# Consumption-Based Asset Pricing with Higher Cumulants: Supplementary Appendix

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## I Regularity conditions for the GMM exercise

I provide a proof that $\rho$ and $\gamma$ are identified in the body of the paper. Here, I check that the remaining regularity conditions hold.

Hansen (1982) requires certain assumptions for the GMM estimator to converge. To be consistent with Hansen's notation, we can rewrite (11) in the main paper, for $i = 1, 2$, as $\mathbb{E}\, f(x_{t+1}, \beta_0) = 0$, for $x_{t+1} = \log C_{t+1}/C_t$, $\beta_0 = (\rho, \gamma)$, and with $f(\cdot, \cdot)$, a vector-valued function defined in the obvious way:

$$f(x_{t+1}, \beta_0) = \begin{pmatrix} e^{-\rho} \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} (R_{t+1} - R_{f,t+1}) \\ e^{-\rho} \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} (1 + R_{f,t+1}) - 1 \end{pmatrix}.$$

Now, $x_{t+1}$ is stationary and ergodic by the i.i.d. property, so Hansen's Assumption 3.1 holds; the parameter space $S$ is all of $\mathbb{R}^2$, so his Assumption 3.2 holds; and $f(\cdot, \beta)$ and $\partial f/\partial\beta(\cdot, \beta)$ are manifestly Borel measurable for all $\beta \in \mathbb{R}^2$, and $\partial f/\partial\beta(x, \cdot)$ is continuous everywhere for each $x \in \mathbb{R}^2$, so Assumption 3.3 holds. First-moment continuity, in Assumption 3.4, is satisfied because in the calibration with Normally distributed jumps, $\boldsymbol{c}(\theta)$ is finite for all $\theta \in \mathbb{R}$.[1] The other aspect of Assumption 3.4 is the requirement that the matrix $\mathbb{E}\, \partial f/\partial\beta(x_{t+1}, \beta_0)$ exists, is finite, and has full rank. This holds, because the

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[1] The proof, which is straightforward, is omitted. At one point, the observation that $|x \log x| \leq 1 + x^2$ is needed.

Jacobian is

$$\mathbb{E}\left[\frac{\partial f}{\partial \beta}\right] = \begin{pmatrix} -\mathbb{E}\left[e^{-\rho}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}(R_{t+1}-R_{f,t+1})\right] & -\mathbb{E}\left[e^{-\rho}\log\frac{C_{t+1}}{C_t}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}(R_{t+1}-R_{f,t+1})\right] \\ -\mathbb{E}\left[e^{-\rho}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}(1+R_{f,t+1})\right] & -\mathbb{E}\left[e^{-\rho}\log\frac{C_{t+1}}{C_t}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}(1+R_{f,t+1})\right] \end{pmatrix}.$$

Existence and finiteness follow because $\boldsymbol{c}(\theta)$ is finite for all $\theta \in \mathbb{R}$. The left column of the Jacobian can be simplified using the Euler equation, giving

$$\mathbb{E}\left[\frac{\partial f}{\partial \beta}\right] = \begin{pmatrix} 0 & -\mathbb{E}\left[e^{-\rho}\log\frac{C_{t+1}}{C_t}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}(R_{t+1}-R_{f,t+1})\right] \\ -1 & -\mathbb{E}\left[e^{-\rho}\log\frac{C_{t+1}}{C_t}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}(1+R_{f,t+1})\right] \end{pmatrix}.$$

Thus, the Jacobian has full rank so long as the upper-right term is nonzero. This holds in the baseline calibration because, by numerical integration,

$$\mathbb{E}\left[e^{-\rho}\log\frac{C_{t+1}}{C_t}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}(R_{t+1}-R_{f,t+1})\right] = 0.044.$$

Assumption 3.5 holds trivially by the i.i.d. property. Since the setup is exactly identified, Assumption 3.6 is satisfied by letting $a_N^*$ equal the identity matrix for all $N$.

# II GMM: further asymptotic results

The estimator of the coefficient of relative risk aversion, $\widehat{\gamma}$, is determined by setting

$$g_T(\widehat{\gamma}) \equiv \frac{1}{T}\sum_{1}^{T}\left(\frac{C_{t+1}}{C_t}\right)^{-\widehat{\gamma}}(R_{t+1}-R_{f,t+1}) = 0.$$

As is well-known, $\sqrt{T}\,(\widehat{\gamma}-\gamma)$ is asymptotically Normally distributed. We can briefly sketch why this is the case. By the strong law of large numbers, $g_T(\widetilde{\gamma}) \to \mathbb{E}\left[(C_{t+1}/C_t)^{-\widetilde{\gamma}}(R_{t+1}-R_{f,t+1})\right]$. In particular, by the Euler equation, $g_T(\gamma) \to 0$. So $\widehat{\gamma}$, chosen such that $g_T(\widehat{\gamma}) = 0$, is a consistent estimator of $\gamma$. To demonstrate asymptotic Normality, we use the mean-value theorem to write

$$g_T(\widehat{\gamma}) = g_T(\gamma) + g_T'(\overline{\gamma})(\widehat{\gamma}-\gamma) \quad \text{for some } \overline{\gamma} \in (\gamma, \widehat{\gamma}).$$

Combining this with the fact that $g_T(\widehat{\gamma}) = 0$, we have

$$\sqrt{T}(\widehat{\gamma}-\gamma) = -\frac{1}{g_T'(\overline{\gamma})}\sqrt{T}g_T(\gamma) = -\frac{v}{g_T'(\overline{\gamma})}\frac{1}{\sqrt{T}}\sum_{1}^{T}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{(R_{t+1}-R_{f,t+1})}{v},$$

where I write $v^2 = \text{var}\left[(C_{t+1}/C_t)^{-\gamma}(R_{t+1} - R_{f,t+1})\right]$ to express the sum over random variables with mean zero (by the Euler equation) and variance 1, so that by the central limit theorem the distribution of

$$Z_T \equiv \frac{1}{\sqrt{T}} \sum_{1}^{T} \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \frac{(R_{t+1} - R_{f,t+1})}{v}$$

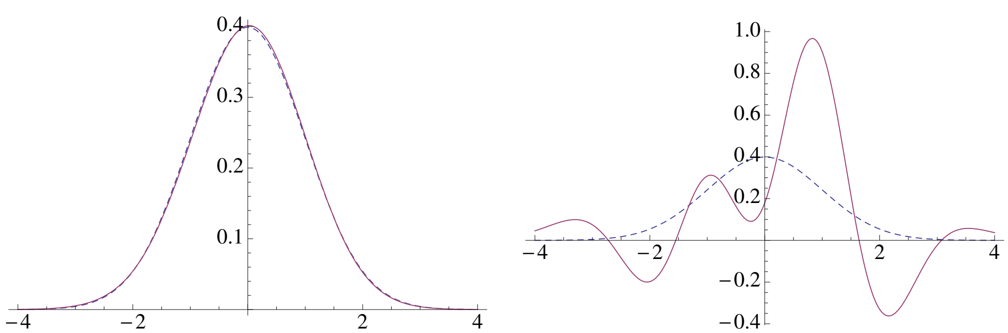
is standard Normal in the limit as $T \to \infty$, which gives the result.

We can explore the non-asymptotic behavior of the probabiliity density function of $Z_T$ under the baseline calibration—call this probability density function $f_T(x)$—using an Edgeworth expansion. This exercise exploits the fact that we know the true distribution of $C_{t+1}/C_t$ in the calibration. Rothenberg (1984, equation 3.3) shows that

$$f_T(x) = \phi(x)\left[1 + \frac{k_3 H_3(x)}{6\sqrt{T}} + \frac{3k_4 H_4(x) + k_3^2 H_6(x)}{72T}\right] + o\left(\frac{1}{T}\right), \tag{1}$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ is the Normal probability density function, $k_n$ is the $n$th cumulant of $(C_{t+1}/C_t)^{-\gamma}[(R_{t+1} - R_{f,t+1})/v]$, and $H_n$ is the $n$th Hermite polynomial: $H_3(x) = x^3 - 3x$, $H_4(x) = x^4 - 6x^2 + 3$, and $H_6(x) = x^6 - 15x^4 + 45x^2 - 15$. As usual, $o(1/T)$ indicates terms of smaller order than $1/T$. Equation (1) provides a measure of how fast $f_T(x) \to \phi(x)$ as $T \to \infty$. Calculating the right-hand side of (1) (ignoring the $o(1/T)$ term) using the known distribution of $C_{t+1}/C_t$ provides a way of assessing whether asymptotic Normality of the GMM estimator is a reasonable assumption using $T = 100$ years of data.

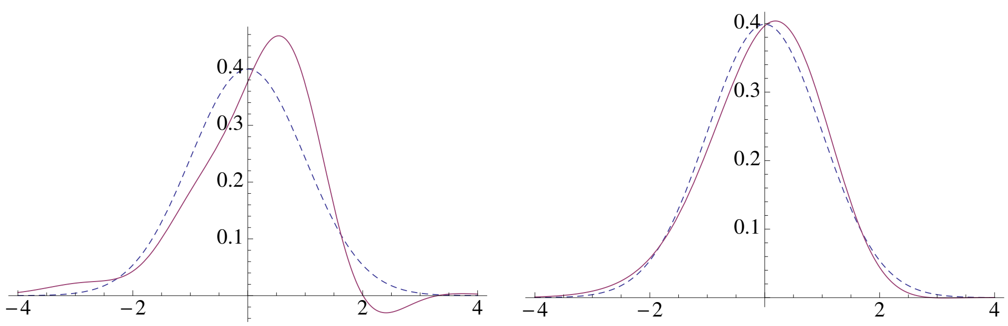
In the baseline lognormal calibration without disasters, we have $k_3 = -0.487$ and $k_4 = 4.73$. In the baseline calibration with disasters, there is far more skewness and kurtosis: we have $k_3 = -36.9$ and $k_4 = 1830$.




(a) No disasters, $T = 100$ (b) Disasters, $T = 100$

Figure 1: Dashed line: the Normal probability density function. Solid line: the Edgeworth expansion for the probability density function of $Z_T$ provided in (1).

This is reflected in Figure 1. Figure 1a plots the Edgeworth approximation to $f_T(x)$ for the calibration without disasters: it is indistinguishable from the Normal probability density function. Figure 1b shows the corresponding plot with disasters. In both cases, I set $T = 100$ years of data, as in the main text. Evidently, the sum is not close to converging in the calibration with disasters.[2] This is why the distribution of the estimates of $(\widehat{\rho}, \widehat{\gamma})$ is so far from Normal in Figures 3 and 4 of the paper. Figure 2 shows that we need *much* more data to be able to rely on asymptotic Normality in the presence of disasters: if $T$ is on the order of 10,000 years of data, then we have a tolerable approximation to the Normal distribution.



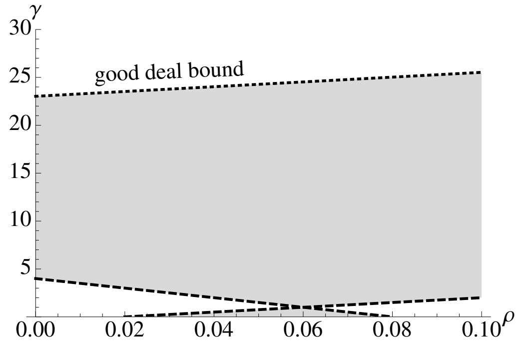

(a) Disasters, $T = 1,000$ (b) Disasters, $T = 10,000$

Figure 2: Dashed line: the Normal probability density function. Solid line: the Edgeworth expansion for the probability density function of $Z_T$ provided in (1).

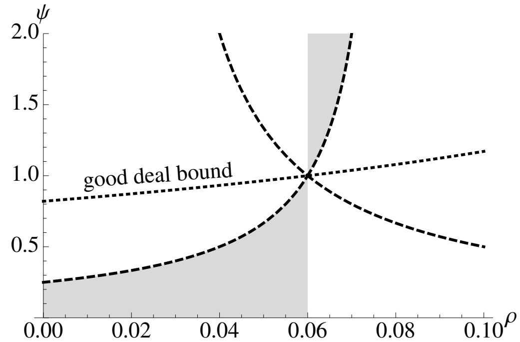
# III Good deal bounds

Figure 3 is identical to Figure 8 of the main paper, except that the maximum Sharpe ratio is assumed to be 1.25 rather than 0.75. As in the paper, the right-hand panel assumes $\gamma = 4$.

[2]See also Jondeau and Rockinger, 2001, "Gram-Charlier Densities," *Journal of Economic Dynamics and Control*, 25:1457–1483. Note, also, that the Edgeworth expansion is not even guaranteed to be positive. See Chernozhukov, Fernández-Val and Galichon, 2010, "Rearranging Edgeworth-Cornish-Fisher expansions," *Economic Theory*, 42:2:419–435.




(a) Power utility case: $\gamma$ and $\rho$




(b) Epstein-Zin case: $\psi$ and $\rho$

Figure 3: Shaded areas indicate admissible parameter values for i.i.d. models with $rp = 6\%$, $r_f = 2\%$, $c/w = 6\%$, and a maximal Sharpe ratio of 1.25.

# IV The non-i.i.d. case

The main paper exploited the fact that the hypothesis that $\log Q_{t+1}$ and $\log(C_{t+1}/C_t)$ are independent cannot be rejected in the data. It remains possible that the consumption growth of marginal investors may look very different from aggregate consumption growth, and may be an important determinant of valuation ratios. For example, one might find that shocks to valuations are driven by the marginal investor's consumption growth via $Q_{t+1} = (C_{t+1}/C_t)^\xi$, for some $\xi$ that I assume is greater than $-1$ to reduce the number of cases that must be considered.[3] We then have the following result. (The coefficient $\xi$ should be determined empirically using micro-level data; but to show in principle how this dependence can arise in equilibrium models, I will exhibit a particular model that generates this relationship after proving the result, below. The exercise also throws light on the factors that determine the value of $\xi$ in equilibrium models.) Other examples of this approach in the context of non-i.i.d. models are provided in Martin (2011a, 2011b).

**Result 1** ($Q_{t+1} = (C_{t+1}/C_t)^\xi$)**.** *The parameter space can be divided into three regions: (i)* $1-\gamma+\xi\vartheta < 1+\xi$*; (ii)* $-\xi-\gamma+\xi\vartheta < 1+\xi < 1-\gamma+\xi\vartheta$*; and (iii)* $1+\xi < -\xi-\gamma+\xi\vartheta$*.*

*In region (i) (which applies with power utility, or if* $\xi = 0$*), we have*

$$\frac{\vartheta(c/w_t-\rho)+r_{f,t}-c/w_t}{\xi+\gamma-\xi\vartheta} \le \frac{\vartheta(c/w_t-\rho)}{\gamma-1-\xi\vartheta} \le \frac{r_{f,t}+rp_t-c/w_t}{1+\xi}.$$

*In region (ii), we have*

$$\frac{\vartheta(c/w_t-\rho)+r_{f,t}-c/w_t}{\xi+\gamma-\xi\vartheta} \le \frac{r_{f,t}+rp_t-c/w_t}{1+\xi} \le \frac{\vartheta(c/w_t-\rho)}{\gamma-1-\xi\vartheta}.$$

[3] However, we no longer need the assumption that $\mathbb{E}_t\, Q_{t+1} = 1$.

*In region (iii), we have*

$$\frac{r_{f,t}+rp_t-c/w_t}{1+\xi} \le \frac{\vartheta(c/w_t-\rho)+r_{f,t}-c/w_t}{\xi+\gamma-\xi\vartheta} \le \frac{\vartheta(c/w_t-\rho)}{\gamma-1-\xi\vartheta}.$$

*The analogue of Result 4 is that*

$$\frac{\gamma-1-\xi\vartheta}{1+\xi}\,(c/w_t-r_{f,t})+\vartheta\,(c/w_t-\rho)\le\log(1+h^2)$$

*if the Sharpe ratio is less than or equal to h.*

*Proof.* Note that $\boldsymbol{c}_t(\theta_1,\theta_2)=\boldsymbol{c}_t(\theta_1+\xi\theta_2,0)\equiv\boldsymbol{c}_t(\theta_1+\xi\theta_2)$; the last identity defines notation. We have

$$\begin{aligned}\vartheta(\rho-c/w_t) &= \boldsymbol{c}_t(1-\gamma+\xi\vartheta) && (2)\\ c/w_t-r_{f,t}+\vartheta(\rho-c/w_t) &= \boldsymbol{c}_t(-\gamma+\xi(\vartheta-1)) && (3)\\ r_{f,t}+rp_t-c/w_t &= \boldsymbol{c}_t(1+\xi) && (4)\end{aligned}$$

The result follows by using these equations together with the fact that $\boldsymbol{c}_t(x)/x$ increases in $x$, by convexity. The proof of the generalized good-deal bound follows as before. □

*An equilibrium model that generates* $Q_{t+1}=(C_{t+1}/C_t)^\xi$. The relationship $Q_{t+1}=(C_{t+1}/C_t)^\xi$ holds asymptotically in the model of Parlour, Stanton and Walden (PSW, "Revisiting Asset Pricing Puzzles in an Exchange Economy", 2011, *Review of Financial Studies*). Their Proposition 3(ii) shows that in their baseline ("above-the-breakpoint") calibration, $Q_{t+1}$ converges to $(D_{t+1}/D_t)^\xi$, where $\xi$ (or $\alpha-1$ in PSW's notation) is determined by preferences (risk aversion $\gamma>1$ and time preference rate $\rho$) and the characteristics of cashflows (mean log dividend growth $\mu$ and dividend volatility $\sigma$, in their lognormal framework).

Since PSW is a special case of the model considered in Martin (2011a)—although PSW focus on a different region of the parameter space—this result can be generalized to allow dividend growth to be an arbitrary Lévy process, subject to conditions that ensure that expected utility is finite. Assuming that $\gamma>2$, it can be shown, by adapting the proof of Proposition 7 of Martin (2011a), that the same result holds in this more general case. As before, $\xi$ is determined by preferences and the properties of dividend growth: it equals the unique root $\xi\in(0,\gamma/2-1)$ of the equation

$$\rho-\boldsymbol{c}(1-\gamma+\xi)=0$$

where $\boldsymbol{c}(\theta)\equiv\log\mathbb{E}[(D_{t+1}/D_t)^\theta]$. (There is a unique root $\xi\in(0,\gamma/2-1)$ so long as we are "above the breakpoint", in PSW's terminology. This follows, once again, by the convexity of the CGF.) If dividend growth of the risky asset is lognormal, as in PSW, then $\boldsymbol{c}(\theta) = \mu\theta + \frac{1}{2}\sigma^2\theta^2$, so the root is $\xi = \gamma - 1 - \frac{\mu + \sqrt{\mu^2 + 2\rho\sigma^2}}{\sigma^2}$. PSW label this quantity $\alpha - 1$: see their Proposition 3(ii) and equation (4).

*Proof.* The proof follows the lines of the proof of Proposition 7 of Martin (2011a). As in PSW, we consider a model with two trees, the first of which is riskless, paying a constant unit dividend stream per unit time, $D_{1t} \equiv 1$, and the second of which pays the risky dividend stream $D_{2t}$. We are interested in the asymptotic regime in which tree 1 is tiny, i.e. $s_t \equiv D_{1t}/(D_{1t} + D_{2t}) = 1/(1 + D_{2t}) \to 0$. As in Martin (2011a), it will be convenient to define the state variable $u$, a monotonic function of $s_t$, via $u_t \equiv \log[(1 - s_t)/s_t]$. Thus we are interested in the asymptotics as $s_t \to 0$ and $u_t \to \infty$. The notation $a \doteq b$ means "$a$ equals $b$ plus higher order terms that tend to zero as $u_t \to \infty$". So, for example, $s_t \doteq e^{-u_t}$ and $C_{t+1}/C_t \doteq e^{u_{t+1} - u_t}$. I will show that $Q_{t+1} \doteq (C_{t+1}/C_t)^\xi$ for appropriately defined $\xi$.

To do so, we start out by calculating $W_t/C_t$. By definition of $s_t$

$$\frac{W_t}{C_t} = s_t \frac{P_{1t}}{D_{1t}} + (1 - s_t)\frac{P_{2t}}{D_{2t}}, \tag{5}$$

where $P_{it}$ is the price of the claim to tree $i$; $W_t = P_{1t} + P_{2t}$; and $C_t = D_{1t} + D_{2t}$.

I will write $\boldsymbol{\kappa}(\theta_1, \theta_2) = \log \mathbb{E}[(D_{1,t+1}/D_{1t})^{\theta_1}(D_{2,t+1}/D_{2t})^{\theta_2}]$; in Martin (2011a), this function was written $\boldsymbol{c}(\theta_1, \theta_2)$. I change the notation here to avoid confusion with $\boldsymbol{c}(\theta)$, which will be reserved for the cumulant-generating function of log dividend growth of tree 2, i.e. $\boldsymbol{c}(\theta) = \log \mathbb{E}[(D_{2,t+1}/D_{2,t})^\theta]$.

The proof of Proposition 7 of Martin (2011a), in Appendix A.5.1, establishes that

$$\frac{P_{1t}}{D_{1t}} \doteq \frac{\mathscr{B}(\gamma/2 - z^*, \gamma/2 + z^*)}{\boldsymbol{\kappa}_1(1 - \gamma/2 + z^*, -\gamma/2 - z^*) - \boldsymbol{\kappa}_2(1 - \gamma/2 + z^*, -\gamma/2 - z^*)} e^{(\gamma/2 - z^*)u_t} \tag{6}$$

where $z^*$ solves $\rho - \boldsymbol{\kappa}(1 - \gamma/2 + z^*, -\gamma/2 - z^*) = 0$ and $\mathscr{B}(\cdot, \cdot)$ is Euler's beta function. I will assume that $z^* < \gamma/2 - 1$; this holds in the special case considered by PSW. I also assume that $\rho - \boldsymbol{\kappa}(1 - \gamma/2, -\gamma/2) > 0$; this is a finiteness condition imposed in Martin (2011a), and it implies that $z^* > 0$.

Exactly the same logic that gives (6) implies that

$$\frac{P_{2t}}{D_{2t}} \doteq \frac{\mathscr{B}(\gamma/2 - \widetilde{z}, \gamma/2 + \widetilde{z})}{\boldsymbol{\kappa}_1(-\gamma/2 + \widetilde{z}, 1 - \gamma/2 - \widetilde{z}) - \boldsymbol{\kappa}_2(-\gamma/2 + \widetilde{z}, 1 - \gamma/2 - \widetilde{z})} e^{(\gamma/2 - \widetilde{z})u_t} \tag{7}$$

where $\widetilde{z}$ solves $\rho - \boldsymbol{\kappa}(-\gamma/2 + \widetilde{z}, 1 - \gamma/2 - \widetilde{z}) = 0$. By comparing this expression with the corresponding expression for $z^*$, it follows that $\widetilde{z} = 1 + z^*$, and hence also (by the assumptions made above) that $\widetilde{z} \in (1, \gamma/2)$. Equations (5)–(7) imply that

$$\frac{W_t}{C_t} \doteq \frac{\gamma - 1}{\gamma/2 + \widetilde{z} - 1} \frac{\mathscr{B}(\gamma/2 - \widetilde{z}, \gamma/2 + \widetilde{z})}{\boldsymbol{\kappa}_1(-\gamma/2 + \widetilde{z}, 1 - \gamma/2 - \widetilde{z}) - \boldsymbol{\kappa}_2(-\gamma/2 + \widetilde{z}, 1 - \gamma/2 - \widetilde{z})} e^{(\gamma/2 - \widetilde{z})u_t},$$

using the property of the beta function that $\mathscr{B}(A,B) + \mathscr{B}(A-1,B+1) = [(A+B-1)/(A-1)] \cdot \mathscr{B}(A,B)$.

It follows that

$$Q_{t+1} \doteq \left(\frac{W_{t+1}}{C_{t+1}}\right) / \left(\frac{W_t}{C_t}\right) \doteq \left(\frac{C_{t+1}}{C_t}\right)^{\gamma/2-\widetilde{z}},$$

as required. Thus the appropriate definition is $\xi = \gamma/2 - \widetilde{z} \in (0, \gamma/2-1)$; we can suppress all mention of $\widetilde{z}$ by defining $\xi$ directly via $\rho - \boldsymbol{\kappa}(-\xi, 1-\gamma+\xi) = 0$. Finally, using the fact that $D_{1t} \equiv 1$ to note that $\boldsymbol{\kappa}(\theta_1, \theta_2) = \boldsymbol{c}(\theta_2)$, this simplifies even further: $\xi$ can be defined directly via $\rho - \boldsymbol{c}(1-\gamma+\xi) = 0$, as claimed. □

## CGFs and CAR(p) processes

Darolles, Gourieroux and Jasiak (DGJ, 2006, "Structural Laplace Transform and Compound Autoregressive Models", *Journal of Time Series Analysis* 27:4:477–503) introduce a class of time series models that they call compound autoregressive (CAR) models. Suppose that we are given a state vector $\boldsymbol{y}_t = (\Delta c_t, \ldots)$ whose first entry is log consumption growth $\Delta c_t = c_t - c_{t-1}$. Then $\boldsymbol{y}_t$ is a CAR(1) process if

$$\boldsymbol{c}_t(\boldsymbol{\theta}) \equiv \log \mathbb{E}_t \, e^{\boldsymbol{\theta}'\boldsymbol{y}_{t+1}} = a(\boldsymbol{\theta})'\boldsymbol{y}_t + b(\boldsymbol{\theta}), \tag{8}$$

where $a(\cdot)$ and $b(\cdot)$ are (respectively vector-valued and scalar-valued) functions that specify the properties of the CAR(1) process.

It is straightforward, if tedious, to verify that

$$\mathbb{E}_t \exp\left\{\boldsymbol{\theta}'(\boldsymbol{y}_{t+1} + \cdots + \boldsymbol{y}_{t+T})\right\} = \exp\left\{\sum_{j=1}^{T} b(\phi_j(\boldsymbol{\theta})) + a(\phi_T(\boldsymbol{\theta}))'\boldsymbol{y}_t\right\},$$

where the functions $\phi_j(\cdot)$ are defined recursively via $\phi_1(\boldsymbol{\theta}) \equiv \boldsymbol{\theta}$ and $\phi_n(\boldsymbol{\theta}) = \boldsymbol{\theta} + a(\phi_{n-1}(\boldsymbol{\theta}))$ for $n > 1$. This expression can be used to find the time-$t$ price, $Z_{t,T}$, of a zero-coupon consumption claim expiring in $T$ periods, under power utility. Defining $\boldsymbol{v} = (1-\gamma, 0, \ldots, 0)'$ for notational convenience, we have

$$\frac{Z_{t,T}}{C_t} = e^{-\rho T}\, \mathbb{E}_t \left(\frac{C_{t+T}}{C_t}\right)^{1-\gamma} = e^{-\rho T}\, \mathbb{E}_t \, e^{\boldsymbol{v}'(\boldsymbol{y}_{t+1}+\cdots+\boldsymbol{y}_{t+T})} = e^{-\rho T} \cdot \exp\left\{\sum_{j=1}^{T} b(\phi_j(\boldsymbol{v})) + a(\phi_T(\boldsymbol{v}))'\boldsymbol{y}_t\right\}.$$

DGJ and Bertholon, Monfort and Pegoraro (2008, "Econometric Asset Pricing Modelling", *Journal of Financial Econometrics*, 6:4:407–458) provide examples of CAR processes and the associated functions $a(\cdot)$ and $b(\cdot)$. Some constraints on $a(\cdot)$ are imposed directly by the fact that the CGF in (8) is convex. As an example, suppose for simplicity that $\boldsymbol{y}_t = \Delta c_t$ is univariate. Then assuming $\Delta c_t$ can take both positive and negative values, convexity of $\boldsymbol{c}_t(\theta)$ forces $a(\theta) = k\theta$ for some scalar $k$.