergence of expectations. We can thus use equation (A17) to find the variance of M_t from the perspective of agent z, as $N \to \infty$, by computing the MGF of a chi-squared random variable. Doing so, we find that

$$\mathbb{E}^{(z)}\left[M_t^2\right] = \frac{\theta}{\sqrt{\theta^2 - \phi^2}} \exp\left[\frac{\left[z\sqrt{\theta\phi} + (\theta + 1)\sigma\sqrt{\phi T}\right]^2}{\theta(\theta - \phi)}\right].$$

If $\theta \leq 1$ and $\phi = 1$ (i.e., $t = T$), we have $\liminf \mathbb{E}^{(z)}\left[\left(M_T^2\right)^{(N)}\right] \geq \mathbb{E}^{(z)}\left[e^{\frac{1}{\theta+1}\xi^2 + B\xi + C}\right]$ for some constants B and C, using Fatou's lemma and equation (A17). As $1/(\theta + 1) \geq 1/2$ and $\xi \sim N(0,1)$, the expectation on the right-hand side is infinite, so $\mathbb{E}^{(z)}\left[M_T^2\right] = \infty$. ∎

PROOF OF RESULT 12:

Write $r_{0\to T}^{(x)} = \log R_{0\to T}^{(x)}$ and $r_{0\to T} = \log R_{0\to T}$. Rearranging (23), we have

$$r_{0\to T}^{(x)} = \frac{1}{2}\log\frac{\theta + 1}{\theta} + \frac{1}{2}(x - z_g)^2 - \frac{1}{2(1 + \theta)} \times \left\{\frac{r_{0\to T} - \mathbb{E}^{(z)} r_{0\to T}}{\sigma\sqrt{T}} + \frac{\mathbb{E}^{(z)} r_{0\to T} - \mathbb{E}^{(x)} r_{0\to T}}{\sigma\sqrt{T}} - \sqrt{\theta}(x - z_g)\right\}^2.$$

As $\mathbb{E}^{(z)} r_{0\to T} - \mathbb{E}^{(x)} r_{0\to T} = (z - x)\sigma\sqrt{T}/\sqrt{\theta}$ and $\frac{r_{0\to T} - \mathbb{E}^{(z)} r_{0\to T}}{\sigma\sqrt{T}}$ is a zero-mean, unit-variance random variable in the opinion of agent z,

$$\mathbb{E}^{(z)} r_{0\to T}^{(x)} = \frac{1}{2}\log\frac{\theta + 1}{\theta} + \frac{1}{2}(x - z_g)^2 - \frac{1}{2(1 + \theta)} \times \left\{1 + \left[\frac{z - x}{\sqrt{\theta}} - \sqrt{\theta}(x - z_g)\right]^2\right\}.$$

Result 7 showed that

$$\mathbb{E}^{(z)} r_{0\to T} = \frac{z\sigma\sqrt{T}}{\sqrt{\theta}} + \frac{\theta + 1}{2\theta}\sigma^2 T = -\frac{z z_g}{\theta + 1} + \frac{1}{2}\frac{z_g^2}{\theta + 1},$$

where we use the definition of z_g in the second equality. It follows that

$$\mathbb{E}^{(z)}\left(r^{(x)}_{0\to T} - r_{0\to T}\right) = \frac{1}{2}\log\frac{\theta+1}{\theta} + \frac{z^2-1}{2(1+\theta)} - \frac{(z-x)^2}{2\theta},$$

which gives the first part of the result because $\log\left(1+\xi^{(z,x)}\right) = \mathbb{E}^{(z)}\left(r^{(x)}_{0\to T} - r_{0\to T}\right)$.

As the asymptotic distribution of types x is standard Normal, $\xi = \int_{-\infty}^{\infty}\xi^{(z,x)}g(x)dx$ where $g(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, and evaluating the integral gives the final part of the result. ∎

PROOF OF RESULT 13:

From the definition (29), we see that $\log(1-A_T) = \frac{1}{N}\sum_{i=1}^{N}\log y_i - \log\mu$, where μ is cross-sectional average wealth. In our setting, with a continuum of investors, this becomes $\log(1-A_T) = \tilde{\mathbb{E}}\log W^{(z)} - \log\tilde{\mathbb{E}}W^{(z)}$. (As before, we use the notation $\tilde{\mathbb{E}}$ to denote a cross-sectional expectation that averages across agents.) As $\tilde{\mathbb{E}}W^{(z)} = p_0R_{0\to T}$ equals aggregate wealth, whereas $\log W^{(z)} = \log\left(p_0R^{(z)}_{0\to T}\right)$ equals the log return chosen by investor z, we have $\log(1-A_T) = \tilde{\mathbb{E}}\log R^{(z)}_{0\to T} - \log R_{0\to T}$. Henceforth, we write $r^{(z)}_{0\to T} = \log R^{(z)}_{0\to T}$ and $r_{0\to T} = \log R_{0\to T}$, so that $\log(1-A_T) = \tilde{\mathbb{E}}r^{(z)}_{0\to T} - r_{0\to T}$.

We can rewrite equation (23) as

$$r^{(z)}_{0\to T} = \frac{1}{2}\log\frac{\theta+1}{\theta} + \frac{1}{2}(z-z_g)^2 - \frac{1}{2(1+\theta)} \times \left[\frac{r_{0\to T} - z\sigma\sqrt{\frac{T}{\theta}} - \frac{\theta+1}{2\theta}\sigma^2T}{\sigma\sqrt{T}} - \sqrt{\theta}(z-z_g)\right]^2.$$

This expression is quadratic in z. As z has zero mean and unit variance, so that $\tilde{\mathbb{E}}z = 0$ and $\tilde{\mathbb{E}}z^2 = 1$, we have (after some algebra)

$$\tilde{\mathbb{E}}r^{(z)}_{0\to T} = \frac{1}{2}\log\frac{\theta+1}{\theta} + \frac{1}{2}\left(1+z_g^2\right) - \frac{1}{2(1+\theta)} \times \left\{\left[\frac{r_{0\to T}}{\sigma\sqrt{T}} - \frac{\theta+1}{2\theta}\sigma\sqrt{T} - (\theta+1)\sigma\sqrt{T}\right]^2 + \left(\sqrt{\theta}+\frac{1}{\sqrt{\theta}}\right)^2\right\}.$$

Using the expression for z_g given in equation (21) and simplifying, we find that

$$\log(1-A_T) = \frac{1}{2}\left(\log\frac{\theta+1}{\theta} - \frac{1}{\theta}\right) - \frac{1}{2(1+\theta)\sigma^2T}\left(r_{0\to T} - \frac{1+\theta}{2\theta}\sigma^2T\right)^2.$$

This is equivalent to the expression given in the text. ∎

PROOF OF RESULT 14:

There are N periods of length T/N. Let us write $t = \phi T$. Suppose there have been $n = q$ down-moves (jumps) and $m = \phi N - q$ up-moves by time t. If $\tilde{q}$ of the remaining $(1-\phi)N$ periods are down-moves and j are up-moves, then we must have $\tilde{q} + j = (1 - \phi)N$. From Lemma 1, the price at time t is $\left[\mathbb{E}\left(e^{(q+\tilde{q})J}\right)\right]^{-1}$, where the expectation is over $\tilde{q} \sim \mathrm{BetaBinomial}\big((1 - \phi)N, q + 1/\omega^2, N/(\omega^2\lambda T) + \phi N - q\big)$. We now use the fact that as $n \to \infty$, a beta binomial distribution with parameters n, α, Cn approaches a negative binomial distribution with $r = \alpha$ and $p = 1/(1 + C)$. Therefore, as $N \to \infty$, $\tilde{q}$ is asymptotically distributed as a negative binomial distribution with parameters $q + 1/\omega^2$ and $\omega^2\lambda T(1 - \phi)/(1 + \omega^2\lambda T)$. Using the formula for the MGF of a negative binomial distribution, the price equals

$$e^{-qJ}\left[\frac{1 - \omega^2\lambda T\left(e^J - 1\right) + \omega^2\lambda t e^J}{1 + \omega^2\lambda t}\right]^{q+\frac{1}{\omega^2}}.$$

Simplifying this expression gives the price (31).

As the riskless rate equals zero, agent z's SDF equals the ratio of the risk-neutral probability of q jumps occurring by time t to the corresponding true probability (which is $\left(\lambda(1 - z\omega)t\right)^q e^{-\lambda(1-z\omega)t}/q!$). As in the proof of Result 8, the risk-neutral probability of $m = \phi N - q$ up-moves having occurred during the first ϕN moves is $\left(p_0/p_{m,\phi N}\right)x_N$, where x_N is the probability of m realizations in a beta-binomial distribution with parameters ϕN, $N/(\omega^2\lambda T)$, and $1/\omega^2$ or, equivalently, the probability of $\phi N - m = q$ realizations in a beta-binomial distribution with parameters $\left(\phi N, 1/\omega^2, N/(\omega^2\lambda T)\right)$. In the limit as $N \to \infty$, using the convergence of this beta binomial to a negative binomial distribution with parameters $1/\omega^2, \frac{\omega^2\lambda\phi T}{1 + \omega^2\lambda\phi T}$, we find that the probability x_N is therefore equal to

$$\frac{\Gamma\left(q + \frac{1}{\omega^2}\right)}{q!\,\Gamma\left(\frac{1}{\omega^2}\right)}\left(\frac{1}{1 + \omega^2\lambda t}\right)^{\frac{1}{\omega^2}}\left(\frac{\omega^2\lambda t}{1 + \omega^2\lambda t}\right)^q.$$

Similarly, as N tends to infinity, $p_0/p_{m,\phi N}$ tends to the reciprocal of the return from 0 to t conditional on q jumps having occurred, as provided in Result 14. The SDF follows as stated. To calculate $\mathbb{E}^{(z)}\log M^{(z)}_{0\to T}$, note that investor z perceives the number of jumps, q, that occur by time T as distributed according to a Poisson distribution with parameter $\lambda(1 - z\omega)T$. As $\Gamma(z) = O\left(z^{z-1/2}e^{-z}\right)$, it follows that $\mathbb{E}^{(z)}\log M^{(z)}_{0\to T}$ is finite but $\mathbb{E}^{(z)}\left[\left(M^{(z)}_{0\to T}\right)^2\right]$ is infinite. ∎

PROOF OF RESULT 15:

The risk-neutral probability inside the limit in (33) is the price of a security with unit payoff if there is at least one jump in $[t, t + \varepsilon]$. As the interest rate is zero, this price equals $1 - x_\varepsilon$, where x_ε is the price of a security with unit payoff if there are no jumps between t and $t + \varepsilon$. A straightforward calculation gives

$$x_\varepsilon = \left[1 + \frac{\varepsilon\omega^2\lambda e^J}{1 - \omega^2\lambda T\left(e^J - 1\right) + \omega^2\lambda t e^J}\right]^{-q-1/\omega^2}.$$

As $\lambda^*_t = \lim_{\varepsilon\to 0}\frac{1 - x_\varepsilon}{\varepsilon}$, the result follows by the binomial theorem. ∎

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