Internet Appendix

Results 4 and 5 of the main text characterized when disasters matter, but at the cost of putting a certain amount of structure on the pricing problem: a conditional lognormality assumption in the case of Result 4, and an assumption about the form of the stochastic discount factor in the case of Result 5. This appendix relaxes these assumptions. The first section remains in the conditionally lognormal framework, but allows for imperfectly correlation between log returns and the log SDF, thereby allowing for unpriced as well as priced risk. The second section presents a result that can be thought of as a litmus test for whether disasters matter in more general frameworks.

Imperfect correlation between $\log R$ and $\log M$

This section presents a result that addresses the case of imperfect correlation between log returns and the log SDF, thereby allowing for some of the asset's risk to be unpriced. In full generality, there are certain obvious constraints on what can be said, since we can always add idiosyncratic noise to a return without affecting pricing. My starting point is a factorization of $M_t R_t$ into an idiosyncratic component I_t and a systematic component S_t. Having done so we find, first, that large values of the systematic component, S_t, can only be attributed to bad news (given the empirically observed Sharpe ratio and volatility of the market) and, second, that the fact that the market's Sharpe ratio appears to be not just higher but *considerably* higher than its volatility places tight limits on the importance of I_t relative to S_t for the edges of the distribution of X_t. Intuitively, the fact that the market's Sharpe ratio is significantly higher than its volatility indicates that most of the market's risk is priced rather than unpriced.

Result 6. Suppose that the asset return and SDF are jointly lognormal, so that we can write $R_t \equiv e^{\mu_{R,t-1}-\frac{1}{2}\sigma^2_{R,t-1}+\sigma_{R,t-1}Z_{R,t}}$ and $M_t \equiv e^{-r_{f,t}-\frac{1}{2}\sigma^2_{M,t-1}-\sigma_{M,t-1}Z_{M,t}}$ where $Z_{R,t}$ and $Z_{M,t}$ are standard Normal random variables with conditional correlation ρ_{t-1}. Then we can factorize $M_t \cdot R_t = I_t \cdot S_t$ where, conditional on time $t-1$ information, (i) I_t and S_t each have unit mean; (ii) I_t and S_t are independent of one another; (iii) I_t is an idiosyncratic component that is independent of M_t; and (iv) $\log S_t$ is a systematic component that is perfectly correlated with $\log M_t$.

If $\lambda_{t-1} > \rho_{t-1}^2 \sigma_{R,t-1}$ then large values of S_t correspond to large values of M_t, i.e. to bad news. An obvious sufficient condition is that $\lambda_{t-1} > \sigma_{R,t-1}$, which holds in the data if the asset in question is the market.

If $\lambda_{t-1} > (\rho_{t-1}^2 + \rho_{t-1}\sqrt{1-\rho_{t-1}^2})\sigma_{R,t-1}$ then S_t has fatter tails than I_t, so sufficiently large disasters are almost always associated with bad news as opposed to idiosyncratic shocks. A sufficient condition is that $\lambda_{t-1} > \frac{1+\sqrt{2}}{2}\sigma_{R,t-1} \approx 1.2\sigma_{R,t-1}$. Again, this holds in the data if the asset in question is the market.

Proof. The pricing equation $\mathbb{E}_t\, M_{t+1}R_{t+1} = 1$ implies that $\lambda_t = \rho_t \sigma_{M,t}$. Since $Z_{M,t}$ and $Z_{R,t}$ are Normal with zero mean, unit variance and correlation ρ_{t-1}, we can write $Z_{R,t} = \rho_{t-1}Z_{M,t} + \sqrt{1-\rho_{t-1}^2}\widetilde{Z}_t$, where $\widetilde{Z}_t$ is a Normal random variable that is independent of $Z_{M,t}$ and has zero mean and unit variance. We can then define

$$I_t = \exp\left\{\sigma_{R,t-1}\sqrt{1-\rho_{t-1}^2}\widetilde{Z}_t - \frac{1}{2}\sigma_{R,t-1}^2(1-\rho_{t-1}^2)\right\}$$

and

$$S_t = \exp\left\{-(\sigma_{M,t-1} - \rho_{t-1}\sigma_{R,t-1})Z_{M,t} - \frac{1}{2}(\sigma_{M,t-1} - \rho_{t-1}\sigma_{R,t-1})^2\right\}.$$

If $\sigma_{M,t-1} > \rho_{t-1}\sigma_{R,t-1}$ then large values of S_t correspond to large values of M_t. Since $\lambda_t = \rho_t\sigma_{M,t}$, this condition is equivalent to $\lambda_{t-1} > \rho_{t-1}^2\sigma_{R,t-1}$.

S_t has fatter fails than I_t if $\sigma_{M,t-1} - \rho_{t-1}\sigma_{R,t-1} > \sigma_{R,t-1}\sqrt{1-\rho_{t-1}^2}$. This is equivalent to $\lambda_{t-1} > (\rho_{t-1}^2 + \rho_{t-1}\sqrt{1-\rho_{t-1}^2})\sigma_{R,t-1}$; finally, note that $\rho_{t-1}^2 + \rho_{t-1}\sqrt{1-\rho_{t-1}^2} \le \frac{1+\sqrt{2}}{2}$. $\square$

A large deviations result

This section presents a result that characterizes whether disasters matter in more general frameworks; the key object is

$$\kappa(\theta_M, \theta_R) \equiv \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[(M_1 \cdots M_t)^{\theta_M} (R_1 \cdots R_t)^{\theta_R} \right] \tag{1}$$

which, in Hansen's (2011) terminology, is the long-term growth rate of the multiplicative functional $(M_1 \cdots M_t)^{\theta_M}(R_1 \cdots R_t)^{\theta_R}$. Hansen (2011) provides extensive discussion of how κ can be calculated in structural models, so I will take κ as given.

A natural index of the extent to which explosions in X_t reflect bad news rather than good news is $P_t(\phi, \psi) \equiv \mathbb{P}\left(M_1 \cdots M_t > e^{\psi t} \middle| M_1 R_1 \cdots M_t R_t > e^{\phi t}\right)$, the conditional probability that $M_1 \cdots M_t > e^{\psi t}$, conditional on the event that $X_t > e^{\phi t}$. Here $\phi > 0$ and $\psi > 0$ are fixed parameters. We can say that bad news dominates consideration in the long run if $P_t(\phi, \psi) \to 1$ as $t \to \infty$. For fixed ϕ, this criterion is more stringent the higher ψ is. We can focus on the most severe disasters by allowing ϕ and ψ to tend to infinity, and saying that *bad news is strongly dominant* if $\lim_{\phi\to\infty} \lim_{t\to\infty} P_t(\phi, \phi) = 1$. This definition is appropriate for risky assets, but is less appropriate for an insurance asset like a put option: when X_t explodes for an asset we will generally have contributions from both $M_1 \cdots M_t$ and $R_1 \cdots R_t$, so $P_t(\phi, \phi)$ will tend to be small. We can therefore make the less stringent definition that *bad news is weakly dominant* if $\lim_{\phi\to\infty} \lim_{t\to\infty} P_t(\phi, \phi/2) = 1$. This definition allows for some contribution from high returns, so is more appropriate for insurance assets.

We need some technical conditions on $\kappa(\cdot, \cdot)$, namely that $\kappa(\theta_M, \theta_R)$ is (i) finite for all $\theta_M, \theta_R \in \mathbb{R}$, (ii) continuously differentiable for all $\theta_M, \theta_R \in \mathbb{R}$ and (iii) steep (see Dembo and Zeitouni (1998) for the definition of a steep function). I write $\kappa_M(\cdot, \cdot)$ and $\kappa_R(\cdot, \cdot)$ for the partial derivatives of κ with respect to its first and second argument respectively. If the vectors $(\log M_t, \log R_t)$ are i.i.d. for all t, then the definition (1) reduces to $\kappa(\theta_M, \theta_R) = \log \mathbb{E}\left(M_t^{\theta_M} R_t^{\theta_R}\right)$, so $\kappa(\cdot, \cdot)$ is the cumulant-generating function of the random vector $(\log M_t, \log R_t)$.

Result 7. Let θ_M^* and θ_R^* solve the equations

$$\begin{aligned}\kappa_M(\theta_M^*, \theta_R^*) &= \psi \\ \kappa_R(\theta_M^*, \theta_R^*) &= \phi - \psi\,.\end{aligned}$$

Then $P_t(\phi,\psi) \to 1$ as $t \to \infty$ if $\theta_M^* < \theta_R^*$ and $P_t(\phi,\psi) \to 0$ as $t \to \infty$ if $\theta_M^* > \theta_R^*$.

Proof. By Bayes' rule,

$$P_t(\phi,\psi) = \frac{\mathbb{P}\left(G_{M,t} > \psi \text{ and } G_{M,t} + G_{R,t} > \phi\right)}{\mathbb{P}\left(G_{M,t} + G_{R,t} > \phi\right)} = \frac{\mathbb{P}(A_t)}{\mathbb{P}(A_t) + \mathbb{P}(B_t)},$$

where $G_{M,t} \equiv \frac{1}{t}\sum_1^t \log M_i$, $G_{R,t} \equiv \frac{1}{t}\sum_1^t \log R_i$, and A_t and B_t are the disjoint events "$G_{M,t} > \psi$ and $G_{M,t} + G_{R,t} > \phi$" and "$G_{M,t} < \psi$ and $G_{M,t} + G_{R,t} > \phi$".

When $\phi > 0$, $\mathbb{P}(A_t)+\mathbb{P}(B_t)$ tends to zero as $t \to \infty$. (To see this, note that $\mathbb{P}(A_t)+\mathbb{P}(B_t) = \mathbb{P}(M_1 \cdots R_t > e^{\phi t})$. Now pick arbitrary $\varepsilon > 0$. As a corollary of the first part of Result **??**, if we take T large enough that $e^{\phi T} > 1/\varepsilon$, then $\mathbb{P}(M_1 \cdots R_t > e^{\phi t}) < \varepsilon$ for all $t > T$. That is, $\mathbb{P}(A_t) + \mathbb{P}(B_t) \to 0$.) Since $\mathbb{P}(A_t) + \mathbb{P}(B_t)$ tends to zero, $\mathbb{P}(A_t)$ and $\mathbb{P}(B_t)$ must each tend to zero.

The goal is now to analyze the rates at which $\mathbb{P}(A_t)$ and $\mathbb{P}(B_t)$ tend to zero. We will have $P_t(\phi,\psi) \to 1$ if $\mathbb{P}(B_t)$ tends to zero at a faster rate than $\mathbb{P}(A_t)$, and conversely $P_t(\phi,\psi) \to 0$ if $\mathbb{P}(A_t)$ tends to zero faster than $\mathbb{P}(B_t)$. So we must find a condition that ensures that $\mathbb{P}(B_t)$ tends to zero faster than $\mathbb{P}(A_t)$ in the sense that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(\overline{B}_t) \leq \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}(A_t). \tag{2}$$

For technical reasons, we work with $\overline{B}_t$, the closure of B_t, i.e. the event "$G_{M,t} \leq \psi$ and $G_{M,t} + G_{R,t} \geq \phi$". The argument for the converse condition, which ensures that $\mathbb{P}(A_t) \to 0$ faster than $\mathbb{P}(B_t) \to 0$, is very similar, so is omitted.

Let $\kappa^*(x_M, x_R) \equiv \sup_{\theta_M,\theta_R \in \mathbb{R}} x_M\theta_M + x_R\theta_R - \kappa(\theta_M,\theta_R)$, the Fenchel-Legendre transform of $\kappa(\cdot,\cdot)$. By the Gärtner-Ellis theorem,[1] the inequality (2) holds if

$$\inf_{\substack{x_M > \psi \\ x_M + x_R \geq \phi}} \kappa^*(x_M, x_R) \leq \inf_{\substack{x_M \leq \psi \\ x_M + x_R \geq \phi}} \kappa^*(x_M, x_R)\,. \tag{3}$$

The function κ^* has the following properties:

(i) it is convex (by Lemma 2.3.9 of Dembo and Zeitouni (1998, p. 46));

(ii) $\kappa^*(x_M, x_R) \geq 0$ (since it is at least as large as $x_M \cdot 0 + x_R \cdot 0 - \kappa(0,0) = 0$);

(iii) $\kappa^*(x_M, x_R) \geq x_M + x_R$ (since it is at least as large as $x_M \cdot 1 + x_R \cdot 1 - \kappa(1,1) = x_M + x_R$);

(iv) $\kappa^*(\mu_M, \mu_R) = 0$ where $\mu_M \equiv \kappa_M(0,0)$ and $\mu_R \equiv \kappa_R(0,0)$, so κ^* attains its global minimum at (μ_M, μ_R).

From (iii) and (iv), $\mu_M + \mu_R \leq 0$, so $(\mu_M, \mu_R) \notin \{(x_M, x_R) : x_M + x_R \geq \phi\}$. It follows by convexity that κ^* attains its minimum over $\{(x_M, x_R) : x_M + x_R \geq \phi\}$ on the boundary of the set, i.e. on the line $\{(x_M, x_R) : x_M + x_R = \phi\}$. The question is then whether the minimum is attained for x_M greater than ψ or less than ψ. Setting $f(x) \equiv \kappa^*(x, \phi - x)$, (3) is satisfied if $f'(\psi) < 0$, or equivalently $\kappa^*_M(\psi, \phi - \psi) < \kappa^*_R(\psi, \phi - \psi)$, where κ^*_M denotes the derivative of κ^* with respect to its first argument, and similarly for κ^*_R. The result follows by the envelope theorem. □

For comparison with the other results of the paper, consider an example in which $\kappa(\theta_M, \theta_R) = \mu_M \theta_M + \mu_R \theta_R + \sigma_{MM}\theta_M^2/2 + \sigma_{MR}\theta_M\theta_R + \sigma_{RR}\theta_R^2/2$. This occurs if (but not only if) the vector $(\log M_t, \log R_t)$ is i.i.d. bivariate Normal with mean (μ_M, μ_R) and covariance matrix $\left(\begin{smallmatrix} \sigma_{MM} & \sigma_{MR} \\ \sigma_{MR} & \sigma_{RR} \end{smallmatrix}\right)$. The fundamental asset pricing equation imposes $\kappa(1,1) = 0$, which implies in this case that $\mu_M + \sigma_{MM}/2 + \mu_R + \sigma_{RR}/2 = -\sigma_{MR}$.

[1] See Dembo and Zeitouni (1998, p. 44) for a proof of the theorem. The simplified version of the theorem outlined in Remark (c) (p. 45) suffices, due to the assumption that $\kappa(\theta_M, \theta_R) < \infty$ for all $\theta_M, \theta_R \in \mathbb{R}$. For another application of the Gärtner-Ellis theorem in finance, see Stutzer (2003).

Applying Result 6, we find that $P_t(\phi, \psi) \to 1$ if

$$\left(\frac{\sigma_{MM} + \sigma_{MR}}{\sigma_{MM} + 2\sigma_{MR} + \sigma_{RR}}\right)\phi - \mu_R - \sigma_{RR}/2 - \sigma_{MR}/2 > \psi\,.$$

Bad news is strongly dominant if this inequality holds in the limit as ψ and ϕ tend to infinity with $\psi = \phi$, i.e. if

$$\frac{\sigma_{MM} + \sigma_{MR}}{\sigma_{MM} + 2\sigma_{MR} + \sigma_{RR}} > 1,$$

or equivalently, if $\sigma_{MR} + \sigma_{RR} < 0$. Bad news is weakly dominant if the inequality holds in the limit as ψ and ϕ tend to infinity with $\psi = \phi/2$, i.e. if

$$\frac{\sigma_{MM} + \sigma_{MR}}{\sigma_{MM} + 2\sigma_{MR} + \sigma_{RR}} > 1/2,$$

or equivalently, if $\sigma_{MM} > \sigma_{RR}$.

In the special case considered in the introduction, the log return and log SDF were perfectly correlated. The condition for bad news to be strongly dominant simplifies to $\lambda > \sigma$, and the condition for bad news to be weakly dominant simplifies to $|\lambda| > \sigma$. This second condition allows for the possibility that the asset is an insurance asset with negative Sharpe ratio.

Bibliography

Dembo, Amir, and Zeitouni, Ofer. 1998. *Large Deviations Techniques and Applications*. New York: Springer.

Hansen, Lars Peter. 2011. "Dynamic Valuation Decomposition Within Stochastic Economies." *Econometrica*, forthcoming.

Stutzer, Michael. 2003. "Portfolio Choice with Endogenous Utility: A Large Deviations Approach." *Journal of Econometrics* 116: 365–386.