Sentiment and speculation in a market with heterogeneous beliefs

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Introduction

- Agents disagree about the probabilities of good/bad news
- Optimists go long; pessimists go short
- If the market rallies, optimists get rich; if the market sells off, pessimists get rich
- So prices embed ex post winners' beliefs
- Sentiment creates volatility, and induces speculation: agents may even trade in the opposite direction to their own fundamental views

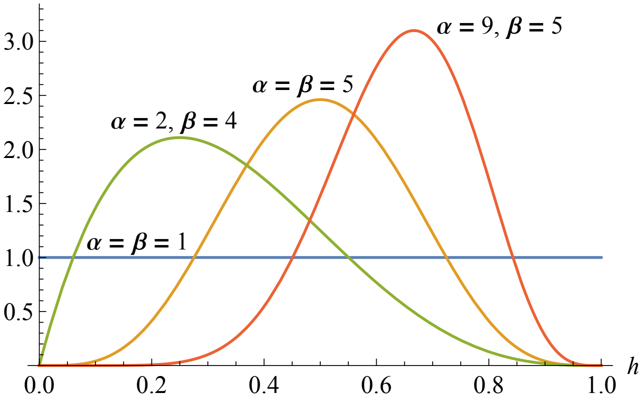
Related literature

An incomplete and somewhat arbitrary list

- Heterogeneous beliefs
  - Keynes (1936, Chapter 12); Harrison and Kreps (1978); Scheinkman and Xiong (2003); Geanakoplos (2010); Simsek (2013); Basak (2005); Banerjee and Kremer (2010); Atmaz and Basak (2018); Zapatero (1998); Jouini and Napp (2007); Bhamra and Uppal (2014); Kogan, Ross, Wang, and Westerfield (2006); Buraschi and Jiltsov (2006); Sandroni (2000); Borovička (2020); Blume and Easley (2006); Cvitanić, Jouini, Malamud, and Napp (2011); Chen, Joslin, and Tran (2012); Ottaviani and Sorenson (2015); ...
- Heterogeneous risk aversion
  - Dumas (1989); Chan and Kogan (2002); Longstaff and Wang (2012); ...

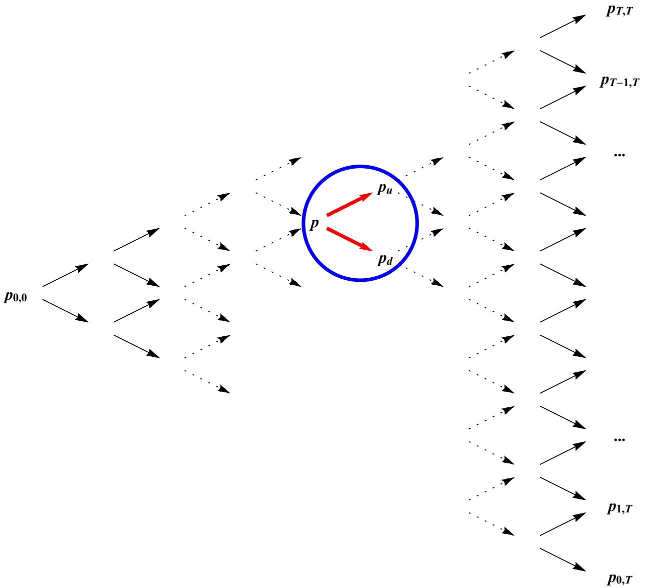
Setup

- All investors are endowed with one unit of a risky asset which evolves on a binomial tree with exogenous terminal payoffs
- Investor $h \in (0, 1)$ thinks the probability of an up-move is h
- Investors have log utility over terminal wealth
- The interest rate is normalized to zero
- No learning (today; see the paper for results with learning)



- Paper handles arbitrary belief distributions
- Today, beta distribution, pdf $f(h) \propto h^{\alpha-1}(1-h)^{\beta-1}$ where $\alpha, \beta > 0$: lets us consider Brownian and Poisson limits

Log investors are myopic



Equilibrium (1): individual optimization

- Solve backwards: the price of the risky asset is p_d or p_u next period
- Agent h has wealth w_h and holds x_h units of the asset (price p)
- So portfolio problem is

$$\max_{x_h} h \log \underbrace{\left[w_h - x_h p + x_h p_u\right]}_{\text{wealth in up state}} + (1-h) \log \underbrace{\left[w_h - x_h p + x_h p_d\right]}_{\text{wealth in down state}}$$

- First order condition:

$$x_h = w_h \left(\frac{h}{p - p_d} - \frac{1-h}{p_u - p} \right)$$

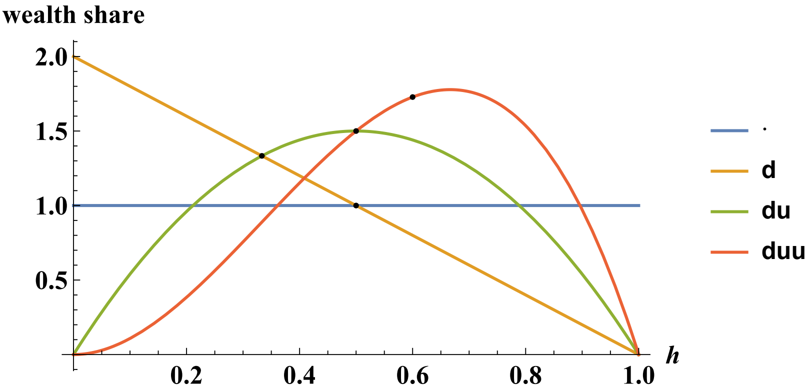
- Helpful to rewrite the FOC in terms of the risk-neutral probability of an up-move, h^*, which is defined via $p = h^* p_u + (1 - h^*) p_d$
- The realized return on wealth, for agent h, is then

$$\frac{h}{h^*} \quad \text{in the up state;} \qquad \frac{1-h}{1-h^*} \quad \text{in the down state}$$

- So after m up and n down steps, agent h's wealth is $\lambda_{\text{path}} h^m (1-h)^n$
- To pin down λ_{path}, note that aggregate wealth equals p, so

$$w_h = \frac{B(\alpha, \beta)}{B(\alpha + m, \beta + n)} p h^m (1-h)^n$$

- The richest agent is $h = m/(m+n)$, who looks right in hindsight



- Figure assumes uniform distribution of beliefs, i.e., $\alpha = \beta = 1$
- Less disagreement $\implies$ smaller shifts in wealth distribution

Equilibrium (2): market clearing

- From the FOC,

$$x_h = \underbrace{\frac{B(\alpha,\beta)}{B(\alpha+m,\beta+n)} p h^m (1-h)^n}_{w_h} \left(\frac{h}{p-p_d} - \frac{1-h}{p_u-p} \right)$$

- The equilibrium price ensures that, in aggregate, agents hold one unit of the asset:

$$p = \frac{(m+n+\alpha+\beta)p_u p_d}{(m+\alpha)p_d + (n+\beta)p_u}$$

A general pricing formula

Result

If the risky asset has terminal payoffs $p_{m,T}$ then its initial price is

$$p_0 = \frac{1}{\displaystyle\sum_{m=0}^{T} \frac{c_m}{p_{m,T}}}$$

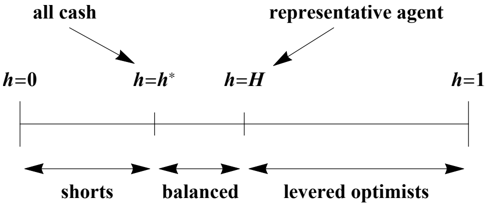
where

$$c_m = \binom{T}{m} \frac{B(\alpha + m, \beta + T - m)}{B(\alpha, \beta)}$$

Result (The effect of sentiment)

The price p_0 falls as disagreement increases if $\frac{1}{p_{m,T}}$ is convex in m (and rises if $\frac{1}{p_{m,T}}$ is concave)

Two special investors



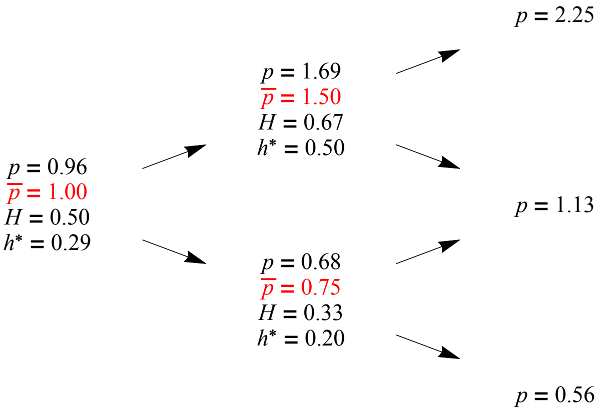
- In equilibrium,

$$\text{risky share of agent } h = \frac{h - h^*}{H - h^*} \quad \text{where} \quad H = \frac{m + \alpha}{m + n + \alpha + \beta}$$

- $h = H$ is the rep agent—"Mr. Market"
- $h = h^*$ is out of the market—a bond investor who's fully in cash
- H and h^*—and hence the identity of Mr. Market—change over time

An example

$\alpha = \beta = 1$



p: price. $\overline{p}$: price in homogeneous economy. H: rep agent

h^*: cash investor (cutoff between longs and shorts)

- Agents disagree on the risk premium

$$\text{agent } h\text{'s perceived risk premium} = \frac{(h-h^*)(H-h^*)}{h^*(1-h^*)}$$

- But they agree on objectively measurable quantities, such as

$$\text{risk-neutral variance} = \frac{(H-h^*)^2}{h^*(1-h^*)}$$

or

$$\text{VIX}^2 = 2\left[h^* \log \frac{h^*}{H} + (1-h^*) \log \frac{1-h^*}{1-H}\right]$$

- Notice that

$$\text{risky share of agent } h = \frac{h-h^*}{H-h^*} = \frac{\text{agent } h\text{'s risk premium}}{\text{risk-neutral variance}}$$

- In particular, the risk premium perceived by Mr. Market equals risk-neutral variance

Example 1: A risky bond

- $T = 50$ periods to go. Uniform beliefs
- Bond defaults (recover 30) in the bottom state. Else pays 100
- In order of increasing pessimism:
  - $h = 0.50$ thinks default prob is less than 10^{-15}
  - $h = 0.25$ thinks default prob is less than 10^{-6}
  - $h = 0.10$ thinks default prob is less than 0.006
  - $h = 0.05$ thinks default prob is less than 8%
  - $h = 0.01$ thinks default prob is more than 60%
- Initially, $h = 0.50$ is the representative agent
- What price does the bond trade at?

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- Who will *stay* short? marginal agent h^* in period $0, 1, 2, \ldots$ is $h = 0.48, 0.31, 0.22, \ldots$

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- What price does the bond trade at? at \$95.63
- Who would go short, at this price? everyone below $h = 0.48$!
- Who will *stay* short? marginal agent h^* in period $0, 1, 2, \ldots$ is $h = 0.48, 0.31, 0.22, \ldots$; only $h < 0.006$ stay short to the bitter end

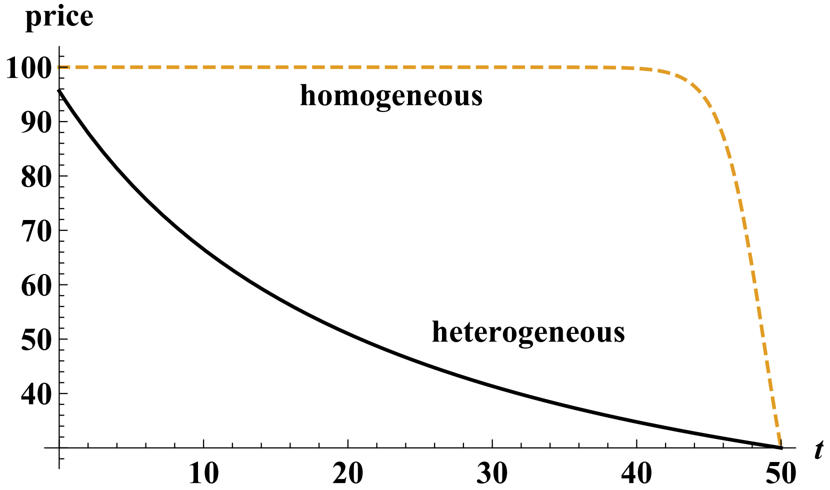


Figure: The risky bond's price over time following consistently bad news

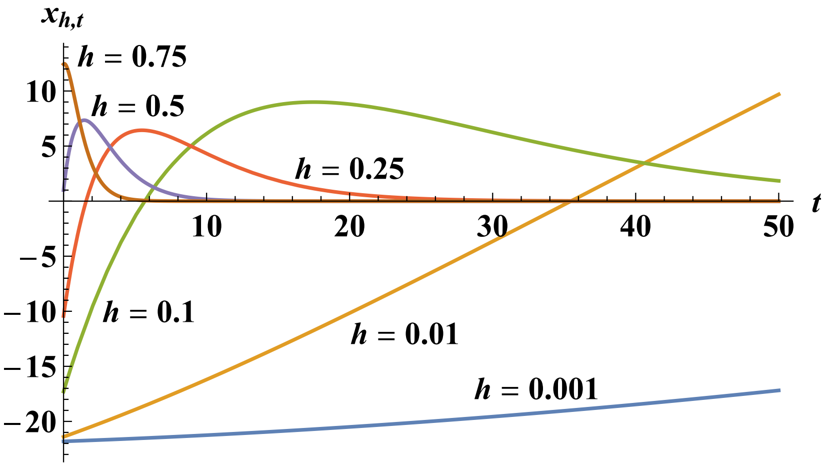


Figure: The number of units of the risky bond held by different agents, $x_{h,t}$, plotted against time

- Price is low at time zero because all investors—even “reasonable” ones—worry about the short-term effect of bad news on sentiment
- The risk-neutral probability of default, δ^*, is 6.25%

$$\delta^* = \frac{1}{1+\varepsilon T} = O(1/T)$$

- In the homogeneous economy, it is less than 10^{-14}

$$\delta^* = \frac{1}{1+\varepsilon\left(2^T-1\right)} = O\left(2^{-T}\right)$$

- Polynomial / exponential dichotomy holds for any finite α, β; and if “recovery value” is greater than 100 (bubbly asset)
- Sentiment makes long-dated extreme securities far more valuable

Speculative strategies vs. fundamental views

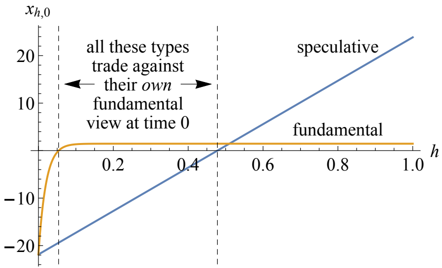


Figure: Positions of different investors at time 0 under dynamic ("speculative") and static ("fundamental") trade

- Investor $h = 0.25$ thinks there's less than a 10^{-6} chance of default, so risky bond is almost sure to deliver an excess return $\sim 5\%$
- Nonetheless, goes short initially to speculate on sentiment

Example 2: Risky bond vs. bubbly asset

Left-skewed (risky bond)

- Sentiment drives price down
- Price drop occurs early
- Volatility declines over time
- Median investor increasingly bullish

Right-skewed (bubbly asset)

- Sentiment drives price up
- Bubble emerges late
- Volatility rises over time
- Median investor bullish, then bearish, then bullish

- Risk drives the price **toward** the worst-case scenario for left-skewed asset, and **away** from the best-case scenario for right-skewed asset
- Result: it's all over more quickly for left-skewed asset. High vol and risk premia late in the game for right-skewed asset

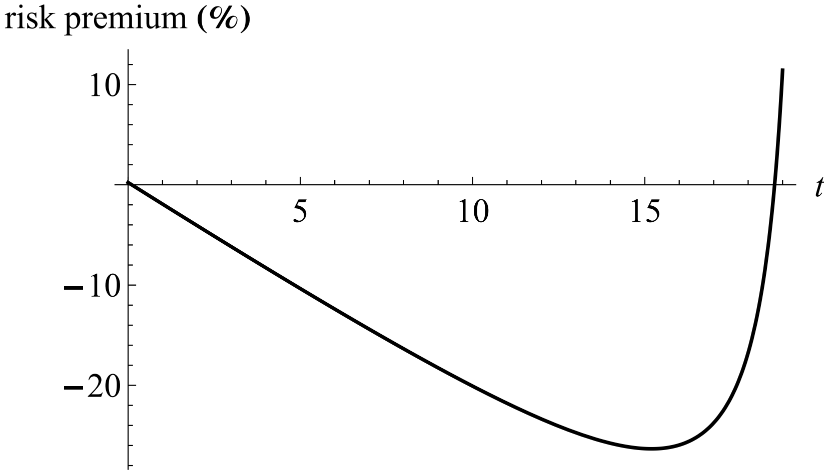


Figure: Median investor's expected excess return on the bubbly asset

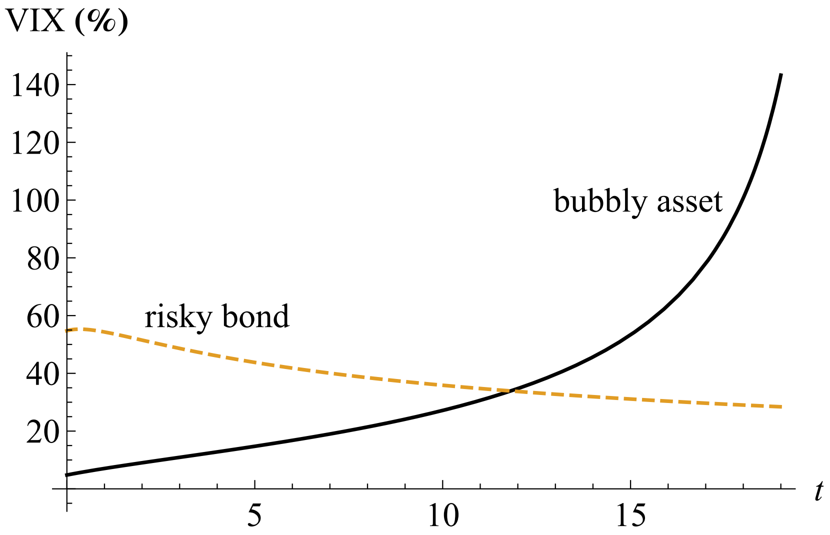


Figure: VIX over time following consistently good/bad news

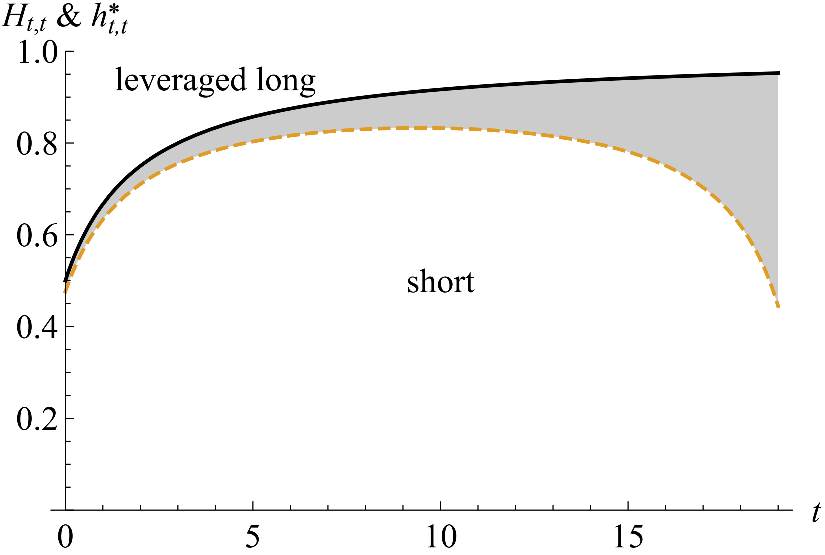


Figure: Identities of the rep investor and cash investor over time

Example 3: A diffusion limit

- Slice the period from 0 to T into $2N$ short periods
- Cox–Ross–Rubinstein terminal payoffs, $p_{m,T} = e^{2\sigma\sqrt{\frac{T}{2N}}(m-N)}$
- Tune down per-period disagreement by parametrizing $\alpha = \beta = \theta N$
- Low θ: lots of disagreement. $\theta \to \infty$: homogeneous economy
- Convenient to index agents by their z-score, the number of s.d. by which they are more/less optimistic than the mean
- As $N \to \infty$, everyone perceives the risky return as lognormal
- This is a world in which people agree on second moments (volatility) but disagree on first moments (the risk premium)

Result (Subjective expectations)

The (annualized) expected return of the asset from 0 to t from the perspective of a trader z is:

$$\frac{1}{t}\log \mathbb{E}^{(z)} R_{0\to t} = \frac{\theta+1}{\theta+\frac{t}{T}}\left[\frac{z\sigma}{\sqrt{\theta T}} + \frac{\theta+1}{\theta}\frac{\theta+\frac{t}{2T}}{\theta+\frac{t}{T}}\sigma^2\right]$$

In particular, the cross-sectional average expected return is

$$\widetilde{\mathbb{E}}\frac{1}{t}\log \mathbb{E}^{(z)} R_{0\to t} = \frac{(\theta+1)^2\left(\theta+\frac{t}{2T}\right)}{\theta\left(\theta+\frac{t}{T}\right)^2}\sigma^2$$

Disagreement is the cross-sectional standard deviation of expected returns:

$$\textit{disagreement} = \frac{\theta+1}{\theta+\frac{t}{T}}\frac{\sigma}{\sqrt{\theta T}}$$

Result (Option pricing and the volatility term structure)

The time 0 price of a call option with maturity t and strike price K obeys the Black–Scholes formula with implied volatility

$$\widetilde{\sigma}_t = \frac{\theta + 1}{\sqrt{\theta(\theta + \frac{t}{T})}}\,\sigma$$

In particular, short-dated options have $\widetilde{\sigma}_0 = \frac{\theta+1}{\theta}\sigma$ *and long-dated options have* $\widetilde{\sigma}_T = \sqrt{\frac{\theta+1}{\theta}}\sigma$*. As all agents agree on true volatility*

$$\sigma_t^{(z)} = \left(\frac{\theta + 1}{\theta + \frac{t}{T}}\right)\sigma\,,$$

there is a variance risk premium $\frac{1}{T}\left(\mathrm{var}^* \log R_{0\to T} - \mathrm{var} \log R_{0\to T}\right) = \frac{\sigma^2}{\theta}$

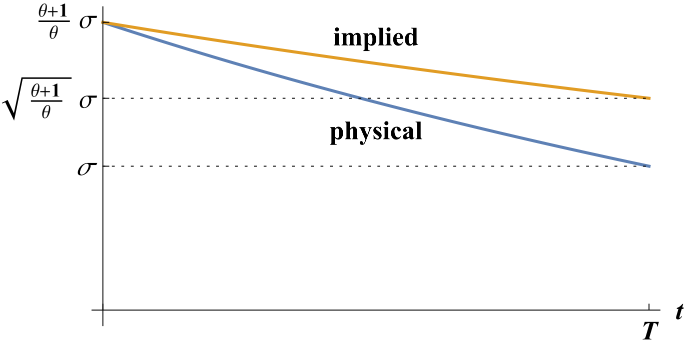


Figure: The term structures of implied and physical volatility

- Variance risk premium $\frac{1}{T}\left(\text{var}^* \log R_{0\to T} - \text{var} \log R_{0\to T}\right) = \frac{\sigma^2}{\theta}$

An illustrative calibration

	Data	Model
1mo implied vol	18.6%	18.6%
1yr implied vol	18.1%	18.2%
2yr implied vol	17.9%	17.7%
1yr cross-sectional mean risk premium	3.8%	3.2%
1yr disagreement	4.8%	4.2%
10yr cross-sectional mean risk premium	3.6%	1.8%
10yr disagreement	2.9%	2.8%

- $T = 10$, $\sigma = 12\%$, $\theta = 1.8$
- Despite being highly stylized, the model generates predictions of broadly the right order of magnitude across multiple dimensions

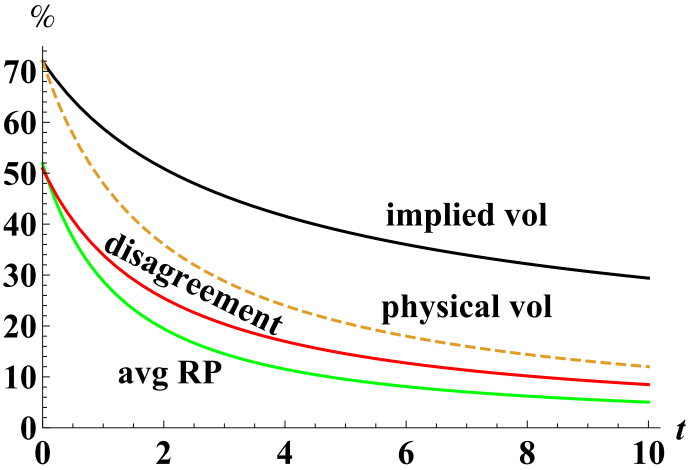


Figure: Volatility term structures in a “crisis” calibration with $\theta = 0.2$

Why is there a variance risk premium?

- We introduce an identity

$$\operatorname{var}^* X - \operatorname{var} X = R_f \operatorname{cov}\left[M, (X-\kappa)^2\right]$$

for any tradable X, where $\kappa = (\mathbb{E} X + \mathbb{E}^* X)/2$ is a constant

- This is a general result, relying only on absence of arbitrage
- In the mind of our median investor, it specializes to

$$\operatorname{var}^* \log R_{0\to T} - \operatorname{var} \log R_{0\to T} = \underbrace{\operatorname{cov}^{(z)}\left[M^{(z)}_{0\to T}, (\log R_{0\to T})^2\right]}_{\text{zero in Black–Scholes, positive here—but why?}}$$

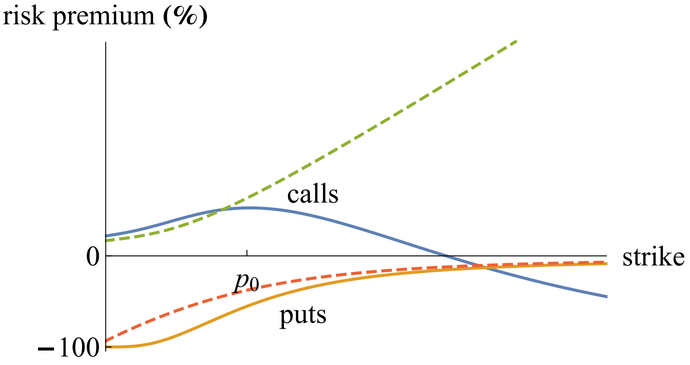
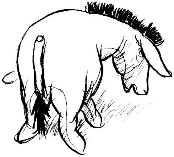


Figure: Expected excess returns on options of different strikes, as perceived by the rep agent. Solid: heterogeneous beliefs. Dashed: homogeneous

- Median agent thinks OTM options are overvalued due to extremists
- Perceives *negative* expected excess returns on deep OTM calls

Speculation in equilibrium

- Our investors use complicated trading strategies to speculate
- These strategies induce different wealth returns for each investor, as a function of the underlying asset return
- Dynamically complete market, so can think about strategies either in time-series terms ("sell if the market rallies, buy if it crashes") or in derivatives terms ("sell options" or "short vol")
- Notation: the gloomy investor, $z = z_g = -\frac{\theta+1}{\sqrt{\theta}}\sigma\sqrt{T}$ is the investor who has lowest expected utility in equilibrium



Speculation in equilibrium

Result

Agent z's equilibrium return on wealth, $R^{(z)}_{0\to T}$, is a function of $R_{0\to T}$:

$$R^{(z)}_{0\to T} = \sqrt{\frac{\theta+1}{\theta}} \exp\left\{ \frac{1}{2}\left(z - z_g\right)^2 - \frac{1}{2(1+\theta)\sigma^2 T}\left[\log\left(R_{0\to T}/K^{(z)}\right)\right]^2 \right\}$$

- Target return for investor z, $K^{(z)}$, is the investor's ideal outcome
- It satisfies

$$\log K^{(z)} = \mathbb{E}^{(z)} \log R_{0\to T} + (z - z_g)\sigma\sqrt{\theta T}$$

- Extremists are happiest if the market moves more than they expect
- Gloomy investor $z = z_g$ wants to be proved right

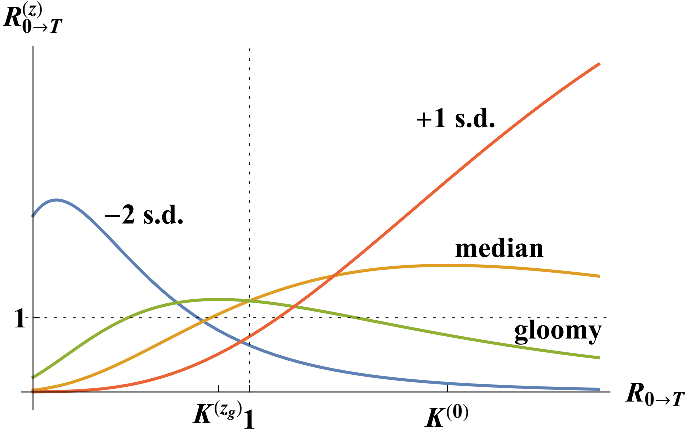


Figure: Return on wealth against return on the market

Investors have U-shaped SDFs

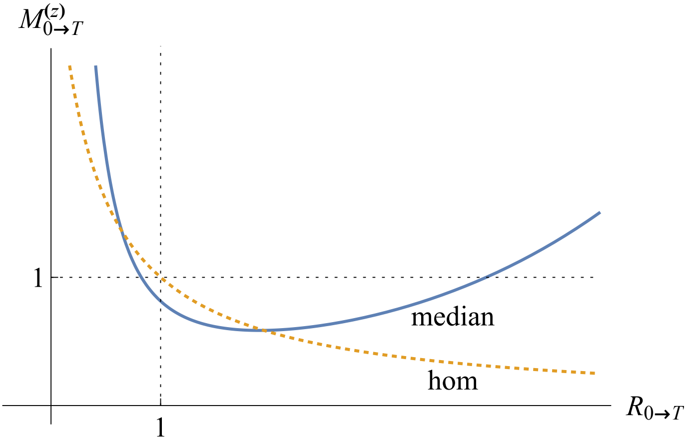


Figure: Median agent's SDF compared to the SDF with homogeneous beliefs

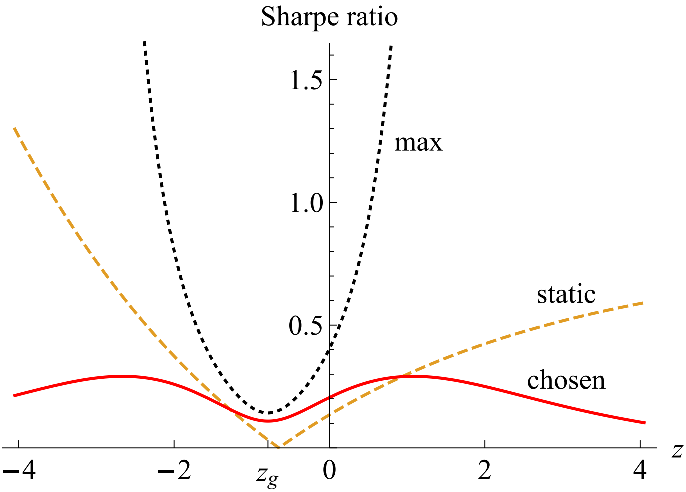
Sharpe ratios

Result

The maximum Sharpe ratio (as perceived by investor z) is finite if $\theta > 1$:

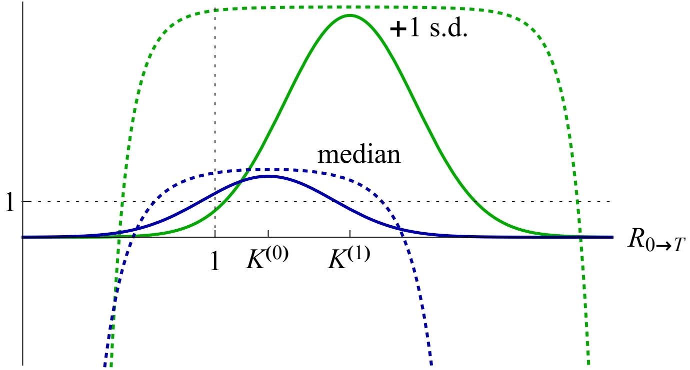
$$MSR_{0\to T}^{(z)} = \sqrt{\frac{\theta}{\sqrt{\theta^2 - 1}} \exp\left\{\frac{\left(z - z_g\right)^2}{\theta - 1}\right\} - 1}$$

- As people have different beliefs but agree on market prices, they have different SDFs, whose properties reflect different views on Sharpe ratios and on the value of speculation
- The gloomy investor perceives the smallest maximum Sharpe ratio, which is $\sqrt{\frac{\theta}{\sqrt{\theta^2-1}} - 1}$ (or infinity if $\theta \leq 1$!)



- Dotted: Perceived max Sharpe ratio achievable by speculating
- Dashed: Perceived static Sharpe ratio of the market
- Solid: Perceived Sharpe ratio on chosen strategy

Max SR strategies are very short OTM options



- Log scale on x-axis
- Dotted: MSR returns, for investors $z = 0$ and 1
- Solid: the returns investors $z = 0$ and 1 actually choose

A cautionary tale for empiricists

- As MSR strategy is mean-variance-efficient, can use it for beta pricing with zero alphas in the usual way
- Conversely, if betas are calculated wrt the market, or to the returns that investors actually choose, then MSR strategies, which load up on tail risk, earn large alphas
- But our investors dislike tail risk and don't do mean-variance analysis!
- They choose portfolios inside the mean–variance frontier
- In fact, they would prefer to stay in cash than to put any money at all in an MSR strategy
- In short: alphas and Sharpe ratios aren't of economic interest here

Example 4: A Poisson limit

- Bad news arrives according to a Poisson process
- If q arrivals occur, terminal payoff is e^{-qJ} (for some constant J)
- Agents disagree on the jump arrival rate ω and hence on all moments of returns
- Optimists perceive low arrival rates and sell insurance to pessimists; like derivative traders inside financial institutions, they do well in quiet times but experience losses at times of turmoil
- As before, we have a representative agent ($\omega_{\text{rep},t}$) and an agent who is out of the market (ω_t^*; and $\omega_t^* =$ the CDS rate)
- Both more pessimistic ($\omega_{\text{rep},t}$ and ω_t^* get larger) following jumps
- All investors think arbitrarily high Sharpe ratios are attainable

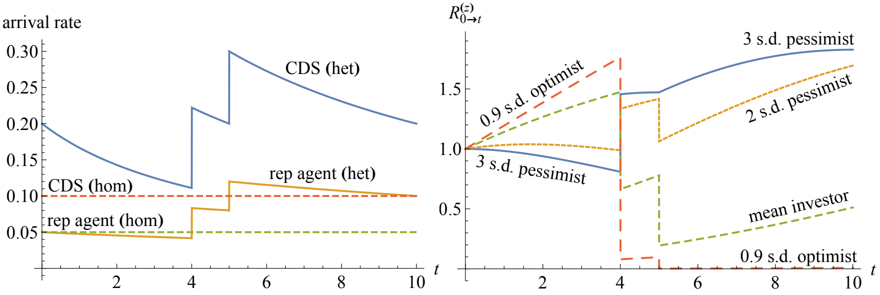


Figure: Left: $\omega_{\text{rep},t}$ and ω^*_t on a sample path with jumps at times $t = 4$ and 5
Right: The cumulative return of four agents along the same sample path

- Even though individuals have stable beliefs, the CDS rate and rep agent's perceived arrival rate spike after a jump

Speculation is a mixed blessing

- All investors think speculation is in their own interest
- But all investors also think that speculation is socially costly
- On the other hand, if speculation (i.e., dynamic trade) is closed down entirely, the market can collapse
- To see what can go wrong, consider the Brownian limit. . .

Collapse of static equilibrium in the Brownian limit

- Given any positive time 0 price, return to maturity is lognormal
- If dynamic trade is banned, all agents choose risky shares $\in [0, 1]$, as short or levered positions risk bankruptcy
- To clear the market, the average risky share must be 1. So *all* agents must choose risky share equal to 1
- But this is impossible! At any fixed positive price, some investors will not wish to invest fully in the risky asset
- Hence static equilibrium does not exist
- Although speculation is socially costly, the ability to trade dynamically means investors can reduce their position sizes to avoid bankruptcy if the market starts to move against them

Summary

- Sentiment creates volatility, ambiguous impact on risk premia
- Extreme scenarios are important for pricing
- Asymmetric effects on right- and left-skewed assets
- Moderate investors are contrarian, “short vol”, liquidity suppliers
- Mean-variance-efficient returns are very short deep-OTM options; they do not interest our investors despite their high Sharpe ratios
- CDS rates spike after jumps, even though all investors perceive constant arrival rates
- Everyone thinks that speculation is socially harmful, but good news for themselves and for people with similar beliefs