The Lucas Orchard

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A Lucas orchard is a collection of Lucas trees

There is more comovement across markets at the level of returns than there is at the level of fundamentals—Shiller (1989) and others

A representative agent with power utility consumes the fruits not of just one fruit tree, but of the entire orchard

I investigate the properties of

- Prices
- Expected returns
- Yield curve

What changes with more than one tree? (1)

Consider the following simple example:

- A rational individual holds two assets in 50:50 proportions

Suppose asset 1 does well

- Higher proportion of wealth is now in asset 1—say, 70:30
- If expected returns/ risk don't change, the individual has no reason to change from original optimized 50:50 portfolio. . .
- . . . so rebalances: sells outperformer to buy underperformer

What changes with more than one tree? (2)

This story doesn't work in the aggregate!

- Expected returns/ risk must change in order to leave the representative investor happy to hold asset 1 as a larger proportion of wealth
- Imposing equilibrium leads to interesting dynamics

In this paper we argue for the importance of explicit recognition of the essential interdependences of markets in theoretical and empirical specifications of financial models. . .
—Brainard and Tobin (AER, 1968)

A correlation decomposition (1)

Ammer and Mei (1996) do a Campbell decomposition to investigate links between US and UK stock markets

$$\begin{aligned} \widetilde{US} &= \widetilde{US}_{CF} - \widetilde{US}_{DR} \\ \widetilde{UK} &= \widetilde{UK}_{CF} - \widetilde{UK}_{DR} \end{aligned}$$

- $\widetilde{US}$: unexpected return on US stock market
- $\widetilde{UK}$: unexpected return on UK stock market
- $\widetilde{US}_{CF}, \widetilde{UK}_{CF}$: US, UK cashflow news
- $\widetilde{US}_{DR}, \widetilde{UK}_{DR}$: US, UK discount rate news

A correlation decomposition (2)

corr	$\widetilde{US}_{CF}$	$\widetilde{US}_{DR}$
$\widetilde{UK}_{CF}$	0.30	0.36
$\widetilde{UK}_{DR}$	-0.23	0.60

Table: Correlations between news variables, 1957:1–1989:12, backed out from Ammer and Mei (1996).

Roadmap

- Theory
- Intuition
- Some empirics
- The small asset limit

Setup (1)

Tastes: the representative investor

Power utility with risk aversion γ and time preference rate ρ

$$\max \mathbb{E} \int_{t=0}^{\infty} e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} \, dt$$

Setup (2)

Technology: dividend processes

- Two assets ("countries") with dividend streams D_{1t}, D_{2t}
- Dividend growth is i.i.d. over time, but may be correlated across assets; formally, log dividends follow a Lévy process
  - Dividend growth is hard to forecast; i.i.d. assumption hard-wires this into the model—like Campbell-Cochrane (1999)
- The framework can handle situations in which, for example,
  - Dividends follow geometric Brownian motions, or
  - Dividends are subject to occasional disasters, or
  - A combination of both, or many other possibilities

Setup (3)

Technology: dividend processes

The technological side of the model (dividend growth) is summarized in *one* object: the cumulant-generating function, $\mathbf{c}(\theta_1, \theta_2)$, defined by

$$\mathbf{c}(\theta_1, \theta_2) = \log \mathbb{E}\left[\left(\frac{D_{1,t+1}}{D_{1,t}}\right)^{\theta_1}\left(\frac{D_{2,t+1}}{D_{2,t}}\right)^{\theta_2}\right]$$

- Example: if the two dividend streams are independent geometric Brownian motions,

$$\mathbf{c}(\theta_1, \theta_2) = \mu_1\theta_1 + \mu_2\theta_2 + \frac{1}{2}\sigma_1^2\theta_1^2 + \frac{1}{2}\sigma_2^2\theta_2^2$$

Setup (4)

Closing the model

- The representative investor holds the market

$$C_t = D_{1t} + D_{2t}$$

Simple inputs

- Representative agent, power utility, consumption $C_t = D_{1t} + D_{2t}$
- i.i.d. dividend growth

Complicated, interesting, empirically relevant outputs

- Price-dividend ratios
- Excess returns
- Riskless rate
- Yield curve
- Volatilities
- Correlation between assets

Related papers (1)

- Lucas (JME, 1982): finds Euler equation in two-country case
- Cole-Obstfeld (JME, 1991): simple representation of uncertainty, focus on welfare calculations, no analytical results except in Cobb-Douglas case
- Brainard-Tobin (OEP, 1992): similar to Cole-Obstfeld—"Our illustrative model is simple and abstract; nevertheless it is not easy to analyze, and numerical simulations will be used"
- Menzly-Santos-Veronesi (JPE, 2004), Santos-Veronesi (RFS, 2005): dividend processes are reverse-engineered to make the model easy to solve
- Kyle-Xiong (JF, 2001): log utility + irrational traders
  Kodres-Pritske (JF, 2002): CARA utility, two periods
  Pavlova-Rigobon (2006, 2007): log-linear utility

Related papers (2)

Most closely related paper is Cochrane, Longstaff & Santa-Clara's "Two Trees" (RFS, 2008). Different solution technique allows me to extend in various directions

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- I solve for the term structure of interest rates
- (Imperfect substitution between goods: $C_t = \left(D_{1t}^{\chi} + D_{2t}^{\chi}\right)^{1/\chi}$)

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- (Imperfect substitution between goods: $C_t = \left(D_{1t}^{\chi} + D_{2t}^{\chi}\right)^{1/\chi}$)
- ("Two Trees, Two Agents")

One asset is easy

Define the CGF $\mathbf{c}(\theta) = \log \mathbb{E}\left[(D_{t+1}/D_t)^\theta\right]$

$$\begin{aligned} P_0 &= \int_{t=0}^{\infty} \mathbb{E}\left[e^{-\rho t}\left(\frac{D_t}{D_0}\right)^{-\gamma} \cdot D_t\right] dt \\ &= D_0 \int_{t=0}^{\infty} e^{-[\rho - \mathbf{c}(1-\gamma)]t}\, dt \\ &= \frac{D_0}{\rho - \mathbf{c}(1-\gamma)} \end{aligned}$$

- Dividend yield $D_0/P_0 = \rho - \mathbf{c}(1-\gamma)$
- Riskless rate $\rho - \mathbf{c}(-\gamma)$
- Risk premium $\mathbf{c}(1) + \mathbf{c}(-\gamma) - \mathbf{c}(1-\gamma)$

Why so difficult with two assets?

Lucas's Euler equation: $P_{1,0} = \mathbb{E} \int_0^\infty e^{-\rho t} \left(\frac{C_t}{C_0} \right)^{-\gamma} \cdot D_{1,t} \, dt$

Consider the log utility case:

$$\begin{aligned} P_{1,0} &= (D_{10} + D_{20}) \int_0^\infty e^{-\rho t} \, \mathbb{E} \left[\frac{D_{1t}}{D_{1t} + D_{2t}} \right] dt \\ &= (D_{10} + D_{20}) \int_0^\infty e^{-\rho t} \mathbb{E} \left[\frac{1}{1 + D_{2t}/D_{1t}} \right] dt \end{aligned}$$

- The expectation is hard (eg, consider lognormal D_{2t}/D_{1t})

A suggestive special case (1)

Suppose $D_{1t} \equiv 1$ and D_{2t} is always smaller than 1 (eg asset 2 is subject to random *downward* jumps at random times). Then,

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{1+D_{2t}}\right] &= \mathbb{E}\left[1 - D_{2t} + D_{2t}^2 - \ldots\right] \\
&= \sum_{n=0}^{\infty} (-1)^n D_{20}^n \, \mathbb{E}\left[\left(\frac{D_{2t}}{D_{20}}\right)^n\right] \\
&= \sum_{n=0}^{\infty} (-1)^n D_{20}^n e^{\mathbf{c}(0,n)t}
\end{aligned}$$

A suggestive special case (2)

Substituting back, we find that

$$P_{1,0}/D_{1,0} = \frac{1}{\sqrt{s(1-s)}} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1-s}{s}\right)^{n+1/2}}{\rho - \mathbf{c}(0,n)} ,$$

where the state variable

$$s \equiv \frac{D_{1,0}}{D_{1,0} + D_{2,0}}$$

is the *dividend share* of asset 1, which by assumption starts out greater than 0.5 and increases towards 1 over time

Solution method (1)

Four tricks are needed to handle the general case

$$\mathbb{E}\left[\frac{D_{1t}}{(D_{1t}+D_{2t})^{\gamma}}\right]$$

Cumulant-generating function

- CGFs are convex—used in many proofs

Symmetrize

- Preparing for the Fourier transform

Change of measure

- Takes care of the dividend; not like the usual "risk-neutral" or "martingale" measure which handles stochastic discount factor

Fourier transform

- Analog of "geometric series" trick that worked in special case

Solution method (2)

Prices: The price-dividend ratio of asset 1 is

$$\frac{P_1}{D_1}(s) = \frac{1}{\sqrt{s^\gamma (1-s)^\gamma}} \int_{-\infty}^{\infty} \frac{\mathscr{F}_\gamma(v) \left(\frac{1-s}{s}\right)^{iv}}{\rho - \mathbf{c}(1-\gamma/2 - iv, -\gamma/2 + iv)} \, dv$$

where

$$\mathscr{F}_\gamma(v) \equiv \frac{1}{2\pi} \cdot B(\gamma/2 + iv, \gamma/2 - iv) \qquad (B \text{ is the } \textit{beta function})$$

- Similar "integral formulas" for expected returns
- Can be evaluated numerically—effectively instantly—or analytically in special cases
- In the one-tree case, we had $P_1/D_1 = 1/[\rho - \mathbf{c}(1-\gamma)]$

Interest rates: The yield to time T, $\mathscr{Y}(T)$, is

$$-\frac{1}{T}\log\left\{\frac{1}{\sqrt{s^\gamma(1-s)^\gamma}}\int_{-\infty}^{\infty}\mathscr{F}_\gamma(v)\left(\frac{1-s}{s}\right)^{iv}e^{-[\rho-\mathbf{c}(-\gamma/2-iv,-\gamma/2+iv)]T}\,dv\right\}.$$

The instantaneous riskless rate is

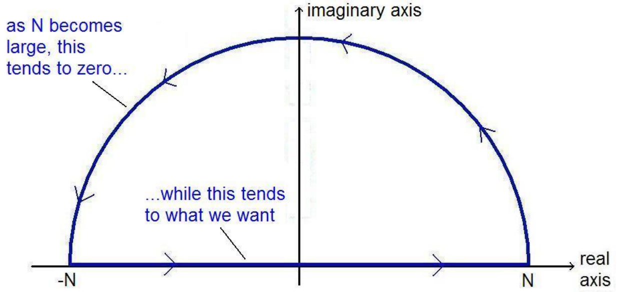
$$\frac{1}{\sqrt{s^\gamma(1-s)^\gamma}}\int_{-\infty}^{\infty}\mathscr{F}_\gamma(v)\left(\frac{1-s}{s}\right)^{iv}[\rho-\mathbf{c}(-\gamma/2-iv,-\gamma/2+iv)]\,dv\,.$$

The long rate is a constant, independent of the current state s:

$$\mathscr{Y}(\infty)=\max_{\theta\in[-\gamma/2,\gamma/2]}\rho-\mathbf{c}(-\gamma/2+\theta,-\gamma/2-\theta).$$

- Symmetric case: $\mathscr{Y}(\infty)=\rho-\mathbf{c}(-\gamma/2,-\gamma/2)$

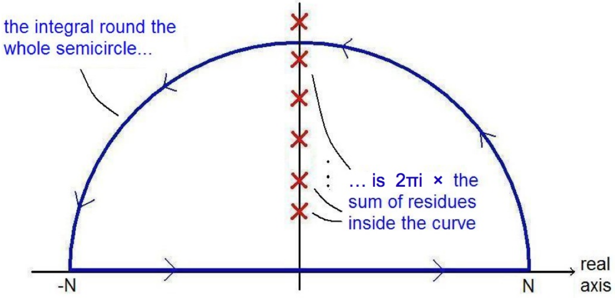
Solution method (3)



Can solve these integrals analytically. . .

- . . . if dividends follow geometric Brownian motion
- . . . or in small asset limit
- "The shortest path between two truths in the real domain passes through the complex domain." (Jacques Hadamard)

Solution method (4)



Solution uses Cauchy's residue theorem

- Residue of $f(\cdot)$ at a: coefficient on $(z-a)^{-1}$ in a series expansion of a function $f(z)$ at a point a where $f(a)=\infty$
- In GBM case, sum of the residues takes on a neat form
- For a very small asset, only nearest residue to real axis matters

The Brownian motion case

$$P/D_1(s) = \frac{1}{B(\lambda_1-\lambda_2)}\left[\frac{1}{(\gamma/2+\lambda_1)\,s^\gamma}\,F\left(\gamma,\frac{\gamma}{2}+\lambda_1;1+\frac{\gamma}{2}+\lambda_1;\frac{s-1}{s}\right)\right.$$
$$\left.+\frac{1}{(\gamma/2-\lambda_2)\,(1-s)^\gamma}\,F\left(\gamma,\frac{\gamma}{2}-\lambda_2;1+\frac{\gamma}{2}-\lambda_2;\frac{s}{s-1}\right)\right]$$

where B,λ_1,λ_2 are constants determined by ρ,γ and dividend processes, and $F(a,b;c;z)$ is Gauss's hypergeometric function

- This expression generalizes the result of Cochrane, Longstaff and Santa-Clara (2008) to allow for $\gamma \geq 1$

A calibration

I'll show some pictures using the following parameter values in an economy with two independent but fundamentally identical assets

time preference rate, ρ	0.03
risk aversion, γ	4
mean log dividend growth	2%
dividend volatility	10%

Implied moments of consumption growth

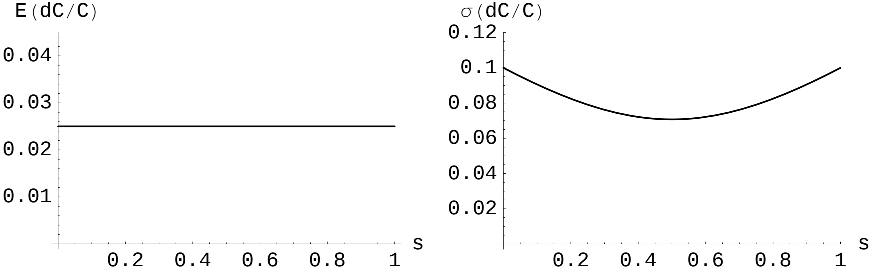


Figure: Left: Mean consumption growth against dividend share, s. Right: standard deviation of consumption growth against dividend share, s.

- In the middle, there is a diversification benefit.
  At the edges, all eggs are in one basket

The riskless rate

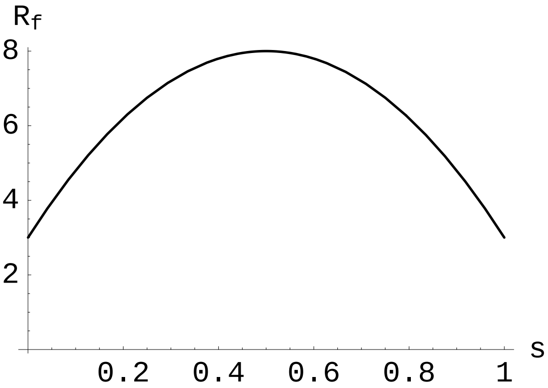


Figure: The riskless rate plotted against s.

- The riskless rate is highest at intermediate values of the dividend share because the diversification effect lowers precautionary saving

Excess returns

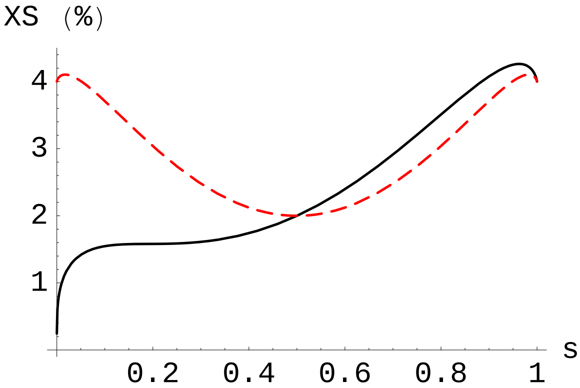


Figure: Excess returns on asset 1 and aggregate market.

- When asset 1 has a small share, it has low correlation with overall consumption—that is, low risk

Price-dividend ratio

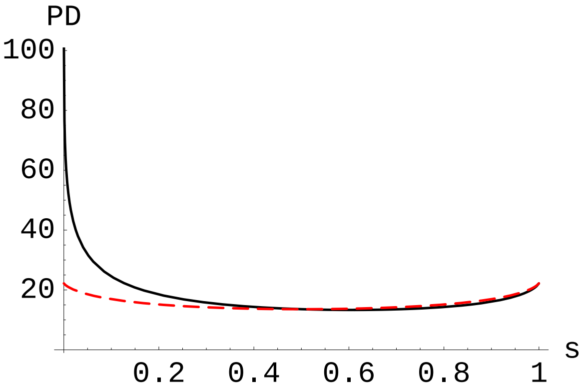


Figure: P/D for asset 1 and aggregate market.

- All else equal, small assets have low risk and high valuations
- The extreme case $s \downarrow 0$ is of particular interest

A value-growth effect

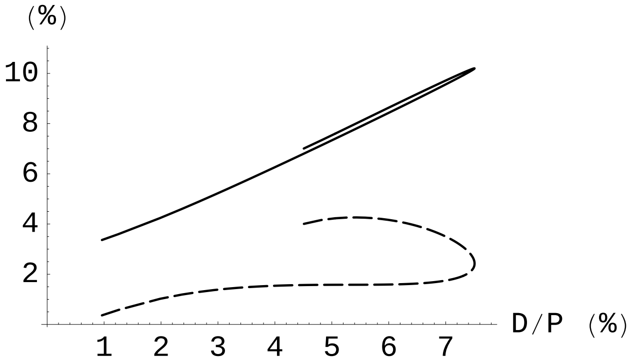


Figure: Expected returns and risk premia against dividend yield

- High D/P is associated with high expected returns (solid) and high expected excess returns (dashed)

The yield spread and excess returns on bonds

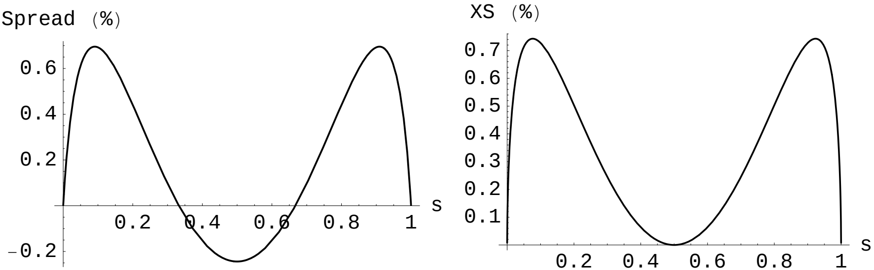


Figure: Left: The yield spread. Right: The risk premium on a perpetuity.

- High yield spreads forecast high excess returns on bonds and on the market

Comovement (1)

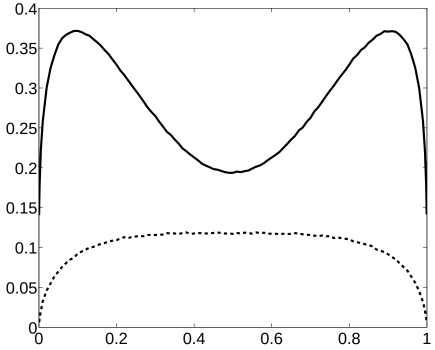


Figure: Correlation between returns of assets 1 and 2. Dotted: $\gamma = 1$.

- The correlation is strikingly high, given that the two assets have independent fundamentals

Comovement (2)

Intuition:

- Suppose asset 2 has good news
- Asset 2 is now a larger share of consumption
- Asset 1 is a smaller share of consumption
- Asset 1 requires a lower expected return
- So asset 1 *also* appreciates now, and will earn a lower expected return in future

Comovement (3)

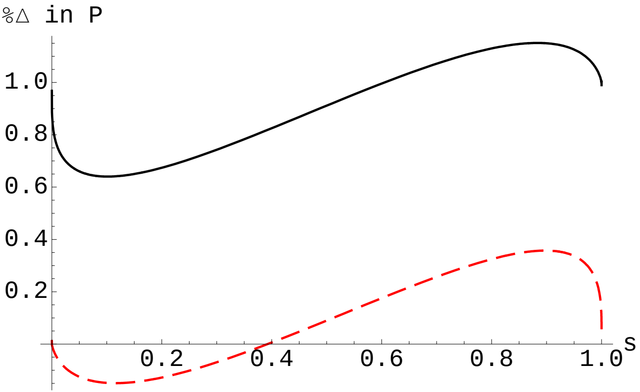


Figure: The response of asset 1 and asset 2 to a +1% increase in the dividend of asset 1.

Excess volatility (1)

$$P = D \times \frac{P}{D}$$

Lucas one-tree model: P/D is constant $\longrightarrow$ volatility puzzle

- Here, P/D is time-varying while dividends are unforecastable—we can expect to see excess volatility

Excess volatility (2)

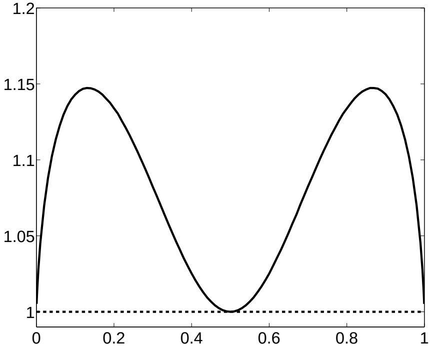


Figure: Ratio of market's return vol to its dividend vol. Dotted: $\gamma = 1$.

- This is a failure of the model: not enough excess volatility

Disasters (1)

Now consider a calibration, based on Barro (2007), which is intended to highlight the effects of disasters

- Preferences as before, $\rho = 0.03, \gamma = 4$
- 2% mean log dividend growth, 2% volatility
- Disasters hit each asset at rate 0.017: shock log dividend by $N(-0.38, 0.25^2)$
- When a disaster occurs, with probability 0.5 it hits both assets simultaneously

Disasters (2)

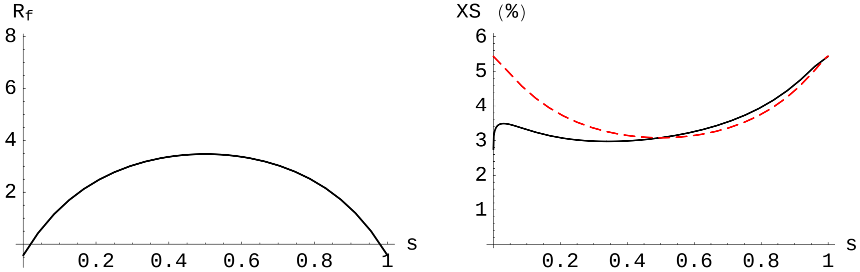
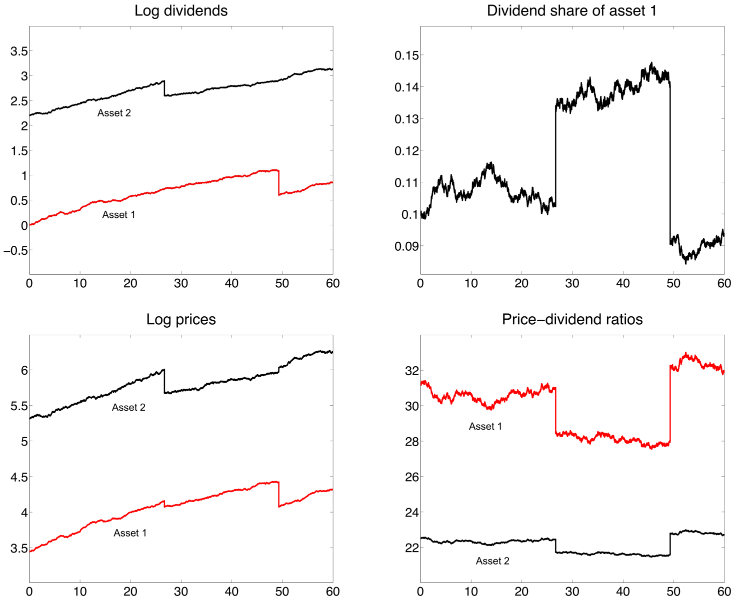


Figure: Left: The riskless rate. Right: Risk premia.

- Disasters can make the riskless rate and equity premium puzzles go away

Disasters (3)



Log dividends
Asset 2
Asset 1
Dividend share of asset 1
Log prices
Asset 2
Asset 1
Price−dividend ratios
Asset 1
Asset 2

Disasters (4)

When a large asset suffers a disaster

Excess return on other asset $\uparrow$, riskless rate $\uparrow\uparrow$

$\Longrightarrow$ other asset's price drops: contagion

When a small asset suffers a disaster

Excess return on other asset $\uparrow$, riskless rate $\downarrow\downarrow$

$\Longrightarrow$ other asset's price increases: flight-to-quality

- Contagion effect is more robust than flight-to-quality effect
- Another source of high risk premia even for assets with stable fundamentals
- Flight-to-quality effect is weaker with more assets

Disasters (5)

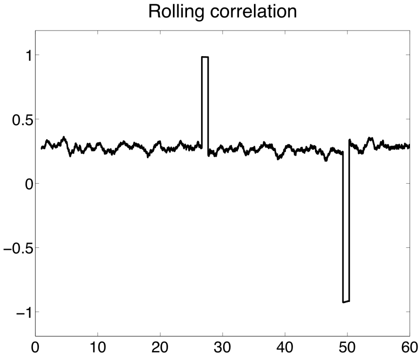


Figure: One-year rolling correlation between assets 1 and 2.

- Correlation spikes endogenously at times of crisis
- $+1$: "contagion". -1: "flight-to-quality"

A correlation decomposition (1)

corr	$\widetilde{US}_{CF}$	$\widetilde{US}_{DR}$
$\widetilde{UK}_{CF}$	0.30	0.36
$\widetilde{UK}_{DR}$	-0.23	0.60

Table: Correlations between news variables, 1957:1–1989:12, backed out from Ammer and Mei (1996).

- Considerable amount of comovement is purely due to comovement in discount rates (bottom right)
- There is an interesting asymmetry in the signs of off-diagonals

A correlation decomposition (2)

- I simulate a three-asset world: US, UK, rest of world
- Dividend volatility of each asset is 11.2%
- In the model, cashflow news = dividend growth
- Pairwise dividend growth correlations are 0.30
- Starting shares: US = 0.4, UK = 0.1, rest of world = 0.5

A correlation decomposition (3)

corr	$\widetilde{US}_{CF}$	$\widetilde{US}_{DR}$	corr	$\widehat{US}_{CF}$	$\widehat{US}_{DR}$
$\widetilde{UK}_{CF}$	0.30	0.36	$\widehat{UK}_{CF}$	0.30	0.26
				(0.06)	(0.10)
$\widetilde{UK}_{DR}$	−0.23	0.60	$\widehat{UK}_{DR}$	−0.14	0.71
				(0.24)	(0.13)

Table: Left: Ammer-Mei estimated correlations. Right: means and (standard deviations) of correlations in simulations.

- Model is calibrated to match top-left correlation (0.30)
- Other correlations emerge endogenously

Momentum across size portfolios

Data	s.d.(Nr)	s.d.(Ncf)	N-Corr	Model	s.d.(Nr)	s.d.(Ncf)	N-Corr
Small	0.202	0.409	0.575	Tree 1	0.182	0.409	0.928
4	0.126	0.289	0.348	Tree 2	0.109	0.286	0.845
7	0.094	0.227	0.226	Tree 3	0.069	0.227	0.649
Big	0.063	0.179	0.026	Tree 4	0.026	0.180	0.021

Table: Variance decomposition as a function of firm size. Left panel: standard deviations and correlations from Vuolteenaho (2002). Right panel: values generated by model.

- $\rho = 0.03$; $\gamma = 2$; all trees have mean log dividend growth $\mu = 0.02$
- Volatility of dividend growth (middle column) differs across trees, chosen to match Vuolteenaho's cashflow volatilities
- Left and right columns are determined endogenously

Small assets (1)

To what extent do these interesting dynamics apply to very small assets, as $s \downarrow 0$? This limit is interesting because

- It allows us to think about emerging technologies
- It is the polar opposite of one-tree, constant-P/D case
- And... does all this have any impact on individual assets?

In the limit as $s \downarrow 0$, can get simple closed-form solutions—and an interesting phenomenon emerges

Small assets (2)

Step back from this particular model. How do you price a tiny, idiosyncratic asset with i.i.d. dividend growth?

- tiny, idiosyncratic $\implies$ discount at riskless rate
- Gordon GM: $D/P =$ riskless rate $-$ mean dividend growth

So, if riskless rate is 6% and mean dividend growth on the tiny asset is 4%, GGM gives $D/P = 6\% - 4\% = 2\%$

- With these parameters, this logic is correct
- But what if the riskless rate is only 2%? GGM breaks down!

Small assets (3)

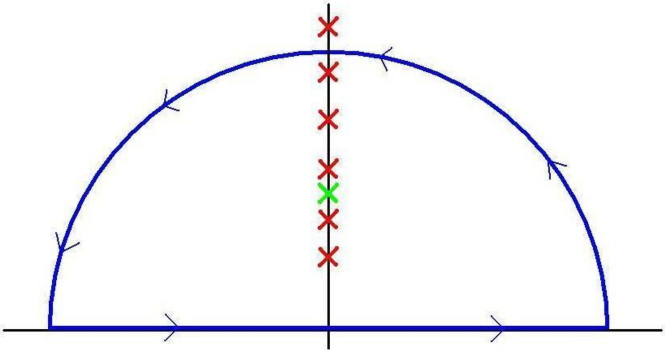


Figure: Impatient: high ρ. "Subcritical" case.

$$\frac{P_1}{D_1}(s) = \frac{1}{\sqrt{s^\gamma(1-s)^\gamma}} \int_{-\infty}^{\infty} \frac{\mathscr{F}_\gamma(v)\left(\frac{1-s}{s}\right)^{iv}}{\rho - \mathbf{c}(1-\gamma/2 - iv, -\gamma/2 + iv)}\, dv$$

Small assets (3)

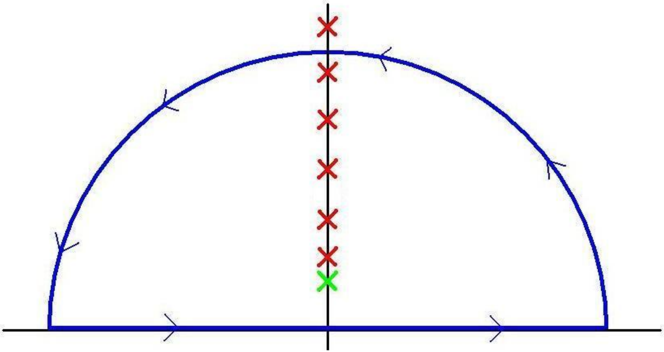


Figure: Patient: low ρ. "Supercritical" case.

$$\frac{P_1}{D_1}(s) = \frac{1}{\sqrt{s^\gamma (1-s)^\gamma}} \int_{-\infty}^{\infty} \frac{\mathscr{F}_\gamma(v) \left(\frac{1-s}{s}\right)^{iv}}{\rho - \mathbf{c}(1-\gamma/2 - iv, -\gamma/2 + iv)}\, dv$$

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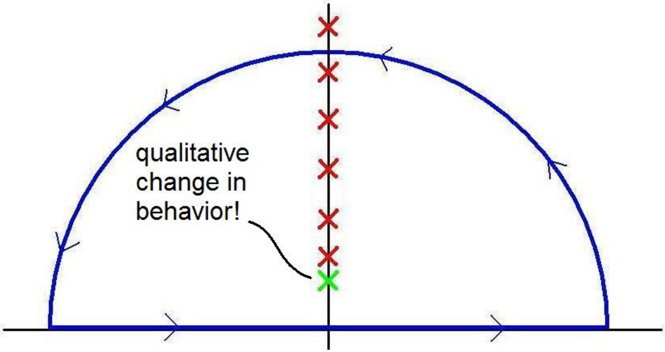

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Small assets (4)

Subcritical—mean dividend growth < real interest rate

When riskless rate − mean dividend growth > 0, we have

$$\begin{aligned} D/P_1 &\rightarrow \text{riskless rate} - \text{mean dividend growth} \\ XS_1 &\rightarrow 0 \end{aligned}$$

- The small asset earns the riskless rate and is priced using GGM

Small assets (5)

Subcritical—mean dividend growth $<$ real interest rate

- $\rho = 0.05, \gamma = 4, \mu = 0.02, \sigma = 0.1$
- Aggregate dividend $D_1 + D_2$ is normalized to 1

D_1	P_1	P/D_1	XS_1 (%)
0.1	1.72	17.2	1.29
0.01	0.288	28.8	0.60
0.001	0.0356	35.6	0.22
0.0001	0.00384	38.4	0.08
0.00001	0.000394	39.4	0.03
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	0	40	0

Small assets (6)

Supercritical—mean dividend growth > real interest rate

When riskless rate − mean dividend growth < 0, the Gordon growth model breaks down. But the asset's price is still well-defined, and we have

$$\begin{aligned} D/P_1 &\rightarrow 0 \\ XS_1 &\rightarrow \mathbf{c}(0,\theta^*) + \mathbf{c}(0,-\gamma) - \mathbf{c}(0,\theta^*-\gamma) > 0 \end{aligned}$$

where $\theta^* \in (0,1)$ is uniquely determined by preferences—ρ, γ—and technology—$\mathbf{c}(\cdot)$—via

$$\rho - \mathbf{c}(1-\theta^*, \theta^* - \gamma) = 0$$

Small assets (7)

Supercritical—mean dividend growth > real interest rate

- $\rho = 0.01, \gamma = 4, \mu = 0.02, \sigma = 0.1$
- Aggregate dividend $D_1 + D_2$ is normalized to 1

D_1	P_1	P/D_1	XS_1 (%)
0.1	4.03	40.3	1.97
0.01	1.28	128.5	1.80
0.001	0.33	332.4	1.52
0.0001	0.077	765.8	1.39
0.00001	0.017	1674.2	1.33
⋮	⋮	⋮	⋮
0	0	∞	1.28

Small assets (8)

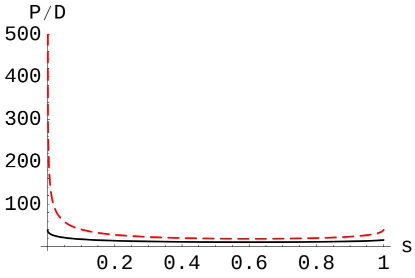

Figure: P/D and excess returns in the subcritical ($\rho = 0.05$) and supercritical ($\rho = 0.01$) cases

- Why the positive excess returns? Even tiny, totally idiosyncratic assets can comove in equilibrium!
- Expected returns are entirely due to expected capital gains in supercritical case

Small assets (9)

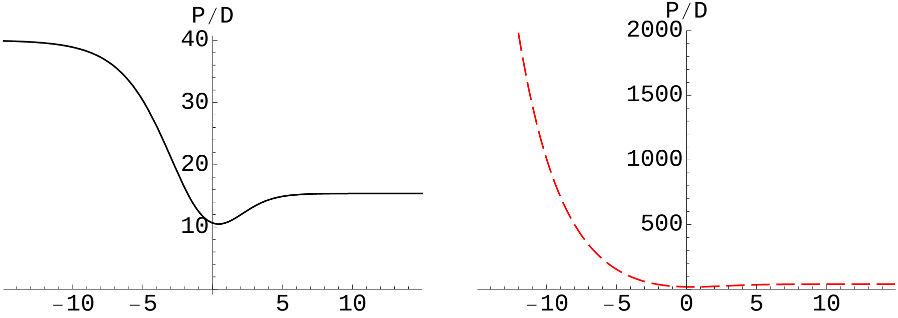

Figure: P/D in the subcritical ($\rho = 0.05$) and supercritical ($\rho = 0.01$) cases

- P/D against a rescaled state variable which moves at constant (expected) "speed" on $(-\infty, \infty)$

Small assets (10)

In subcritical case, can analyze behavior *near* the small-asset limit, $s \approx 0$. To first order in s, we have

$$\begin{aligned} P/D &= A - B \cdot s^{\alpha} \\ \text{riskless rate} &= C + D \cdot s \\ \text{excess return} &= E + F \cdot s^{\alpha} \end{aligned}$$

where A, B, C, D, E, F and $\alpha \in (0, 1)$ are constants determined by preferences—ρ, γ—and the technological environment—$\mathbf{c}(\cdot)$

- When s is tiny, s^{α} is much bigger than s: much more movement in excess returns and P/D than in riskless rate
- In the time series, we have $P/D = G - H \cdot \text{excess return}$

Small assets (11)

Cochrane (in *Asset Pricing*, p. 400):

> *It is nonetheless an uncomfortable fact that almost all variation in price/dividend ratios is due to variation in expected excess returns. How nice it would be if high prices reflected expectations of higher future cashflows. Alas, that seems not to be the case. If not, it would be nice if high prices reflected lower interest rates. Again, that seems not to be the case. High prices reflect low risk premia, lower expected excess returns.*

Conclusions

- Comovement is a robust feature of the neoclassical model
- Betas are determined endogenously
- Disasters spread across assets
- Return correlations spike endogenously at times of disaster
- Interesting pricing effects even for tiny, idiosyncratic assets

Methodology allows extensions to:

- Imperfect substitution between goods
- Cole-Obstfeld (1991) with jumps + non-Cobb-Douglas case
- Heterogeneous agents
- "Two Trees, Two Agents"